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# Some six dimensional compact symplectic and complex solvmanifolds

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RIASSUNTO – *Si costruisce una famiglia di varietà compatte simplettiche di dimensione 6 che hanno la stessa comologia di una varietà compatta Kähleriana e struttura simpletttica e complessa.*

ABSTRACT – *We construct a family of compact symplectic solvmanifolds of dimension 6 which have the same cohomology ring as a compact Kähler manifold. They have complex structures, but we cannot determine whether or not these solvmanifolds admit positive definite Kähler metrics.*

KEY WORDS – *Symplectic manifolds - Complex manifolds - Indefinite Kähler metrics - Solvmanifolds.*

A.M.S. CLASSIFICATION: 53C15 - 53C55

## 1 – Introduction

It is well-known [12,5] that there are strong topological conditions for a compact manifold $M$ of dimension $2n$ to admit a positive definite Kähler metric:

(1) the Betti numbers $b_{2i}(M)$ are non-zero for $1 \leq i \leq n$ ;

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(2) the Betti numbers $b_{2i-1}(M)$ are even ;

(3) $b_i(M) \geq b_{i-2}(M)$ for $1 \leq i \leq n$ ;

(4) the strong Lefschetz theorem holds for $M$ ;

(5) the minimal model of $M$ is formal (so in particular all Massey products of $M$ vanish).

On the other hand, the additional structure of a complex manifold leads to the Frölicher spectral sequence $\{E_r\}$ (see [7]). For a compact manifold with positive definite Kähler metric this spectral sequence satisfies:

(6) $E_1 \cong E_2 \cong \cdots \cong E_\infty$.

If $M$ is a compact nilmanifold, not a torus, the conditions (4) and (5) always fail (see [1,3,9]). Thus $M$ carries no positive definite Kähler metrics. In contrast to the case of compact nilmanifolds there are compact solvmanifolds non-nilmanifolds that satisfy both conditions (4) and (5) ([6,2]). There, the examples described are 4 and 8-dimensional, respectively.

In the present paper we construct a new family of compact solvmanifolds $M^6(k)$ of dimension six each of which satisfies the conditions (1)–(6). Whereas in [3] is proved that the minimal model of compact nilmanifolds is not formal by showing that there are non-zero Massey products, all Massey products in the spaces $M^6(k)$ vanish. For the compact solvmanifolds considered in [2] and [6] is proved that the corresponding minimal model is formal by computing such a model, but the computation of the minimal model of $M^6(k)$ is very long. Therefore we resort to the definition of formal manifold given in ([8], p. 158) to show that $M^6(k)$ satisfies condition (5).

Moreover we prove that $M^6(k)$ possesses indefinite Kähler metrics and so $M^6(k)$ has symplectic and complex structures. Finally, we also study the Frölicher spectral sequence associated to the (natural) complex structure on $M^6(k)$, and we show that this spectral sequence satisfies condition (6). But, as the example of Benson and Gordon [2], we do not know whether or not $M^6(k)$ admit (positive definite) Kähler metrics.

## 2 – The compact solvmanifolds $M^6(k)$

Let $G(k)$ be the connected solvable (non-nilpotent) Lie group of dimension 5 consisting of matrices of the form

$$\mathbf{a} = \begin{pmatrix} e^{kz} & 0 & 0 & 0 & 0 & x_1 \\ 0 & e^{-kz} & 0 & 0 & 0 & y_1 \\ 0 & 0 & e^{kz} & 0 & 0 & x_2 \\ 0 & 0 & 0 & e^{-kz} & 0 & y_2 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x_i, y_i, z \in \mathbb{R}$, $1 \leq i \leq 2$, and $k$ is a real number different from 0. Then, a global system of coordinates $\{x_1, y_1, x_2, y_2, z\}$ for $G(k)$ is given by

$$x_i(\mathbf{a}) = x_i\,, \quad y_i(\mathbf{a}) = y_i\,, \quad z(\mathbf{a}) = z\,, \qquad 1 \leq i \leq 2\,;$$

and a standard computation shows that a basis for the right invariant 1-forms on $G(k)$ consists of

$$\{dx_1 - kx_1dz,\ dy_1 + ky_1dz,\ dx_2 - kx_2dz,\ dy_2 + ky_2dz,\ dz\}\,.$$

This Lie group $G(k)$ can be easily described as the semidirect product $\mathbb{R} \times_\varphi \mathbb{R}^4$, where

$$\varphi\colon \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{R}^4)$$

is the representation defined by

$$\varphi(z) = \begin{pmatrix} e^{kz} & 0 & 0 & 0 \\ 0 & e^{-kz} & 0 & 0 \\ 0 & 0 & e^{kz} & 0 \\ 0 & 0 & 0 & e^{-kz} \end{pmatrix}, \qquad z \in \mathbb{R}.$$

Therefore, $G(k)$ possesses a discrete subgroup $\Gamma(k)$ such that the quotient space $G(k)/\Gamma(k)$ is compact. Hence the forms $dx_i - kx_idz, dy_i + ky_idz,\ dz$, $1 \leq i \leq 2$, descend to 1-forms $\alpha_i$, $\beta_i$, $\gamma$, $1 \leq i \leq 2$, on $G(k)/\Gamma(k)$.

Now, let us consider the product $M^6(k) = (G(k)/\Gamma(k)) \times S^1$. Then there are 1-forms $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$, $\gamma$, $\eta$ on $M^6(k)$ such that

$$d\alpha_i = -k\alpha_i \wedge \gamma\,, \quad d\beta_i = k\beta_i \wedge \gamma\,, \quad d\gamma = d\eta = 0, \quad 1 \leq i \leq 2\,.$$

Now, we notice that $G(k)$ is *completely solvable*, that is $ad_X : \mathfrak{g}(k) \longrightarrow \mathfrak{g}(k)$ has only real eigenvalues for each $X \in \mathfrak{g}(k)$, where $\mathfrak{g}(k)$ denotes the Lie algebra of $G(k)$. In fact, $\mathfrak{g}(k)$ is a Lie subalgebra of real upper triangular matrices in $\mathfrak{gl}(6, \mathbb{R})$. A theorem of Hattori[9] asserts that the de Rham cohomology ring $H^*(G(k)/\Gamma(k), \mathbb{R})$ is isomorphic to the cohomology ring $H^*(\mathfrak{g}(k))$ of the Lie algebra $\mathfrak{g}(k)$ of $G(k)$. Using this result we compute the real cohomology of $M^6(k)$ :

$$
\begin{aligned}
H^0(M^6(k), \mathbb{R}) &= \{1\}, \\
H^1(M^6(k), \mathbb{R}) &= \{[\gamma], [\eta]\}, \\
H^2(M^6(k), \mathbb{R}) &= \{[\alpha_1 \wedge \beta_1], [\alpha_1 \wedge \beta_2], [\beta_1 \wedge \alpha_2], [\alpha_2 \wedge \beta_2], [\gamma \wedge \eta]\}, \\
H^3(M^6(k), \mathbb{R}) &= \{[\alpha_1 \wedge \beta_1 \wedge \gamma], [\alpha_1 \wedge \beta_2 \wedge \gamma], [\beta_1 \wedge \alpha_2 \wedge \gamma], \\
&\quad [\alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \eta], \\
&\quad [\beta_1 \wedge \alpha_2 \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \eta]\}, \\
H^4(M^6(k), \mathbb{R}) &= \{[\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2], [\alpha_1 \wedge \beta_1 \wedge \gamma \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \gamma \wedge \eta], \\
&\quad [\beta_1 \wedge \alpha_2 \wedge \gamma \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta]\}, \\
H^5(M^6(k), \mathbb{R}) &= \{[\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \eta]\}, \\
H^6(M^6(k), \mathbb{R}) &= \{[\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta]\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
b_0(M^6(k)) &= b_6(M^6(k)) = 1 , \\
b_1(M^6(k)) &= b_5(M^6(k)) = 2 , \\
b_2(M^6(k)) &= b_4(M^6(k)) = 5 , \\
b_3(M^6(k)) &= 8 .
\end{aligned}
$$

Hence $M^6(k)$ satisfies conditions (1)–(3).

THEOREM 1. *The minimal model of $M^6(k)$ is formal.*

PROOF. The manifold $M^6(k)$ is formal if the homotopy type of the exterior algebra of differential forms $(\Lambda^* M^6(k), d)$ is the same as the homotopy type of the cohomology ring $(H^*(M^6(k)), d = 0)$ (see [8], p. 158). In other words, $M^6(k)$ is formal if $(\Lambda^* M^6(k), d)$ and $(H^*(M^6(k)), d = 0)$ have the same minimal model.

Now, we define a map of cochain complexes

$$\Phi: (H^*(M^6(k)), d = 0) \longrightarrow (\Lambda^* M^6(k), d)$$

by linearly choosing closed forms representatives for each cohomology class; that is, $\Phi[\gamma] = \gamma$, etc. One easily proves that $\Phi$ is multiplicative and then it is a homomorphism of differential graded algebras which induces the identity on cohomology. Now, let $\Psi: (\mathbf{M}, \mathbf{d}) \longrightarrow (H^*(M^6(k)), d = 0)$ be the minimal model of $(H^*(M^6(k)), d = 0)$. Then $\Phi \circ \Psi: (\mathbf{M}, \mathbf{d}) \longrightarrow (\Lambda^* M^6(k), d)$ is the minimal model of $(\Lambda^* M^6(k), d)$. Thus, the manifold $M^6(k)$ is formal. □

Next, we show that $M^6(k)$ satisfies the strong Lefschetz theorem.

Let $\{X_1, Y_1, X_2, Y_2, Z, T\}$ be a basis of (global) vector fields on $M^6(k)$ dual to the basis of 1-forms $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \eta\}$. Then

$$[X_i, Z] = kX_i, \quad [Y_i, Z] = -kY_i, \quad 1 \leq i \leq 2;$$

and the other brackets being zero.

The collection $\{\alpha_1 \wedge \beta_1,\ \alpha_1 \wedge \beta_2,\ \beta_1 \wedge \alpha_2,\ \alpha_2 \wedge \beta_2,\ \gamma \wedge \eta\}$ is a basis for the closed 2-forms on $M^6(k)$. Thus, any cohomology class $[\omega]$ of a closed 2-form $\omega$ is given by

$$[\omega] = r[\gamma \wedge \eta] + s[\alpha_1 \wedge \beta_1] + t[\alpha_1 \wedge \beta_2] + u[\beta_1 \wedge \alpha_2] + v[\alpha_2 \wedge \beta_2]$$

for some constants $r, s, t, u, v$. It is clear that $\omega$ is non-degenerate if and only if $r \neq 0$ and $sv + tu \neq 0$. The Lefschetz maps $\wedge[\omega]: H^2(M^6(k)) \longrightarrow H^4(M^6(k))$ and $\wedge[\omega^2]: H^1(M^6(k)) \longrightarrow H^5(M^6(k))$ are isomorphisms and so $M^6(k)$ satisfies the strong Lefschetz theorem.

Remark. In [11] a compact symplectic $2n$-dimensional manifold $(M, \omega)$ is said to be **Lefschetz manifold** if $\wedge[\omega]^{n-1}: H^1(M) \longrightarrow H^{2n-1}(M)$ is an isomorphism. Then we have proved that $(M^6(k), \omega)$ are Lefschetz manifolds.

As we see, the manifold $M^6(k)$ satisfies the conditions (1)–(5). Next we shall recall the following theorem due to Benson and Gordon [2]:

Theorem 2. ([2]) *If $G$ is a completely solvable Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma/G$ is a solvmanifold which admits a Kähler structure, then*

*(1) there is an Abelian complement* $\mathfrak{a}$ *in* $\mathfrak{g}$ *of the derived algebra* $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$,
*(2)* $\mathfrak{a}$ *and* $\mathfrak{n}$ *are even dimensional,*
*(3) the center of* $\mathfrak{g}$ *intersects* $\mathfrak{n}$ *trivially,*
*(4) the Kähler form is cohomologous to a left invariant symplectic form* $\omega = \omega_0 + \omega_1$, *where* $\mathfrak{n} = \ker\, \omega_0$ *and* $\mathfrak{a} = \ker\, \omega_1$,
*(5) both* $\omega_0$ *and* $\omega_1$ *are closed but not exact in* $\mathfrak{g}$ *(and also in* $\mathfrak{a}$ *and* $\mathfrak{n}$*),*
*(6) the adjoint action of* $\mathfrak{a}$ *on* $\mathfrak{n}$ *is by infinitesimal symplectic automorphisms of* $(\mathfrak{n}, \omega)$.

For the manifold $M^6(k)$ the derived algebra $\mathfrak{n}$ of $\mathfrak{g}(k) \times \mathbb{R}$ is generated by $X_1, Y_1, X_2, Y_2$. Then there exists an Abelian complement $\mathfrak{a}$ of $\mathfrak{n}$ in $\mathfrak{g}(k) \times \mathbb{R}$ generated by $Z, T$. Hence $\mathfrak{a}$ and $\mathfrak{n}$ satisfy the conditions (1)–(3) of Theorem 2. Furthermore, $M^6(k)$ satisfies conditions (4) and (5) of Theorem 2.

Next, we shall construct an indefinite Kähler metric on $M^6(k)$. Define an almost complex structure $J$ on $M^6(k)$ by

$$JX_1 = X_2, \quad JY_1 = Y_2, \quad JZ = T. \tag{1}$$

A direct computation shows that the Nijenhuis tensor of $J$ vanishes. Consequently, $J$ is complex. A basis $\{\lambda, \mu, \nu\}$ for the 1-forms of bidegree (1,0) is given by

$$\lambda = \alpha_1 + \sqrt{-1}\, \alpha_2,$$
$$\mu = \beta_1 + \sqrt{-1}\, \beta_2,$$
$$\nu = \gamma + \sqrt{-1}\, \eta.$$

Thus, we have

$$\begin{cases} d\lambda = -\dfrac{k}{2}\, \lambda \wedge (\nu + \bar{\nu}), \\[2mm] d\mu = \dfrac{k}{2}\, \mu \wedge (\nu + \bar{\nu}), \\[2mm] d\nu = 0\,. \end{cases}$$

Define

$$\Omega = \lambda \wedge \bar{\mu} + \bar{\lambda} \wedge \mu + \sqrt{-1}\, \nu \wedge \bar{\nu}.$$

Then $\Omega$ is closed, and it is easy to see that $\Omega$ has maximal rank. Furthermore $\Omega$ is a symplectic form of bidegree (1,1) on $M^6(k)$; and so the metric $g$ given by $g(U,V) = \Omega(U, JV)$ for vector fields $U, V$ on $M^6(k)$ is an indefinite Kähler metric.

Therefore we have

THEOREM 3. *$M^6(k)$ possesses an indefinite Kähler metric. Hence, $M^6(k)$ has symplectic and complex structures.*

REMARK. If the coordinate functions $x_i$, $y_i$, $1 \leq i \leq 2$ of the Lie group $G(k)$ are such that $x_1 = x_2$ and $y_1 = y_2$ on $G(k)$, then the compact solvmanifold $M(k) = (G(k)/\Gamma(k)) \times S^1$ is 4-dimensional. This manifold was studied in [6], and there was proved that $M(k)$ carries no complex structures.

Next, we shall consider the Frölicher spectral sequence $\{E_r\}$ associated to the complex structure on $M^6(k)$ defined by (1). Let $\bigwedge^*(M^6(k), \mathbb{C}) = \bigoplus_{p,q \geq 0} \bigwedge^{p,q}$ be the usual decomposition of the complex valued differential forms, and $d = \partial + \bar{\partial}$ with $\partial$ of type $(1,0)$ and $\bar{\partial}$ of type $(0,1)$, $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

Then it is known [7] that

$$\begin{aligned} E_1^{p,q} &= \frac{\{\alpha \in \bigwedge^{p,q}\,;\ \bar{\partial}\alpha = 0\}}{\{\alpha \in \bigwedge^{p,q}\,;\ \alpha = \bar{\partial}\beta \ \text{ for some } \ \beta \in \bigwedge^{p,q-1}\}} \\ &= H_{\bar{\partial}}^{p,q}(M^6(k)), \end{aligned}$$

$$E_2^{p,q} = H^{p,q}\left(H_{\bar{\partial}}^*(M^6(k)), \partial\right),$$

that is, the term $E_1$ of the spectral sequence is the cohomology of the differential algebra $\left(\bigwedge^{p,q}, \bar{\partial}\right)$ and the term $E_2$ is the cohomology of the differential algebra ($\bar{\partial}$-cohomology, $\partial$).

From (2), we have

$$
\begin{aligned}
&H^{0,1}_{\bar\partial}(M^6(k)) = \{[\bar\nu]\},\\
&H^{0,2}_{\bar\partial}(M^6(k)) = \{[\bar\lambda\wedge\bar\mu]\},\\
&H^{0,3}_{\bar\partial}(M^6(k)) = \{[\bar\lambda\wedge\bar\mu\wedge\bar\nu]\},\\
&H^{1,0}_{\bar\partial}(M^6(k)) = \{[\nu]\},\\
&H^{1,1}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\bar\mu],[\mu\wedge\bar\lambda],[\nu\wedge\bar\nu]\},\\
&H^{1,2}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\bar\mu\wedge\bar\nu],[\mu\wedge\bar\lambda\wedge\bar\nu],[\nu\wedge\bar\lambda\wedge\bar\mu]\},\\
&H^{1,3}_{\bar\partial}(M^6(k)) = \{[\nu\wedge\bar\lambda\wedge\bar\mu\wedge\bar\nu]\},\\
&H^{2,0}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu]\},\\
&H^{2,1}_{\bar\partial}(M^6(k)) = \{[\mu\wedge\nu\wedge\bar\lambda],[\lambda\wedge\nu\wedge\bar\mu],[\lambda\wedge\mu\wedge\bar\nu]\},\\
&H^{2,2}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu\wedge\bar\lambda\wedge\bar\mu],[\mu\wedge\nu\wedge\bar\lambda\wedge\bar\nu],[\lambda\wedge\nu\wedge\bar\mu\wedge\bar\nu]\},\\
&H^{2,3}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu\wedge\bar\lambda\wedge\bar\mu\wedge\bar\nu]\},\\
&H^{3,0}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu\wedge\nu]\},\\
&H^{3,1}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu\wedge\nu\wedge\bar\nu]\},\\
&H^{3,2}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu\wedge\nu\wedge\bar\lambda\wedge\bar\mu]\},\\
&H^{3,3}_{\bar\partial}(M^6(k)) = \{[\lambda\wedge\mu\wedge\nu\wedge\bar\lambda\wedge\bar\mu\wedge\bar\nu]\}.
\end{aligned}
$$

Now it is easy to see that $\partial\colon H^{p,q}_{\bar\partial}\longrightarrow H^{p+1,q}_{\bar\partial}$ is the zero mapping, and hence $E_1\cong E_2\cong\cdots\cong E_\infty$.

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