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Some remarks about the thermoelastic theory of materials with voids

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RIASSUNTO – *Si formula una teoria per materiali termoelastici porosi che include tra le variabili costitutive la derivata temporale della funzione "grado di vuoto". Per questa teoria si dimostra un teorema di unicitá e per evidenziare la differenza con le teorie proposte precedentemente si studia la propagazione di onde di accelerazione di dilatazione.*

ABSTRACT – *In this paper we formulate a theory of thermoelastic materials with voids which includes voidage time derivative among the independent constitutive variables. A uniqueness theorem is obtained and to test the difference between this theory and the previous ones the growth and decay of dilatational acceleration waves is used.*

KEY WORDS – *Elasticity with voids - Thermoelasticity - Acceleration waves.*

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1 – Introduction

Structural materials for numerous applications in industry, aerospace and other areas of technology may be required to provide satisfactory service at elevated temperatures. For example this is the case of: process vessels and crucibles in the metal-working and chemical process industry, heat exchangers, devices such as turbine engines, components and structure for high-speed aerospace purposes and many others. In addition to high temperature the materials used in many of the above applications also are subjected to corrosion, erosion and mechanical loads. Obviously for long term performance, the materials selected for these applications should exhibit high melting point, good corrosion and erosion resistance, excellent retention of load-bearing capability as well as resistance to creep at high temperature. The only class of materials which can satisfy the above requirements is known as "structural ceramics". Such materials include the metal oxides, carbides, nitrides and silicides. The advantageous properties of these materials at high temperatures derive from the high strength of the bond (usually of ionic and/or covalent type) between atoms, but this type of atomic bonding also causes these ceramics materials to be exceedingly brittle [1].

On the other hand materials which operate at elevated temperatures will invariably be subjected to heat flow at some time during normal use. Such heat flow will involve a non-linear temperature distribution which will inevitably give rise to thermal stresses. This behaviour in combination with the structural properties renders the ceramics highly subscetible to thermal stress failure [1]. For these reasons, the development, design and selection of materials for high temperature applications requires a great deal of care. The role of the pertinent material properties and other variables which can affect the magnitude of thermal stress must be well understood and all possible mode of failure must be considered.

The purpose of this paper is to establish a linear thermoelastic theory of the elastic material with voids as a first step to a better understanding of thermal stress in ceramic materials. Obviously this theory can be useful also in other fields of application which deal with porous materials as geological materials, solid packed granular materials and many others.

The elastic theory of materials with voids has been proposed by COWIN and NUNZIATO [2,3] to deal with manufactured porous materials like ceramics and pressed powders non conductors of heat. The basic premise underlying this theory is the concept of a material for which the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This theory has been developed from the Goodman and Cowin theory of granular materials proposed in 1971 [4].

It is quite surprising that by means of a straightforward generalization of the concept of linear elastic body Cowin and Nunziato have been able to describe a material which can exhibit a wide variety of qualitative behaviors, which occur in real ceramics and are not foreseen by classical elasticity [5].

In 1986 IESAN [6] has proposed a theory of thermoelastic materials with voids, but in this theory he neglected the experimental evidence that changes in the volume fraction result in internal dissipation in the material [1, 7]. In this paper we improve the theory of Iesan by adding into the set of constitutive variables the time derivative of the voidage to include the inelastic effects.

The plan of the paper is the following. The non linear theory of thermoelastic materials with voids is described in section 2 in a form that is intended to be easy reading for someone familiar with the papers of Nunziato and Cowin. The process of specializing the non linear theory to the linear situation is described in section 3. In section 4 we consider the boundedness of solutions to establish the uniqueness to the mixed boundary-value problem and in section 4 we study the evolution of acceleration waves to show the difference with the theory of Iesan.

Throughout this paper we refer the motion of the continuum to a fixed system of rectangular Cartesian axes Ox_i $(i = 1,2,3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1,2,3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

2 – Non Linear Theory

As we have pointed out in the introduction the basic concept underlying the theory of the elastic materials with voids is that the bulk density $\bar{\rho}$ is written as the product of two fields, the density field of the matrix material γ and the volume fraction field ν :

$$\bar{\rho} = \gamma\nu . \tag{2.1}$$

This representation introduces an additional kinematical degree of freedom with respect to the theory of classical elasticity because γ and ν are independent [2,3]. Now we develop a thermoelastic nonlinear theory that will be manipulated in the next section to deal with small changes from a reference configuration of a porous body. In a reference configuration (2.1) can be written as $\rho = \gamma_R \nu_R$ and we assume here that γ_R and ν_R are spatially constant. This reference configuration, as it is customary, is assumed to be strain and stress free.

The independent variables are the displacement field $u_i(\mathbf{x}, t)$ from the reference configuration, the change in temperature from the reference absolute temperature T_0,

$$\theta(\mathbf{x}, t) = T(\mathbf{x}, t) - T_0 \tag{2.2}$$

and the change in volume fraction from the reference volume fraction:

$$\phi(\mathbf{x}, t) = \nu(\mathbf{x}, t) - \nu_R \, . \tag{2.3}$$

The fundamental equations are [2]:
Balance of linear and angular momentum

$$\rho \ddot{u}_i = [T_{kj}(\delta_{ij} + u_{i,j})]_{,k} + \rho b_i \quad , \quad T_{ij} = T_{ji} \, . \tag{2.4}$$

Balance of equilibrated force

$$\rho \overline{k\dot{\phi}} = h_{i,i} + g + \rho l \tag{2.5}$$

Balance of energy

$$\rho \dot{\epsilon} = T_{ij} \dot{E}_{ij} - [g - \frac{1}{2} \dot{k} \dot{\phi}] \dot{\phi} + h_i \dot{\phi}_{,i} + q_{i,i} + \rho s \tag{2.6}$$

Entropy inequality

$$\rho \dot{\eta} T \geq q_{i,i} - \frac{\theta_{,i} q_i}{T} + \rho s \, . \tag{2.7}$$

Here T_{ij} is the second Piola stress tensor, E_{ij} is the Green strain tensor, b_i is the body force vector, ρ the material density in the reference configuration, k the equilibrated inertia, h_i the equilibrated stress, g is the intrinsic equilibrated body force, l the extrinsic equilibrated body force, ϵ the internal energy per unit of mass, η the specific entropy, q_i the heat flux and s the extrinsic heat supply.

The equations (2,4) and (2.6) are analogous to the classical balance equations, the new balance of equilibrated forces (2.5) can be motivated by a variational argument [8]. Here we notice that this theory can also be viewed as a subcase of more complicated microstructure theory [9] which can explain in a more "rational" than "phenomenological" way the equation (2.5).

We restrict our attention to the theory of thermoelastic materials where the constitutive variables are E_{ij}, θ, $\theta_{,i}$, ϕ, $\phi_{,i}$, $\dot{\phi}$. The first constitutive variables characterize the elastic state and they are the variable used by Iesan to formulate his thermoelastic theory, it is the dependence of constitutive equations on the last variable, $\dot{\phi}$, that allow us to account of the well known inelastic effects associated with changes in volume of the voids.

Introducing the free energy function:

$$\psi = \epsilon - T\eta \,, \tag{2.8}$$

and using equation (2.6) we can rewrite the entropy inequality (2.7) as:

$$0 \geq \rho\dot{\psi} + \rho\eta\dot{\theta} - T_{ij}\dot{E}_{ij} + \left[g - \frac{1}{2}\dot{k}\dot{\phi}\right]\dot{\phi} - h_i\dot{\phi}_{,i} - \frac{\theta_{,i}q_i}{T} \,. \tag{2.9}$$

We can assume as in [2] that the equilibrated inertia k depends only on ϕ and then from (2.9) we obtain

$$\begin{aligned} 0 \geq & \left[\rho\frac{\partial\psi}{\partial\phi} + g - \frac{1}{2}\frac{\partial k}{\partial\phi}\dot{\phi}^2\right]\dot{\phi} + \left[\rho\frac{\partial\psi}{\partial E_{ij}} - T_{ij}\right]\dot{E}_{ij} + \\ & + \left[\rho\frac{\partial\psi}{\partial\phi_{,i}} - h_i\right]\dot{\phi}_{,i} + \rho\frac{\partial\psi}{\partial\dot{\phi}}\ddot{\phi} + \rho\left[\frac{\partial\psi}{\partial\theta} + \eta\right]\dot{\theta} + \rho\frac{\partial\psi}{\partial\theta_{,i}}\dot{\theta}_{,i} - \frac{\theta_{,i}q_i}{T} \,. \end{aligned} \tag{2.10}$$

The inequality (2.10) must be satisfied for all independent thermo-kinetic processes, i.e. for every choice of the functions E_{ij}, ϕ and θ. This fact implies some restrictions on the constitutive relations for the quantities ψ, T_{ij}, η, $h_{,i}$, g, $q_{,i}$ and k. If in an arbitrary point, $\mathbf{x}$, for an arbitrary value of time, t, we choose the values of the constitutive variables it is possible to construct a thermo-kinetic process such that $\dot{E}_{ij}$, $\ddot{\phi}$, $\dot{\phi}_{,i}$, $\dot{\theta}$, $\dot{\theta}_{,i}$ are arbitrary. Then to maintain (2.10) for all the values of the constitutive variables is necessary that the coefficient of $\dot{E}_{ij}$, $\ddot{\phi}$, $\dot{\phi}_{,i}$, $\dot{\theta}$, $\dot{\theta}_{,i}$ vanish obtaining the relations:

$$
\begin{aligned}
&T_{ij} = \rho\frac{\partial\psi}{\partial E_{ij}}, \qquad && h_i = \rho\frac{\partial\psi}{\partial\phi_{,i}}, \\
&\eta = -\frac{\partial\psi}{\partial\theta}, && \frac{\partial\psi}{\partial\dot{\phi}} = \frac{\partial\psi}{\partial\theta_{,i}} = 0.
\end{aligned}
\tag{2.11}
$$

Now the (2.10) is simplified into the following inequality:

$$0 \geq f\dot{\phi} - \frac{\theta_{,i}q_i}{T}, \tag{2.12}$$

where $f = \rho\frac{\partial\psi}{\partial\phi} + g - \frac{1}{2}\frac{\partial k}{\partial\phi}\dot{\phi}^2$ is the dissipation function which takes into account of the inelastic behavior of voids [3].

3 – Linear theory

We now deduce from the equations of section 2, the field equations of the linearized theory. We assume that the displacement gradients $u_{i,j}$ and ϕ, $\dot{\phi}$, $\phi_{,j}$, θ, $\theta_{,j}$ are sufficiently small that their squares can be neglected, then the strain tensor is approximated by the infinitesimal strain tensor, $\overline{E}_{ij}$ and the free energy ψ can be expanded in a multiple Taylor series about the reference configuration. We suppose the initial body to be: stress-free, with zero intrinsic equilibrated body force and zero heat flux rate. Then taking note of (2.11,4) we have the quadratic expression:

$$
\begin{aligned}
\rho\psi = {} & \frac{1}{2}\xi\phi^2 + \frac{1}{2}A_{ij}\phi_i\phi_j + \frac{1}{2}C_{ijkm}\overline{E}_{ij}\overline{E}_{km} - \frac{1}{2}a\theta^2 + B_{ij}\overline{E}_{ij}\phi + \\
& + d_i\phi\phi_{,i} + D_{ijk}\phi_{,i}\overline{E}_{jk} - a_i\phi_{,i}\theta - m\theta\phi - \beta_{ij}\theta\overline{E}_{ij}.
\end{aligned}
\tag{3.1}
$$

Using (2.11) we obtain the following linear constitutive equations for T_{ij}, h_i and η:

$$
\begin{aligned}
T_{ij} &= C_{ijrs}\overline{E}_{rs} + B_{ij}\phi + D_{ijk}\phi_{,k} - \beta_{ij}\theta, \\
h_i &= A_{ij}\phi_{,j} + D_{rsi}\overline{E}_{rs} + d_i\phi - a_i\theta, \\
\rho\eta &= \beta_{ij}\overline{E}_{ij} + a\theta + m\phi + a_i\phi_{,i}.
\end{aligned}
\tag{3.2}
$$

Here: $C_{ijrs} = C_{rsij} = C_{jirs}$ $\beta_{ij} = \beta_{ji}$, $D_{ijk} = D_{jik}$, $A_{ij} = A_{ji}$, $B_{ij} = B_{ji}$.

As in the theory of Nunziato and Cowin we assume for the dissipative function f the form:

$$f = -\omega\dot{\phi}\,, \tag{3.3}$$

where ω is a positive constant [3]. For the constitutive equation of the heat flux we assume the celebrated Fourier law:

$$q_i = k_{ij}\theta_j\,. \tag{3.4}$$

where k_{ij} is the thermal conductivity which is positive definite [10].

Now the inequality (2.12) reads:

$$0 \geq -\omega\dot{\phi}^2 - k_{ij}\frac{\theta_{,i}\theta_{,j}}{T}\,, \tag{3.5}$$

which is verified for all thermo-kinetic processes, and the constitutive equation for g is:

$$g = -B_{ij}\overline{E}_{ij} - \xi\phi - \omega\dot{\phi} - d_i\phi_{,i} + m\theta\,. \tag{3.6}$$

Moreover, in this framework, the energy equation is reduced to:

$$\rho\dot{\eta}T_0 = q_{i,i} + \rho s\,. \tag{3.7}$$

To obtain from (3.2), (3.3) and (3.6) the constitutive relations of the thermoelastic theory of Iesan it suffices to set $\omega = 0$ and to obtain the constitutive relations of Nunziato and Cowin we must set $\theta \equiv 0$ and $a_i = m = \beta_{ij} = 0$.

In the case of an isotropic material for the constitutive constants we have:

$$C_{ijrs} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) \quad , \quad A_{ij} = \alpha\delta_{ij} \quad , \quad D_{ijk} = 0\,,$$

$$d_i = 0 \quad , \quad a_i = 0 \quad , \quad k_{ij} = K\delta_{ij} \quad , \quad B_{ij} = b\delta_{ij} \quad , \quad \beta_{ij} = \beta\delta_{ij}$$

and then the equations of motion read:

$$c_\tau^2 u_{i,ii} + (c_\delta^2 - c_\tau^2)u_{j,ji} + \frac{b}{\rho}\phi_{,i} - \frac{\beta}{\rho}\theta_{,i} + b_i = \ddot{u}_i\,, \tag{3.8}$$

$$c_\nu^2\phi_{,ii} - \frac{b}{\rho k}u_{j,j} - \frac{\xi}{\rho k}\phi - \frac{\omega}{\rho k}\dot{\phi} + \frac{m}{\rho k}\theta + \frac{1}{k} = \ddot{\phi}, \tag{3.9}$$

$$K\theta_{,ii} - \beta T_0\dot{u}_{j,j} - mT_0\dot{\phi} - aT_0\dot{\theta} = -\rho s\,, \tag{3.10}$$

where:

$$c_\delta^2 = \frac{\lambda + 2\mu}{\rho} \quad , \quad c_\tau^2 = \frac{\mu}{\rho} \quad , \quad c_\nu^2 = \frac{\alpha}{\rho k}\,. \tag{3.11}$$

The boundary conditions for this theory can be deduced from the work of ATKIN, COWIN and FOX [11]. In particular they are as those reported in COWIN and NUNZIATO [3]. Now we must give the additional data for the surface continuous temperature field on the boundary ∂B of the geometry of the body B and for the time interval for which the solution is desired. Obviously we must also add the initial temperature field $\theta(\mathbf{x},0) = \theta^0(\mathbf{x})$ for $\mathbf{x} \in \overline{B}$. For example for a mixed boundary, initial problem the conditions read as:

$$\begin{aligned} & h_i(\mathbf{x},t)n_i = 0\,, && \forall\,(\mathbf{x},t) \in \partial B \times [0,t_0]\,, \\ & q_i(\mathbf{x},t)n_i = 0\,, && \forall\,(\mathbf{x},t) \in \partial B \times [0,t_0]\,, \\ & u_i(\mathbf{x},t) = \tilde{u}_i(\mathbf{x},t)\,, && \forall\,(\mathbf{x},t) \in \partial B_M \times [0,t_0]\,, \\ & t_{ji}(\mathbf{x},t)n_i = \tilde{t}_i(\mathbf{x},t)\,, && \forall\,(\mathbf{x},t) \in \partial B_T \times [0,t_0]\,, \end{aligned} \tag{3.12}$$

and

$$u_i(\mathbf{x},0) = u_i^0(\mathbf{x}) \quad , \quad \phi(\mathbf{x},0) = \phi^0(\mathbf{x}) \quad , \quad \theta(\mathbf{x},0) = \theta^0(\mathbf{x})\,, \tag{3.13}$$

where ∂B_M is the portion of the boundary on which the displacements are specified, ∂B_T the portion on which the tractions are specified, $\partial B_M \cup \partial B_T = \partial B$, $\partial B_M \cap \partial B_T = \emptyset$ and n_i is the unit outward normal to ∂B.

4 – Uniqueness

In this section we use a simple energetic method to investigate the uniqueness for our dynamic theory.

We consider two solutions ϕ, u_i, θ and ϕ', u_i', θ' of the system of equations (2.4), (2.5) and (2.6) with constitutive equations (3.2), (3.3) and (3.6), for the same body B subjected to the same body force b_i, the same extrinsic equilibrated body force l and the same extrinsic heat supply s. For each solution we have an appropriate set of boundary and initial conditions of the same kind of (2.24) and (2.25). We introduce the circumflex notation to denote the difference from the prime and unprimed solution; for example:

$$\hat{u}_i(\mathbf{x}, t) = u_i(\mathbf{x}, t) - u_i'(\mathbf{x}, t) . \tag{4.1}$$

Then as result of the linearity of the field equations the differential equations governing the difference solutions are:

$$\rho\ddot{\hat{u}}_i = \hat{T}_{ij,j} , \tag{4.2}$$

$$\rho(k\ddot{\hat{\phi}}) = \hat{h}_{i,i} + \hat{g} , \tag{4.3}$$

$$\rho\dot{\hat{\eta}}T_0 = \hat{q}_{i,i} , \tag{4.4}$$

$$\hat{T}_{ij} = C_{ijrs}\hat{E}_{rs} + B_{ij}\hat{\phi} + D_{ijk}\hat{\phi}_{,k} - \beta_{ij}\hat{\theta} , \tag{4.5}$$

$$\hat{h}_i = A_{ij}\hat{\phi}_{,j} + D_{rsi}\hat{E}_{rs} + d_i\hat{\phi} - a_i\hat{\theta} , \tag{4.6}$$

$$\rho\hat{\eta} = \beta_{ij}\hat{E}_{ij} + a\hat{\theta} + m\hat{\phi} + a_i\hat{\phi}_{,i} , \tag{4.7}$$

$$\hat{g} = -B_{ij}\hat{E}_{ij} - \xi\hat{\phi} + \hat{f} - d_i\hat{\phi}_{,i} + m\hat{\theta} , \tag{4.8}$$

where $\hat{E}_{ij} = \frac{1}{2}(\hat{u}_{i,j} + \hat{u}_{j,i})$.

It is fundamental for our purposes to introduce the Biot' s potential [10] defined as:

$$\mathcal{B} = \rho(\epsilon - T_0\eta) . \tag{4.9}$$

Eliminating ϵ from this equation and (2.8), we obtain the following expression for $\mathcal{B}$:

$$\mathcal{B} = \frac{1}{2}\xi\phi^2 + \frac{1}{2}A_{ij}\phi_i\phi_j + \frac{1}{2}C_{ijkm}\overline{E}_{ij}\overline{E}_{km} + \frac{1}{2}a\theta^2 + \\ + B_{ij}\overline{E}_{ij}\phi + d_i\phi\phi_{,i} + D_{ijk}\phi_{,i}\overline{E}_{jk}\,, \tag{4.10}$$

With $\mathcal{K}$ we denotes the kinetic energy per unit mass i.e.:

$$\mathcal{K} = \frac{1}{2}\rho(\dot{u}_i\dot{u}_i + k\dot{\phi}^2)\,. \tag{4.11}$$

Using the field equations, the constitutive equations and the divergence theorem we obtain in this way the energy equation:

$$\frac{d}{dt}\int_B (\mathcal{K}+\mathcal{B})dV = \int_B \Big(\rho b_i\dot{u}_i + \rho l\dot{\phi} + \frac{\rho}{T_0}s\theta - \frac{1}{T_0}q_i\theta_{,i} + f\dot{\phi}\Big)dV + \\ + \int_{\partial B} \Big(t_{ij}\dot{u}_j + h_i\dot{\phi} + \frac{1}{T_0}q_i\theta\Big)n_i\, dA\,. \tag{4.12}$$

Now we are in position to prove the following:

THEOREM. *If in addition of the constitutive restrictions which have been formulated in* (3.2), (3.3) *and* (3.4) *we assume that* $\mathcal{B}$ *is non-negative, then there exists at most one solution for the problem given by the equations* (3.8)-(3.10) *and* (3.12), (3.13).

PROOF. Let (u_i, ϕ, θ) and (u_i', ϕ', θ') two solutions of equations (2.4), (2.5) and (2.6) with constitutive equations (3.2), (3.3) and (3.4) subject to the same b_i, l and s and subject to boundary conditions of the mixed type as in (3.12). In this case for the difference solutions (4.12) reads:

$$\frac{d}{dt}\int_B (\hat{\mathcal{K}}+\hat{\mathcal{B}})dV = \int_B \Big(-\frac{1}{T_0}\hat{q}\hat{\theta}_{,i} - \hat{f}\dot{\hat{\phi}}\Big)dV\,. \tag{4.13}$$

Since the right side of (4.13) is non-positive for (3.5), integration of both sides from 0 to t yields:

$$\int_B \Big(\hat{\mathcal{K}}(0) + \hat{\mathcal{B}}(0)\Big)dV \geq \int_B \Big(\hat{\mathcal{K}}(t) + \hat{\mathcal{B}}(t)\Big)dV\,. \tag{4.14}$$

Then for the initial-value problem the values of the quantities involved in this theory are bounded by their values at time $t = 0$. This relationship proves a sort of weak stability for the theory of the elastic materials with voids.

Obviously if the primed and the unprimed solutions satisfy the same initial and boundary value data then $(\hat{u}_i, \hat{\phi}, \hat{\theta})$ corresponds to the null data, i.e. $\hat{h}_i = \hat{u}_i = \hat{t}_{ij} = 0$ on $\partial B \times [0, t_0]$ and (3.14) requires:

$$0 \geq \int_B \left(\hat{\mathcal{K}}(t) + \hat{\mathcal{B}}(t)\right) dV, \tag{4.15}$$

but $\hat{\mathcal{K}}(t)$ and $\hat{\mathcal{B}}(t)$ are positive definite and then they must be zero everywhere in B. This means that the difference solutions must vanish everywhere in B for all times and this complete the proof. □

5 – Acceleration Waves

To test the difference between the theory proposed by Iesan and our theory we examine the differences in the propagation of acceleration waves.

Without enter the details of the methodology and the algebra necessary to determine and study acceleration waves, for which we refer to [12], let Σ be a moving surface defined by:

$$x_i = x_i(y_1, y_2, t), \tag{5.1}$$

where y_1 and y_2 are curvilinear coordinates on the surface and the functions in (5.1) and Σ fulfill all the necessary regularity conditions. To make easy the comparison we will use the same notation of Iesan paper [6].

The propagation wave it is said to be an acceleration wave if:

1) u_i, ϕ, θ and all their first derivatives are functions continuous everywhere;
2) The second derivatives of u_i, ϕ, and θ have, at most, jump discontinuities across Σ but are continuous everywhere else.

Let h be one of the field variables. The discontinuity on Σ of the second derivatives of h are defined by a single scalar function as follows:

$$[h_{,ij}] = Cn_in_j \quad , \quad [\dot{h}_{,i}] = -CVn_i \quad , \quad [\ddot{h}] = Cn_in_j \, ,$$

where n_i is the unit normal to Σ and V is the speed of propagation of Σ in the direction of the normal.

Introduced the notation:

$$\lambda_i = [u_{i,rs}]n_rn_s \quad , \quad \eta = [\phi_{,rs}]n_rn_s \quad , \quad \xi = [\theta_{,rs}]n_rn_s \, ,$$

as first point we can note that in both the theories the waves can be only longitudinal, transverse or dilatational and the admissible velocities of propagation are the same. Exactly longitudinal waves, $\lambda_i = \Lambda n_i$, propagate with speed c_δ, transverse waves, $\lambda_i n_i = 0$, with speed c_τ and compaction or distension waves, $\eta \neq 0$, with speed c_ν, where c_δ, c_τ, c_ν are defined by (3.11).

It is obvious that the velocity of propagation are not affected introducing a new term that consists in a first derivative, and it is natural that the behavior of longitudinal and transverse waves will be the same in both theories. It is the growth and decay of compaction waves which is quite different in the two theories. Always without to report the algebra that is similar of the one in [6] the equation for the growth and decay of compaction waves, $V = c_\nu$, $\lambda_i = \xi = 0$, are:

$$\{(c_\nu^2 - c_\delta^2)\delta_{ij} - (c_\delta^2 - c_\tau^2 n_j n_i\}\mu_j = \frac{b}{\rho}\eta n_i \, , \tag{5.2}$$

$$-2c_\nu\frac{\delta\eta}{\delta t} + 2c_\nu^2 H\eta - \frac{\omega}{\rho k}\eta c_\nu = 0 \, , \tag{5.3}$$

$$\beta T_0 c_\nu \mu_j n_j + K\gamma + mT_0 c_\nu \eta = 0 \, , \tag{5.4}$$

where $\mu_i = [u_{i,pqr}]n_pn_qn_r$, $\gamma = [\theta_{,ijk}]n_in_jn_k$, $\dfrac{\delta}{\delta t} = \dfrac{\partial}{\partial t} + Vn_i\dfrac{\partial}{\partial x_i}$ and H is the mean curvature i.e.:

$$H = \frac{H_0 - nK_0}{1 - 2nH_0 + n^2K_0} \, ,$$

with H_0 and K_0 respectively the mean and Gaussian curvature at $t = t_0$.

The solution of (5.3) then is:

$$\eta = \eta_0(1 - 2nH_0 + n^2)^{-\frac{1}{2}} \exp\left(-\frac{\omega}{2\rho k c_\nu} n\right), \tag{5.5}$$

$$\mu_i = \frac{b n_i \eta_0}{\rho(c_\nu^2 - c_\delta^2)}(1 - 2nH_0 + n^2)^{-\frac{1}{2}} \exp\left(-\frac{\omega}{2\rho k c_\nu} n\right), \tag{5.6}$$

$$\gamma = -\frac{m c_\nu T_0 \eta_0}{\mathcal{K}}\left(1 + \frac{\beta b}{m(c_\nu^2 - c_\delta^2)}\right) \cdot \\ \cdot (1 - 2nH_0 + n^2)^{-\frac{1}{2}} \exp\left(-\frac{\omega}{2\rho k c_\nu} n\right), \tag{5.7}$$

where $\eta = \eta_0$ when $t = t_0$.

The exponential factor ensures that the discontinuities η, μ_i and γ tend to zero as t tends to infinity, obviously setting $\omega \equiv 0$ we obtain the same expression of Iesan [6] and the difference is striking.

It is very important to note that the study of acceleration waves provides a way to determine experimentally the constitutive parameters. In particular we have that from (5.5) we are able, by means of an experiment, to compute ω.

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