Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 99-125

Nodal solutions of some elliptic problems with critical nonlinearities

M.V. MARCHI

RIASSUNTO – *Si dimostra che l'equazione semilineare ellittica con esponente critico e con condizione al bordo di tipo misto (1) ammette soluzioni che cambiano segno: se $\lambda > 0$, per ogni dominio limitato Ω di $\mathbb{R}^n$, se $\lambda = 0$, sotto opportune ipotesi di carattere geometrico su quella parte di frontiera su cui è data la condizione di Neumann.*

ABSTRACT – *We study the semilinear elliptic problem with critical exponent and mixed boundary conditions (1). We prove the existence of nodal solutions for any bounded domain $\Omega \subset \mathbb{R}^n$, when $\lambda > 0$, and, under some geometrical assumptions on Ω, also when $\lambda = 0$.*

KEY WORDS – *Critical Sobolev exponent - Mixed boundary problem - Nodal solutions.*

A.M.S. CLASSIFICATION: 35J65

– Introduction

Let us consider the problem

$$(1) \qquad \begin{cases} \Delta u = \lambda u + |u|^{2^*-2}u & \text{on} \quad \Omega \\ u = 0 & \text{on} \quad \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with regular boundary, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ and ν is the outer normal to Γ_1.

It is easy to see that the weak solutions of (1) in the space $V(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_0\}$ correspond to the critical points of the functional

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2}\int_\Omega |u|^{2^*}. \tag{2}$$

As it is well known for this type of problems (see [4], [5] and [7]), the difficulty arising in studying problem (1) is that 2^* is the critical esponent for the Sobolev embedding $V(\Omega) \to L^q(\Omega)$, so that the compactness condition of Palais-Smale fails for the functional (2).

Analogously to the results of BREZIS, NIREMBERG, LIONS and STRUWE for the Dirichlet problem (see [7], [4], [5], [16], [20]), LIONS, PACELLA, TRICARICO and GROSSI prove in [18] and [14] that the Palais-Smale condition for the functional F_λ fails only at certain levels. Starting from this result GROSSI proves in [13] the existence of positive solutions of (1), for any $\lambda \in [\lambda^*, \lambda_1[$, where λ_1 is the first eingevalue of $-\Delta$ in $V(\Omega)$ and λ^* is 0 if $n \geq 4$ and is a positive constant depending on Ω if $n = 3$.

In this paper, using some techniques introduced for the Dirichlet problem in the papers [8], [9], [10], we study the existence of solutions of (1) which change sign, both in the case $\lambda > 0$ or $\lambda = 0$ (at least for some particular domains).

Let us remark esplicitely that in the Dirichlet problem the case $\lambda = 0$ has not been treated in the above papers since the techniques there applied do not work in this case. Instead, in the mixed boundary problem (1), using the results of [18] for positive solutions, we are able to extend the method of [10] to find nodal solutions also in the case $\lambda = 0$, under some geometrical assumptions on the domain Ω.

The outline of the paper is the following.

In section 1 we give some preliminaries. In section 2 we prove some compactness results. In section 3, following [8], we prove that (1) has at least two solutions for any $\lambda > 0$. In the same section we also show, as in [9], that, for λ in a suitable neighborhood of the eigenvalue λ_m of $-\Delta$, problem (1) has at least $2m$ solutions, where m is the multiplicity of λ_m.

Since it is easy to see (see for example [7]) that for $\lambda \geq \lambda_1$, (1) does not have positive solutions, we have implicitely proved in this way the existence of nodal solutions for $\lambda \geq \lambda_1$.

For $0 < \lambda < \lambda_1$ we prove in section 4 that (1) has at least two nodal solutions, using the method of [10]. Finally in section 5 we treat the case $\lambda = 0$.

1 – Notations and preliminaries

Let G be an open set in $\mathbb{R}^n$. We define

$$H^1(G) = \Big\{u \in L^2(G) \quad \text{such that} \quad |\nabla u| \in L^2(G)\Big\}$$

$$D(G) = \Big\{u \in L^{2^*}(G) \quad \text{such that} \quad |\nabla u| \in L^2(G)\Big\}, \qquad 2^* = \frac{2n}{n-2}$$

$$H_0^1(G) = \Big\{u \in H^1(G) \quad \text{such that supp}\, u \subset\subset G\Big\}.$$

By the Sobolev embedding theorem $H^1(G) \hookrightarrow D(G)$ and, if G has finite measure, $H^1(G) = D(G)$.

The usual scalar product in $H^1(G)$ is

$$(u,v) = \int_G \nabla u \cdot \nabla v + \int_G uv, \qquad u, v \in H^1(G). \tag{1.1}$$

From now on we will denote by Ω a bounded domain in $\mathbb{R}^n$ with regular boundary and set $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $H_{n-1}(\Gamma_0) > 0$ and $H_{n-1}(\Gamma_1) > 0$, H_{n-1} being the $(n-1)$ dimensional Hausdorff measure. For such a domain Ω we define

$$V(\Omega) = \Big\{u \in H^1(\Omega) \colon u \equiv 0 \text{ on } \Gamma_0\Big\}.$$

If H is either $H_0^1(G)$,$D(G)$ or $V(\Omega)$, we consider the following infimum:

$$S_H = \inf_{\substack{u \in H \\ u \neq 0}} \frac{\int_G |\nabla u|^2}{\left(\int_G |u|^{2^*}\right)^{2/2^*}} \tag{1.2}$$

and set $S = S_{D(\mathbb{R}^n)}$, $\Sigma = S_{D(\mathbb{R}^n_+)}$, $S(\Omega) = S_{V(\Omega)}$ where

$$\mathbb{R}^n_+ = \left\{(x_1, y) \in \mathbb{R} \times \mathbb{R}^{n-1}, x_1 > 0\right\}.$$

The following results hold:

THEOREM 1.1.
i) $S > 0 \quad \Sigma > 0$ *(Sobolev's inequalities)*
ii) $S = \Sigma 2^{2/n}$ □

The proof of i) can be found in [3] while ii) follows from symmetrization arguments used in [21].

THEOREM 1.2.
i) $S_{H_0^1}(\Omega) > 0$ *(Poincaré's inequality)*
ii) $S_{H_0^1}(\Omega) = S$ □

Again the proof of i) can be found in [3], while ii) follows from symmetrization and rescaling arguments (see [1], [5] and [21]).

THEOREM 1.3. ([18])
$0 < S(\Omega) \leq \Sigma$. □

From Theorem 1.2 and 1.3 we immediately deduce that $\|u\| = (\int_\Omega |\nabla u|^2)^{1/2}$ is a norm equivalent to that induced by the scalar product (1.1) and, from now on, we will use this norm.

By $|u|_s$ instead we will mean the norm $(\int_\Omega |u|_s)^{1/s}, s > 0$, whenever this norm is defined for the function u.

Let us define in $V(\Omega)$ the functional

$$(1.3) \qquad F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}, \ \lambda \in \mathbb{R}.$$

It is easy to see that F_λ is of class C^1 and that we have

$$(1.4) \quad \langle dF_\lambda(u), \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\varphi - \lambda \int_\Omega u\varphi - \int_\Omega |u|^{2^*-2} u\varphi, \ \forall\, \varphi \in V(\Omega).$$

It is useful to observe that

$$F_\lambda(u) = \frac{1}{2}\langle dF_\lambda(u), u\rangle + \Big(\frac{1}{2} - \frac{1}{2^*}\Big)\int_\Omega |u|^{2^*}. \tag{1.5}$$

When $\lambda = 0$ it is also possible to define $F_0(u)$ in $D(\mathbb{R}^n)$ and $D(\mathbb{R}^n_+)$ substituting obviously Ω with $\mathbb{R}^n$ or $\mathbb{R}^n_+$, in this case (1.4) and (1.5) also hold.

2 – Compactness theorems

Let F_λ be defined as in the previous section.

LEMMA 2.1. *Let λ belong to $[0, \lambda_1[$ and $u_m \in V(\Omega)$ be a sequence such that*

$$\langle dF_\lambda(u_m), u_m\rangle \to 0. \tag{2.1}$$

Then either

i) $u_m \to 0$ *in* $V(\Omega)$ *(up to a subsequence)*

or

ii) $|u_m|_{2^*}^{2^*} \geq \Big(1 - \frac{\lambda}{\lambda_1}\Big)^{n/2} [S(\Omega)]^{n/2} + o(1)$

PROOF. We have

$$\begin{aligned} F_\lambda(u_m) &\geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u_m\|^2 - \frac{1}{2^*}|u_m|_{2^*}^{2^*} \geq \\ &\geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_1}\Big)S(\Omega)|u_m|_{2^*}^2 - \frac{1}{2^*}|u_m|_{2^*}^{2^*}. \end{aligned}$$

From (2.1), (1.5) we get

$$|u_m|_{2^*}^{2^*} \geq \Big(1 - \frac{\lambda}{\lambda_1}\Big)S(\Omega)|u_m|_{2^*}^2 + o(1). \tag{2.2}$$

Since

$$\|u_m\|_2^2 = \langle dF_\lambda(u_m), u_m\rangle + |u_m|_{2^*}^{2^*} + \lambda|u_m|_2^2 \tag{2.3}$$

then $u_m \to 0$ in $V(\Omega)$ iff $u_m \to 0$ in $L^{2^*}(\Omega)$.

If i) does not hold, from (2.2) we obtain:

$$|u_m|_{2^*}^{2^*-2} \geq \Big(1 - \frac{\lambda}{\lambda_1}\Big) S(\Omega) + o(1)$$

which implies ii). □

COROLLARY 2.1. *Let λ belong to $[0, \lambda_1[$, $u \in V(\Omega)$ and $\langle dF_\lambda(u), u\rangle = 0$. Then either $u = 0$ or*

$$F_\lambda(u) \geq \frac{1}{n}\Big(1 - \frac{\lambda}{\lambda_1}\Big)^{n/2} S(\Omega)^{n/2}.$$

PROOF. It follows immediately from (1.5) and Lemma 2.1. □

COROLLARY 2.2. *Let λ belong to $[0, \lambda_1[$, $u \in V(\Omega)$ changes sign and $\langle dF_\lambda(u^\pm), u^\pm\rangle = 0$. Then:*

$$F_\lambda(u) \geq \frac{2}{n}\Big(1 - \frac{\lambda}{\lambda_1}\Big)^{n/2} S(\Omega)^{n/2}.$$

PROOF. It is a consequence of Corollary 2.1 observing that

$$F_\lambda(u) = F_\lambda(u^+) + F_\lambda(u^-).$$

□

REMARK 2.1. If in the Lemma 2.1 and Corollary 2.1 and 2.2 we consider the functional F_0 on $D(\mathbb{R}^n)$ or $D(\mathbb{R}^n_+)$, we get the same results with S or Σ instead of $S(\Omega)$.

The same is true with the constant Σ if the functional F_0 is considered on the space $H = D(\mathbb{R}^n_+) \cap \{u$ such that $u \equiv 0$ on $\Gamma_0\}$, where $\Gamma_0 = \{(x_1, x_2, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} x_1 = 0 x_2 > 0\}$. In fact in this case it is obvious that $S_H \geq \Sigma$.

Let us now consider the following problems:

$$
(2.4) \qquad \begin{cases} w \in D(\mathbb{R}^n) \\ -\Delta w = |w|^{2^*-2} w & \text{on } \mathbb{R}^n \end{cases}
$$

$$
(2.5) \qquad \begin{cases} w \in D(\mathbb{R}^n_+) \\ -\Delta w = |w|^{2^*-2} w & \text{on } \mathbb{R}^n_+ \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } (\partial \mathbb{R}^n_+) \end{cases}
$$

$$
(2.6) \qquad \begin{cases} w \in D(\mathbb{R}^n_+) \cap \{u \text{ such that } u \equiv 0 \text{ on } \Gamma_0\} \\ -\Delta w = |w|^{2^*-2} w & \text{on } \mathbb{R}^n_+ \\ w = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}
$$

where Γ_0 is as in Remark 2.1 and $\Gamma_1 = \partial \mathbb{R}^n_+ \setminus \Gamma_0$.

THEOREM 2.1. *Let $u_m \in V(\Omega)$ be such that*

$$
F_\lambda(u_m) \to c
$$

$$
dF_\lambda(u_m) \to 0 \text{ in } V^*(\Omega).
$$

Then there exist $u \in V(\Omega)$ such that $u_m \to u$ weakly in $V(\Omega)$ and u is a solution of (1) *and $w_1, w_2, \ldots, w_h$, solutions of problems* (2.4)-(2.6), *such that:*

$$
c = F_\lambda(u) + \sum_{j=1}^{h} F_0(w_j).
$$

Moreover $h > 0$ iff u_m does not converge to u in $V(\Omega)$.

PROOF. The proof is implicitely contained in that of Theorem 2.1 of [14]. □

COROLLARY 2.3. *Let u_m and u be as in Theorem 2.1. Then we have*

i) *if $c < \frac{1}{n}\Sigma^{n/2}$, then $u_m \to u$ in $V(\Omega)$.*

ii) *If $\lambda \in [0, \lambda_1[$ and $c < \frac{1}{n}\left(1 - \frac{\lambda}{\lambda_1}\right)^{n/2} S(\Omega)^{n/2} + \frac{1}{n}\Sigma^{n/2}$ and $u \neq 0$, then $u_m \to u$ in $V(\Omega)$.*

iii) *Let λ and c be as in* ii) *and there exists $\alpha > 0$ such that $\|u_m^{\pm}\| > \alpha$, then $u_m \to u$ in $V(\Omega)$.*

PROOF.
i) and ii) follow immediately from Theorem 2.1 Corollary 2.2 and Remark 2.1.
iii) From Theorem 1.3, Theorem 2.1, Corollary 2.2 and Remark 2.1 it follows that, if u_m does not converges to u, then $h = 1$ and $w_1 \geq 0$ (or $w_1 \leq 0$). By the proof of Theorem 2.1 of [14] that, for brevity, we do not repeat, starting from the function $w_1 \geq 0$, it is possible to construct a sequence $w_m \geq 0$ in $V(\Omega)$, such that $w_m - v_m \to 0$ in $L^{2^*}(\Omega)$, where $v_m = u_m - u$. Therefore setting

$$\Omega_+ = \left\{x \in \Omega\colon v_m(x) = v_m^+(x)\right\} \text{ and } \Omega_- = \Omega \setminus \Omega_+ ,$$

we obtain

$$o(1) = |v_m - w_m|_{2^*}^{2^*} = \int_{\Omega_+} |v_m^+ - w_m|^{2^*} + \int_{\Omega_-} |-v_m^- - w_m|^{2^*} \geq$$

$$\geq \int_{\Omega_-} (v_m^- + w_m^-)^{2^*} \geq \int_{\Omega_-} (v_m^-)^{2^*} = |v_m^-|_{2^*}^{2^*} .$$

From this we get that $v_m^- \to 0$ in $L^{2^*}(\Omega)$. Thus, by hypothesis on $\|u_m^-\|$ it follows that $u \neq 0$. Thus iii) follows from ii). □

3 – Existence and multiplicity theorems for $\lambda_1 \leq \lambda$

Let us start by recalling a critical point theorem proved in [2].

THEOREM 3.1. *Let H be an Hilbert space with norm $\|\cdot\|$ and $I\colon H \to \mathbb{R}$ an even functional satisfying*

I$_1$) $I \in C^1(H, \mathbb{R})$, $I(0) = 0$.

I$_2$) *There exists $\beta > 0$ such that the Palais-Smale condition holds for I in $]-\infty, \beta[$.*

I$_3$) *There exists two closed subspaces V, W, numbers $\delta > 0$, $\rho > 0$, $0 < \varepsilon < \beta$ such that*

i) Codim $V < +\infty$.

ii) $I(u) > \delta$ *for any* $u \in V$ *with* $\|u\| = \rho$.

iii) $I(u) < \varepsilon$ *for any* $u \in W$.

Then I has at least $m = \dim W - \operatorname{codim} V$ pairs of critical points. □

Let F_λ be the functional on $V(\Omega)$ considered in the previous sections. The following estimates are easily deduced.

(*) If $\|u\|^2 - \lambda|u|_2^2 > 0$ then the function $F_\lambda(tu)$ is increasing, with rispect to t in $[0, t_0]$, decreasing to $-\infty$ in $[t_0 + \infty[$, where

$$t_0 = t_0(u) = \left(\frac{\|u\|^2 - \lambda|u|_2^2}{|u|_{2^*}^{2^*}}\right)^{(n-2)/4}. \tag{3.1}$$

Thus

$$|t_0 u|_{2^*}^{2^*} = \left(\frac{\|u\|^2 - \lambda|u|_2^2}{|u|_{2^*}^2}\right)^{n/2}. \tag{3.2}$$

Moreover observing that for $t > 0$

$$(dF_\lambda(tu), tu) = 0 \text{ iff } t = t_0(u) \tag{3.3}$$

from (1.5), (3.2) and (3.3) we get:

$$\max\{F_\lambda(tu)\colon t \in \mathbb{R}\} = F_\lambda(t_0 u) = \frac{1}{n}\left(\frac{\|u\|^2 - \lambda|u|_2^2}{|u|_{2^*}^2}\right)^{n/2}. \tag{3.4}$$

Finally let us point out that, since $F_\lambda(tu) < 0$ iff $t > \left(\frac{2^*}{2}\right)^{(n-2)/4} t_0(u)$

$$F_\lambda(u) \leq 0 \text{ iff } t_0(u) \leq \left(\frac{2}{2^*}\right)^{(n-2)/4} \tag{3.5}$$

and

$$\begin{aligned} &F_\lambda \text{ is bounded from above on any finite} \\ &\text{dimensional subspace of } V(\Omega)\,. \end{aligned} \tag{3.6}$$

Let us remark that in (3.6) we have also used the continuity of $t_0(u)$ with respect to $u \in V(\Omega)$, $u \neq 0$.

Denoting by $\lambda_1 < \lambda_2 < \ldots \lambda_n < \ldots$ the eigenvalues of $-\Delta$ in $V(\Omega)$, let (M_{λ_j}) be the eingenspace corresponding to λ_j. Moreover for any $\lambda > 0$ we set:

$$\lambda_+ = \min\{\lambda_j \text{ such that } \lambda_j > \lambda\}$$

$$M_+ = \bigoplus_{\lambda_j \geq \lambda_+} M(\lambda_j)$$

$$M_- = \bigoplus_{\lambda_j < \lambda_+} M(\lambda_j)$$

Denoting by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}^n$, we have:

LEMMA 3.1.

a) *There exist* $\delta > 0$, $\rho > 0$ *such that* $F_\lambda(u) \geq \delta$, *for any* $u \in M_+$ *with* $\|u\| = \rho$.

b) $F_\lambda(u) \leq \frac{1}{n}(\lambda_+ - \lambda)^{n/2}|\Omega|$ *for any* $u \in M_- \oplus M(\lambda_+)$.

PROOF. If $u \in M_+$ then $F_\lambda(u) \geq \left(1 - \frac{\lambda}{\lambda_+}\right)\|u\|^2 - \frac{S(\Omega)}{2^*}^{-2^*/2}\|u\|^{2^*}$ from which a) follows choosing ρ and δ in a suitable way. To prove b) we observe that for $u \in M_- \oplus M(\lambda_+)$ we have

$$\frac{\|u\|^2 - \lambda|u|_2^2}{|u|_{2^*}^2} \leq \frac{(\lambda_+ - \lambda)|u|_2^2}{|u|_{2^*}^2} \leq (\lambda_+ - \lambda)|\Omega|^{2/n}$$

and therefore b) follows from (3.4) if u satisfies (*). If u does not satisfy (*), $F_\lambda(u) \leq 0$. This ends the proof. □

THEOREM 3.2. *If* $\lambda \in]\lambda_j - \Sigma|\Omega|^{-2/n}, \lambda_j]$ *there exist at least* m_j *pairs of solutions of* (1) *where* m_j *is the dimension of* $M(\lambda_j)$.

PROOF. As in [9] we prove that F_λ satisfies the hypotheses of Theorem 3.1, choosing $V = M_+, W = M_- \oplus M(\lambda_+)$ and $\beta = \frac{1}{n}\Sigma^{n/2}$.

In fact I_1) and i) of I_3) are immediately verified. I_2) follows from Corollary 2.3, while ii) of I_3) is deduced from Lemma 3.1 a).

Finally inequality b) of Lemma 3.1 implies iii) of I_3) as soon as $(\lambda_+ - \lambda) < \frac{1}{n}\Sigma|\Omega|^{-2/n}$. □

The following lemma will also be used in the next sections.

LEMMA 3.2. *Let* M *be a finite dimensional subspace of* $V(\Omega) \cap W^{2,\infty}(\Omega)$. *If* $\varphi \in V(\Omega)$ *and*

$$|\varphi|_{2^*}^{2^*} - c\left(|\varphi|_1^{2^*} + |\varphi|_{2^*-1}^{2^*}\right) > 0 \tag{3.7}$$

for some $c > 0$.

Then there exist two positive constants $l_1(\varphi)$ *and* $l_2(\varphi)$ *such that*

$$\text{(3.8)}\quad \begin{cases} \text{i)} & \sup\{F_\lambda(w + t\varphi) \colon w \in M,\ t \in \mathbb{R}\} \le \\ & \le \sup\{F_\lambda(w + t\varphi) \colon w \in M,\ t \in \mathbb{R},\ |w|_{2^*}^{2^*} \le \\ & \le l_1(\varphi),\ |t|^{2^*} \le l_1(\varphi)\}. \\ \text{ii)} & F_\lambda(v + t\varphi) \le F_\lambda(v) + F_\lambda(t\varphi) + \\ & + l_2(\varphi)\left[\left(|v|_{2^*} + |v|_{2-1^*}^{2-1^*}\right)|\varphi|_1 + |v|_{2^*}|\varphi|_{2^*-1}^{2^*-1}\right] \\ & \textit{for any } v \in M,\ t \in \mathbb{R},\ |v|_{2^*}^{2^*} \le l_1(\varphi),\ |t|^{2^*} \le l_1(\varphi). \end{cases}$$

PROOF.

i)

$$
\begin{aligned}
&|w+t\varphi|_{2^*}^{2^*} - |w|_{2^*}^{2^*} - |t\varphi|_{2^*}^{2^*} = \\
&= 2^* \int_\Omega \int_0^1 \left[|w+\sigma t\varphi|^{2^*-2}(w+\sigma t\varphi) - |\sigma t\varphi|^{2^*-2}\sigma t\varphi \right] t\varphi d\sigma\, dx = \\
&= 2^*(2^*-1) \int_\Omega \int_0^1 |\vartheta w + \sigma t\varphi|^{2^*-2} w t\varphi d\sigma\, dx \leq \\
&\leq 2^*(2^*-1) \int_\Omega \left(|w|^{2^*-1}|t\varphi| + |w|\,|t\varphi|^{2^*-1} \right)
\end{aligned} \tag{3.9}
$$

where $\theta \in]0,1[$.

Since M is finite dimensionale all the norms in M are equivalent, thus, from (3.9), using, twice Young's inequality with $p = p^*$ and $p = p^*/(p^*-1)$ we get

$$
\begin{aligned}
&\left| |w+t\varphi|_{2^*}^{2^*} - |w|_{2^*}^{2^*} - |t\varphi|_{2^*}^{2^*} \right| \leq \\
&\leq 2^*(2^*-1)\left(|w|_\infty^{2^*-1} |t\varphi|_1 + |w|_\infty |t\varphi|_{2^*-1}^{2^*-1} \right) \leq \\
&\leq c(M)\left(|w|_{2^*}^{2^*} |t\varphi|_1 + |w|_{2^*} |t\varphi|_{2^*-1}^{2^*-1} \right) \leq \\
&\leq \frac{2^*-1}{2^*}\frac{1}{4}|w|_{2^*}^{2^*} + |t|^{2^*} c(M) |\varphi|_1^{2^*} + \frac{1}{2^*}\frac{1}{4}|w|_{2^*}^{2^*} + |t|^{2^*} c(M)|\varphi|_{2^*-1}^{2^*} = \\
&= \frac{1}{2}|w|_{2^*}^{2^*} + |t|^{2^*} c(M)\left(|\varphi|_1^{2^*} + |\varphi|_{2^*-1}^{2^*} \right).
\end{aligned}
$$

Therefore

$$
|w+t\varphi|_{2^*}^{2^*} \geq \frac{1}{2}|w|_{2^*}^{2^*} + |t|^{2^*}|\varphi|_{2^*}^{2^*} - |t|^{2^*} c(M)\left(|\varphi|_1^{2^*} + |\varphi|_{2^*-1}^{2^*} \right).
$$

From here it follows that, whenever $|w+t\varphi|_{2^*}^{2^*} \leq$ const. and φ satisfies

(3.7), then:

$$\text{(3.11)}\qquad \begin{aligned} &\text{i)}\quad |w|_{2^*}^{2^*} \le 2\ \text{const.}\\ &\text{ii)}\quad |t|^{2^*} \le \frac{\text{const.}}{|\varphi|_{2^*}^{2^*} - c(M)\Big(|\varphi|_1^{2^*} + |\varphi|_{2^*-1}^{2^*}\Big)}\,. \end{aligned}$$

Thus we prove i) whenever we prove that there exists $l_1(\varphi) > 0$ such that

$$\begin{aligned} &\sup\{F_\lambda(w+t\varphi)\colon w \in M,\ t \in \mathbb{R}\} \le \\ &\le \sup\{F_\lambda(w+t\varphi)\colon w \in M,\ t \in \mathbb{R},\ |w+t\varphi|_{2^*}^{2^*} \le l_1(\varphi)\}\,. \end{aligned}$$

Suppose, by contradiction, that this is not true, then, by (3.3) and (3.4), There exists a sequence $u_m = w_m + t_m\varphi$ such that

$$\begin{aligned} \|u_m\|^2 - \lambda|u_m|_2^2 &= |u_m|_{2^*}^{2^*}\\ |u_m|_{2^*}^{2^*} &> m\,. \end{aligned}$$

In this case $v_m = u_m/|u_m|_{2^*} = w'_m + t'_m\varphi$ satisfies

$$|v_m|_{2^*}^{2^*} = 1 \tag{3.12}$$

$$\|v_m\|^2 - \lambda|v_m|_2^2 > m\,. \tag{3.12'}$$

Since

$$\begin{aligned} &\|v_m\|^2 - \lambda|v_m|_2^2 = \\ &= \|w'_m\|^2 - \lambda|w'_m|_2^2 + \|t'_m\varphi\|^2 - \lambda\|t'_m\varphi\|_2^2 + \\ &+ 2\Big(\int_\Omega \nabla w'_m \nabla t'_m\varphi - \lambda\int_\Omega w'_m t'_m\varphi\Big) \le \\ &\le F_\lambda(w'_m) + F_\lambda(t'_m\varphi) + \frac{1}{2^*}\Big[|w'_m|_{2^*}^{2^*} + |t'_m\varphi|_{2^*}^{2^*}\Big] + \\ &+ c(M)|w'_m|_{2^*}|t'_m\varphi|_1 \end{aligned} \tag{3.13}$$

by (3.6), (3.12) and (3.11) $\|v_m\|^2 - \lambda|v_m|_2^2$ is bounded. This contradicts (3.12').

ii). By (3.9) and by i) we get

$$F_\lambda(w+t\varphi)=$$

$$F_\lambda(w)+F_\lambda(t\varphi)+\int_\Omega \nabla w\nabla(t\varphi)-\lambda\int_\Omega wt\varphi+$$

$$+\frac{1}{2^*}\left[|w|_{2^*}^{2^*}+|t\varphi|_{2^*}^{2^*}-|w+t\varphi|_{2^*}^{2^*}\right]\le$$

$$\le F_\lambda(w)+F_\lambda(t\varphi)+l_2(\varphi)\Big[\Big(|w|_{2^*}+|w|_{2^*-1}^{2^*-1}\Big)|\varphi|_1+|w|_{2^*}|\varphi|_{2^*-1}^{2^*-1}\Big]. \quad \square$$

REMARK 3.1. We observe that from the extimates (3.11) and (3.13) it follows that the functions $l_1(\varphi)$ and $l_2(\varphi)$ are bounded on a subset H of $V(\Omega)$ as soon as there exist $L_1>0$ and $L_2>0$ such that $\sup\{F_\lambda(t\varphi)\colon t\in\mathbb{R}\}<L_1$, $|\varphi|_1<L_1$ and $|\varphi|_{2^*}^{2^*}-c\Big(|\varphi|_1^{2^*}+|\varphi|_{2^*-1}^{2^*}\Big)>L_2$ for any $\varphi\in H$.

Let $x_0\in\Gamma_1$. Let $\rho>0$ and $\tilde\psi\in C_0^\infty(B_\rho(x_0))$, $\tilde\psi\equiv 1$ on $B_{\rho/2}(x_0)$. Define for $\mu>0$:

$$\varphi_\mu(x)=\varphi_{\mu,x_0,\rho}(x)=\psi(x)\cdot\frac{n(n-2)\mu^{(n-2)/4}}{\Big(\mu+|x-x_0|^2\Big)^{(n-2)/2}}, \tag{3.14}$$

where $\psi=\tilde\psi\chi_\Omega$ and χ_Ω is the characteristic function of Ω.

Whenever ρ is sufficiently small φ_μ belongs to $V(\Omega)$.

In the following lemma we list some results about the functions φ_μ (see [7]).

LEMMA 3.3. *Let $n\ge 4$. There exists $K>0$*

$$\begin{aligned}
&\|\varphi_\mu\|^2=\Sigma^{n/2}+0(\mu^{(n-2)/4})\\
&|\varphi_\mu|_1\le K\mu^{(n-2)/4}\\
&|\varphi_\mu|_{2^*-1}^{2^*-1}\le K\mu^{(n-2)/4}\\
&|\varphi_\mu|_{2^*}^{2^*}=\Sigma^{n/2}+0(\mu^{n/2})
\end{aligned} \tag{3.15}$$

Let $n\ge 5$, then

$$|\varphi_\mu|_2^2=K\mu+0(\mu^{(n-2)/2}). \quad \square$$

From the estimates (3.15) and (1.3) it follows:

$$F_\lambda(\varphi_\mu) = \frac{1}{n}\Sigma^{n/2} - \lambda\frac{K}{2}\mu + 0(\mu^{(n-2)/2}) \tag{3.16}$$

$$\sup\left\{F_\lambda(t\varphi_\mu)\colon t \in \mathbb{R}\right\} = \frac{1}{n}\Sigma^{n/2} - \lambda\frac{K}{2}\mu + 0(\mu^{(n-2)/2}) \tag{3.17}$$

and also that, fixed $c > 0$ there exist $\bar{\mu} > 0$, $L_1 > 0$ and $L_2 > 0$ such that for any $\mu < \bar{\mu}$:

$$\begin{aligned} &\sup\{F_\lambda(t\varphi_\mu)\colon t \in \mathbb{R}\} < L_1\,, \quad |\varphi|_1 < L_1 \text{ and} \\ &|\varphi_\mu|_{2^*}^{2^*} - c\left(|\varphi_\mu|_1^{2^*} + |\varphi_\mu|_{2^*-1}^{2^*}\right) > L_2\,. \end{aligned} \tag{3.18}$$

If μ is sufficiently small, from (3.16) it follows that $F_\lambda(\varphi_\mu) > 0$, thus $\varphi_\mu \notin M_-$.

Therefore

$$\dim W_\mu - \operatorname{codim} M_+ = 1 \tag{3.19}$$

where

$$W_\mu = \{u \in V(\Omega)\colon u = w + t\varphi_\mu\ w \in M_- t \in \mathbb{R}\}\,.$$

Theorem 3.3. *Let $n \geq 5$. Then for any $\lambda > 0$ there exists at least a pair of solutions of the problem* (1).

Proof. As in [8] we apply Theorem 3.1 to F_λ, with $\beta = \frac{1}{n}\Sigma^{n/2}$ and $V = M_+$. Thus I_1), I_2), I_3ii) hold as in Theorem 3.2. We set $W = W\mu$ choosing μ sufficiently small in such a way that (3.19), i.e. I_3 i), and (3.7) hold. Setting in Lemma (3.2) $M = M_-$ and $\varphi = \varphi_\mu$, we get, by Remark 3.1, (3.16), (3.17), (3.18):

(3.20)
$$F_\lambda(v + t\varphi_\mu) \leq -\frac{1}{2^*}|v|_{2^*}^{2^*} + \frac{1}{n}\Sigma^{n/2} - \lambda\frac{K}{2}\mu + L_2\left(|v|_{2^*} + |v|_{2^*-1}^{2^*-1}\right)\mu^{(n-2)/4}$$

where $v \in M_-$, $|v|_{2^*} < L_1$.

If $n \geq 7$ from (3.20) it follows I_3iii) for μ sufficiently small.

Finally let us point out that if $\lambda \neq \lambda_j$

$$(3.21) \qquad A(\mu) = -\frac{1}{2^*}|v|_{2^*}^{2^*} + L_2\Big(|v|_{2^*} + |v|_{2^*}^{2^*-1}\Big)\mu^{(n-2)/4} < c\mu^{\frac{n-2}{4}\frac{2^*}{2^*-1}}\,.$$

In fact either $A(\mu) < 0$ or $|v|_{2^*} \leq L\mu^{\frac{n-2}{4}\frac{1}{2^*-1}}$ therefore by (3.20) we obtain I_3iii) for μ sufficiently small.

If $\lambda = \lambda_j$ we get (3.20) and (3.21) with $\tilde{w} = w - \pi_j w$ instead of w, where π_j is the projector on the subspace $M(\lambda_j)$. □

From the previous theorems we immediately have

COROLLARY 3.2.

If $\lambda \geq \lambda_1$ there exist at least two nodal solutions of problem (1).

If $\lambda \geq \lambda_1$ and $\lambda \in]\lambda_j - \Sigma|\Omega|^{-2/n}, \lambda_j]$, there exist at least $2m_j$ nodal solutions of (1). □

4 – Existence of nodal solutions. Case $\lambda \in]0, \lambda_1[$

Let us start by giving some results that we will also use in the next section. Let $\lambda \in [0, \lambda_1[$, by Lemma 2.1, the nodal solutions of (1) belong to

$$(4.1) \qquad U = \{u \in V(\Omega) \colon \langle dF_\lambda(u^\pm), u^\pm\rangle = 0 \quad \text{and} \quad u^\pm \neq 0\}\,.$$

Setting

$$c = \inf\{F_\lambda(u) \colon u \in U\}$$

from Corollary 2.2 it follows that

$$(4.2) \qquad c \geq \frac{2}{n}\Big(1 - \frac{\lambda}{\lambda_1}\Big)^{n/2} S(\Omega)^{n/2} > 0\,.$$

Let $P = \{u \in V(\Omega): u \geq 0\}$ and $\mathcal{M}$ be the set of maps σ such that

$$
(4.3)\qquad
\begin{array}{lll}
\text{i)} & \sigma \in C(Q, V(\Omega)) & Q = [0,1] \times [0,1] \\
\text{ii)} & \sigma(s,0) = 0 & \text{for any } s \in [0,1] \\
\text{iii)} & [1/(t_0(\sigma(s,1))]^{4/(n-2)} \geq 2^* & \text{for any } s \in [0,1] \\
\text{iv)} & \sigma(0,t) \in P & \text{for any } t \in [0,1] \\
\text{v)} & -\sigma(1,t) \in P & \text{for any } t \in [0,1]
\end{array}
$$

We have the following

LEMMA 4.1.

$$c = \inf_{\sigma \in \mathcal{M}} \max \{F_\lambda(v): v \in \sigma(Q)\}$$

PROOF. We observe that by (3.5) for any $u > 0$, $v > 0$ linearly independent, the map $\sigma_{u,v}$ defined by

$$\sigma_{u,v}(s,t) = t(2^*)^{(n-2)/4} t_0^s[(1-s)u - sv] \tag{4.3'}$$

where $t_0^s = t_0[(1-s)u - sv]$ as in (3.1), belongs to $\mathcal{M}$. Thus $\mathcal{M} \neq \emptyset$ and in particular for any $u \in U$, the map $\sigma_u = \sigma_{u^+,u^-}$ belongs to $\mathcal{M}$. For such a map we have:

$$F_\lambda(u) = \max \{F_\lambda(v): v \in \sigma_u(Q)\}\,. \tag{4.4}$$

In fact $u = \sigma_u\left(\frac{1}{2}, \left(\frac{1}{2^*}\right)^{(n-2)/4}\right)$ and, from (3.3) and (4.1), for any $\alpha, \beta > 0$

$$F_\lambda(\alpha u^+ - \beta u^-) = F_\lambda(\alpha u^+) + F_\lambda(\beta u^-) \leq F_\lambda(u^+) + F_\lambda(u^-) = F_\lambda(u)\,.$$

From (4.4) it follows

$$
\begin{aligned}
&\inf_{\sigma \in \mathcal{M}} \max \{F_\lambda(v): v \in \sigma(Q)\} \leq \\
&\inf_{u \in U} \max \{F_\lambda(v): v \in \sigma_u(Q)\} = \qquad (4.5)\\
&\inf \{F_\lambda(u): u \in U\}\,.
\end{aligned}
$$

On the other hand, fixed $\sigma \in \mathcal{M}$, $U \cap \sigma(Q) \neq \emptyset$. In fact we observe that

$$\left[\frac{1}{t_0(u)}\right]^{4/(n-2)} = \frac{|u|_{2^*}^{2^*}}{\|u\|^2 - \lambda |u|_2^2} \leq \frac{S(\Omega)^{-2^*/2}}{1 - \frac{\lambda}{\lambda_1}} \|u\|^{2^*-2}.$$

Thus, setting

$$f_1(s,t) = \left[\frac{1}{\bar{t}_0(\sigma^+(s,t))}\right]^{4/(n-2)} - \left[\frac{1}{\bar{t}_0(\sigma^-(s,t))}\right]^{4/(n-2)}$$

$$f_2(s,t) = \left[\frac{1}{\bar{t}_0(\sigma^+(s,t))}\right]^{4/(n-2)} + \left[\frac{1}{\bar{t}_0(\sigma^-(s,t))}\right]^{4/(n-2)} - 2$$

where

$$\frac{1}{\bar{t}_0(u)} = \begin{cases} \dfrac{1}{t_0(u)} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0, \end{cases}$$

we can prove that f_1 and f_2 belong to $C(Q, \mathbb{R})$.

Moreover from (4.3) iv), v) and i) we get

$$f_1(0,t) \geq 0, \; f_1(1,t) \leq 0 \text{ for any } t \in [0,1] \text{ and } f_2(s,0) < 0 \text{ for any } s \in [0,1].$$

Furthermore from

$$\frac{|u|_{2^*}^{2^*}}{\|u\|^2 - \lambda|u|_2^2} \leq \frac{|u^+|_{2^*}^{2^*}}{\|u^+\|^2 - \lambda|u^+|_2^2} + \frac{|u^-|_{2^*}^{2^*}}{\|u^-\|^2 - \lambda|u^-|_2^2}$$

and from (4.3) iii) and (3.5) it follows

$$f_2(s,1) \geq \left[\frac{1}{t_0(\sigma(s,1))}\right]^{4/(n-2)} - 2 \geq 2^* - 2 > 0.$$

Thus from a fixed point theorem of C. Miranda [19] it follows that there exists a point $(\bar{s}, \bar{t}) \in Q$ such that

$$f_1(\bar{s}, \bar{t}) = f_2(\bar{s}, \bar{t}) = 0.$$

Obviously for such a point

$$t_0\Big(\sigma(\bar{s},\bar{t})^+\Big) = t_0\Big(\sigma(\bar{s},\bar{t})^-\Big) = 1\,,$$

from which, by (3.3) and Lemma 2.1, we deduce

$$u_\sigma = \sigma(\bar{s},\bar{t}) \in U \cap \sigma(Q)\,. \tag{4.6}$$

Thus

$$c = \inf\left\{F_\lambda(u) \colon u \in U\right\} \le \inf\left\{F_\lambda(u_\sigma) \colon \sigma \in \mathcal{M}\right\} \le$$
$$\le \inf_{\sigma\in\mathcal{M}} \max\left\{F_\lambda(v) \colon v \in \sigma(Q)\right\}$$

This ends the proof. □

LEMMA 4.2. *There exists $\alpha > 0$ and a sequence $v_m \in V(\Omega)$ such that*

$$F_\lambda(v_m) \to c$$
$$dF_\lambda(v_m) \to 0 \textit{ in } V^*(\Omega)$$
$$\|v_m^\pm\|^2 > \alpha\,.$$

PROOF. Let $u_m \in U$ be a minimizing sequence for $F_\lambda(u)$. Setting, for brevity, $\sigma_m = \sigma_{u_m}$ as in (4.3)′, we claim that there exist two sequences (v_m) and (w_m) in $V(\Omega)$, such that

$$\begin{aligned} &\text{i)} && F_\lambda(v_m) \to c \\ &\text{ii)} && dF_\lambda(v_m) \to 0 \text{ in } V^*(\Omega) \\ &\text{iii)} && w_m = \alpha u_m^+ - \beta u_m^- \in \sigma_m(Q) \quad \alpha,\beta > 0 \\ &\text{iv)} && \|w_m - v_m\| \to 0\,. \end{aligned} \tag{4.7}$$

In fact if (4.7) are not true, there exists $\delta > 0$ such that we get definitively

$$\sigma_m(Q) \cap A_\delta = \emptyset \tag{4.8}$$

where

$$A = \Big\{ v \in V(\Omega) \colon \|v - u\| < \delta \text{ for some } u \in V(\Omega) \text{ such that } |F_\lambda(u) - c| < \delta$$
$$\text{and } |dF_\lambda(u)| < \delta \text{ in } V^*(\Omega) \Big\}.$$

Using a deformation lemma by H. HOFER [15], we can construct a continuous map

η: $[0,1] \times V(\Omega) \to V(\Omega)$ such that, for some $\varepsilon \in]0, c/2[$

a) $\eta(0,u) = u$ for any $u \in V(\Omega)$

b) if $F_\lambda(u) > c + \varepsilon$ or $F_\lambda(u) \leq c - \varepsilon$ then

$$\eta(t,u) = u \qquad \forall\, t \in [0,1]$$

c) if $F_\lambda(u) < c + \varepsilon/2$ and $u \notin A$ then

$$F_\lambda(\eta(1,u)) < c - \varepsilon/2 .$$

d) Moreover, in the same hypothesis of c), if $u \in P$ then $\eta(1,u) \in P$

e) $\eta(t,-u) = -\eta(t,u)$ for any $t \in [0,1]$.

Setting

$$\sigma^*_m(s,t) = \eta(1, \sigma_m(s,t))$$

from (4.3) ii), (4.2) and b) it follows

$$\sigma^*_m(s,0) = \sigma_m(s,0)$$

and from (4.3) iii), (3.5), (4.2) and b)

$$\sigma^*_m(s,1) = \sigma_m(s,1) .$$

Thus σ^*_m satisfies (4.3) ii) and iii). Furthermore from d) and e) it follows that σ^*_m satisfies also (4.3) iv) and v). Thus $\sigma^*_m \in \mathcal{M}$.

On the other hand, from (4.8) and c) it follows that for m sufficientely large

$$\max\Big\{F_\lambda(v) \colon v \in \sigma^*_m(Q)\Big\} < c - \frac{\varepsilon}{2}, \tag{4.9}$$

which contradicts Lemma 4.1. Therefore (4.7) holds and, by continuity of F_λ, we get:

$$F_\lambda(w_m) = F_\lambda(u_m) + o(1). \tag{4.10}$$

We claim that neither w_m^+ nor w_m^- converges to 0 in $V(\Omega)$. In fact assume that w_m^+ converges to 0, then from (4.7) iii), (4.10) and Corollary (2.1), it follows, for m sufficiently large:

$$\begin{aligned} F_\lambda(\beta u_m^-) &= F_\lambda(w_m^-) = F_\lambda(u_m) + o(1) = \\ &= F_\lambda(u_m^+) + F_\lambda(u_m^-) + o(1) \geq \\ &\geq \left(1 - \frac{\lambda}{\lambda_1}\right)^{n/2} S(\Omega)^{n/2} + o(1) + F_\lambda(u_m^-) > F_\lambda(u_m^-) \end{aligned}$$

which contradicts (3.3) since $u_m \in U$.

Thus there exists $\alpha > 0$ such that $\|w_m^\pm\|^2 \geq \alpha$. By (4.7) iv) the same holds for v_m. □

We recall that

THEOREM 4.1. [13] *Let $n \geq 4$ and $\lambda \in]\lambda^*, \lambda_1[$ where $\lambda^* \leq 0$ is a constant depending on the geometry of Ω. Then there exists a positive solution u_0 of* (1) *such that*

$$F_\lambda(u_0) = \frac{1}{n}\left(1 - \frac{\lambda}{\lambda_1}\right)^{n/2} S(\Omega)^{n/2}. \tag{4.11}$$

□

We observe that from (3.3) it follows that $F_\lambda(u_0) = \sup\{F_\lambda(tu_0)\ t \in \mathbb{R}\}$.

LEMMA 4.3. *Let $n \geq 7$, $0 < \lambda < \lambda_1$. Then there exists $\sigma \in \mathcal{M}$ such that*

$$\max\{F_\lambda(v) : v \in \sigma(Q)\} < \frac{1}{n}\left(1 - \frac{\lambda}{\lambda_1}\right)^{n/2} S(\Omega)^{n/2} + \frac{1}{n}\Sigma^{2/n}. \tag{4.12}$$

PROOF. From (4.11), (3.16) and Theorem 1.3 it follows that, for μ sufficiently small, $F_\lambda(\varphi_\mu) > F_\lambda(u_0)$, where φ_μ is defined as in (3.14) and u_0 as in Theorem 4.1. Thus φ_μ and u_0 are linearly independent. Set $\sigma = \sigma_{u_0,\varphi_\mu}$, assuming that μ is sufficiently small in such a way that (3.7) holds. Then, setting in Lemma 3.2 $M = \{tu_0 \colon \in \mathbb{R}\}$ and $\varphi = \varphi_\mu$ from Remark 3.1, (3.18), (3.17) and (4.11) we get

$$F_\lambda(\alpha u_0 - \beta\varphi_\mu) \le F_\lambda(\alpha u_0) + F_\lambda(\beta\varphi_\mu) + d\mu^{(n-2)/4} \le$$
$$\le \frac{1}{n}\left(1 - \frac{\lambda}{\lambda_1}\right)^{n/2} S(\Omega)^{n/2} + \frac{1}{n}\Sigma^{2/n} - \lambda K\mu + d\mu^{(n-2)/4},$$

from which (4.12) follows for μ sufficiently small and $n \ge 7$. □

Finally, from Lemma 4.2, Lemma 4.3 and Corollary 2.3 iii), we get:

THEOREM 4.2. *Let $n \ge 7$, $0 < \lambda < \lambda_1$, then there exists a least a pair of nodal solutions of the problem* (1). □

5 – Existence of nodal solutions in the case $\lambda = 0$

In this section we will assume $n \ge 4$. When $\lambda = 0$ the mixed boundary problem is quite different from the Dirichlet problem. In fact in [17] it is proved that, whenever Ω is regular and $S(\Omega) < \Sigma$, the infimum in (1.2) is achieved. This implies the existence of a positive solution u_0 of (1) such that:

$$F_0(u_0) = \frac{1}{n} S(\Omega)^{n/2}. \tag{5.1}$$

Therefore we could think of repeating the same procedure of the previous section in case $\lambda = 0$.

Using the same definition for the family $\mathcal{M}$ and the value c as in the previous section, we prove that Lemma 4.1 and Lemma 4.2 hold also in the case $\lambda = 0$.

To prove the analogue of Lemma 4.3, instead, we need to have two positive functions v_0, ϕ_0, linearly independent, to be replaced in the proof of Lemma 4.3, i.e. such that:

$$\max\{F_0(v) \colon v \in \sigma_{v_0,\phi_0}(Q)\} < \frac{1}{n} S(\Omega)^{n/2} + \frac{1}{n}\Sigma^{n/2}. \tag{5.2}$$

If this is true, then the existence of a pair of nodal solutions of the problem (1) follows from Corollary 2.3.

It is easy to see that we can prove (5.2) whenever we find v_0, ϕ_0 and $\eta > 0$ such that:

$$
(5.3) \qquad \begin{cases} \max\{F_0(\alpha v_0) \colon \alpha \in \mathbb{R}\} < \dfrac{1}{n} S(\Omega)^{n/2} + \eta \\ \max\{F_0(\beta\phi_0) \colon \beta \in \mathbb{R}\} \leq \dfrac{1}{n}\Sigma^{n/2} - \eta \\ \operatorname{supp} v_0 \cap \operatorname{supp} \phi_0 = \emptyset \end{cases}
$$

We recall that by "isoperimetric constant of Ω relative to Γ_1" we mean

$$
Q(\Gamma_1, \Omega) = \sup \frac{|E|^{1-1/n}}{P_\Omega(E)}
$$

where the supremum is taken over all measurable subsets E of Ω such that $\partial E \cap \Gamma_0$ does not contain any set of positive $(n-1)$-dimensional Hausdorff measure and P_Ω represents the perimeter of E relative to Ω, that is

$$
P_\Omega(E) = \sup\left\{ \Big| \int_E \operatorname{div}\psi \, dx \Big|, |\psi| \leq 1 \quad \psi \in \left[C_0^\infty(\Omega)\right]^n \right\}.
$$

Let Σ_α be an open cone in $\mathbb{R}^n$ with vertex in the origin and solid angle $\alpha \in]0, \omega_{n-1}]$, where ω_{n-1} is the $(n-1)$ dimensional Hausdorff measure of the unit sphere S^{n-1}. We denote by $\Sigma(\alpha, R)$ the open sector with solid angle α and radius $R > 0$, that is $\Sigma(\alpha, R) = \Sigma_\alpha \cap B_R(0)$. By the symbol α_n we mean the measure of any unitary sector $\Sigma(\alpha, 1)$ with solid angle α.

Define ε_{α_n} the class of all open sets $\Omega \subset \mathbb{R}^n$ such that $Q(\Gamma_1, \Omega) = (n\alpha_n^{1/n})^{-1}$. We list some result contained in [17] and [18].

THEOREM 5.1.

i) *Any convex sector $\Sigma(\alpha, R)$ such that $|\Sigma(\alpha, 1)| = \alpha_n$ belongs to ε_{α_n}*

ii) *Let Ω belong to ε_{α_n}, then*

$$
(5.4) \qquad S(\Omega) \geq B^{-1/2^*} n\alpha_n^{1/n}
$$

where $B > 0$ is a constant depending only on the dimension n. Moreover in (5.4) the equality holds whenever $\Omega = \Sigma(\alpha, R)$ as in i). □

We observe that, given a smooth domain Ω, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, using the definition and the property of $Q(\Gamma_1, \Omega)$, [18], we can deform Γ_1 adding a small convex angle of amplitude α in such a way that the new domain Ω' and the convex sector $\Sigma(\alpha, R)$ belong to the same class ε_{α_n}. Analogously, if $S(\Omega) < \Sigma$, we can deform Γ_1 adding a convex angle of amplitude β in such a way that, $\beta < \left|\frac{\omega_{n-1}}{2}\right|$ and $\left|S(\Omega) - S(\Omega')\right| < \varepsilon$, $\varepsilon > 0$.
We give now an example of a domain on which F_0 has a nodal solution.

Let Ω be a smooth domain. Let us change Γ_1 by adding two disjoint convex angles with amplitude $\alpha < \beta$ as above and denote by Ω' the new domain, by x_α and x_β the vertices of the angles. Set $\chi_\mu(x) = \phi_{\mu, x_\alpha, \rho}(x)$ and $\psi_\lambda(x) = \phi_{\lambda, x_\beta, \rho}(x)$ where the functions ϕ are defined as in (3.14) and $\rho > 0$ is such that

$$B_\rho(x_\alpha) \cap B_\rho(x_\beta) = \emptyset\,. \tag{5.5}$$

Then, given $\varepsilon > 0$ and $\eta > 0$ sufficiently small, we can choose μ and λ sufficiently small in such a way that:

$$\frac{\int_{\Omega'} \left|\nabla \chi_\mu\right|^2}{\int_{\Omega'} \left|\chi_\mu\right|^{2^*}} \le B^{-1/2^*} n \alpha_n^{1/n} + \varepsilon \le S(\Omega) + \varepsilon \text{ (because of (5.4))}$$

$$\frac{\int_{\Omega'} \left|\nabla \psi_\lambda\right|^2}{\int_{\Omega'} \left|\psi_\lambda\right|^{2^*}} \le B^{-1/2^*} n \beta_n^{1/n} + \varepsilon < \Sigma - 2\eta\,.$$

Finally take a smooth open set $\Omega'' \subseteq \Omega'$, such that $\Gamma_0 = \Gamma_0''$ and $\left|\Omega'' - \Omega'\right|$ is small enough. Then:

$$\left|\frac{\int_{\Omega''} \left|\nabla \chi_\mu\right|^2}{\int_{\Omega''} \left|\chi_\mu\right|^{2^*}} - B^{-1/2^*} n \alpha_n^{1/n}\right| \le 2\varepsilon \tag{5.6}$$

$$\frac{\int_{\Omega''} \left|\nabla\psi_\lambda\right|^2}{\int_{\Omega''} \left|\psi_\lambda\right|^{2^*}} \leq \Sigma - 2\eta \tag{5.7}$$

and

$$\left|S(\Omega'') - B^{-1/2^*} n\alpha_n^{1/n}\right| < 2\varepsilon \tag{5.8}$$

Choosing ε and η in a suitable way, we get:

LEMMA 5.1.

$$\max\left\{F_0(v) \colon v \in \sigma_{\chi_\mu,\psi_\lambda}(Q)\right\} < \frac{1}{n} S(\Omega'')^{n/2} + \frac{1}{n}\Sigma^{n/2}.$$

PROOF. Fron (5.5), (5.6), (5.7), (5.8) and (3.17) it follows

$$F_0(\alpha\chi_\mu - \beta\psi_\lambda) = F_0(\alpha\chi_\mu) + F_0(\beta\psi_\lambda) \leq \frac{1}{n} S(\Omega'')^{n/2} + \frac{1}{n}\Sigma^{n/2}.$$

Therefore (5.3) follows and hence (5.2). □

Finally from Lemma 4.1, Lemma 4.2, Lemma 5.1 and Corollary 2.3 iii) we have:

THEOREM 5.2. *Let Ω, $\partial\Omega = \Gamma_0 \cup \Gamma_1$ be a domain in $\mathbb{R}^n$ as in the previous sections. For any $\varepsilon > 0$ there exists a domain Ω'' with $\partial\Omega'' = \Gamma_0'' \cup \Gamma_1''$, $\left|H_{n-1}(\Gamma_1'') - H_{n-1}(\Gamma_1)\right| < \varepsilon$, $\left|\Omega'' - \Omega\right| < \varepsilon$ such that the problem (1), with $\lambda = 0$, has a pair of nodal solutions.* □

REFERENCES

[1] A. AMBROSETTI - M. STRUWE: *A note on the problem* $-\Delta u = \lambda u + u|u|^{2^*-2}$, Manuscripta Math. 54 (1986) 373-379.

[2] P. BARTOLO - V. BENCI - D. FORTUNATO: *Abstract critical point theorems and applications to some nonlinear problems with "strong resonance" at infinity*, Journal of nonlinear Anal. T.M.A. 7, (1983), 981-1012.

[3] H. BREZIS: *Analyse Fonctionelle*, Masson Paris (1983).

[4] H. BREZIS: *Elliptic equations with limiting Sobolev exponents. The impact of topology*, Comm. Pure Appl. Mat. XXXIX, (1986), 17-39.

[5] H. BREZIS: *Nonlinear elliptic equations involving the critical Sobolev exponent-survey and perspective*, Directions in Part. Diff. Eq. - Acad. Press., (1987), 17-36.

[6] H. BREZIS - J.M. CORON: *Convergence of solutions of H-Systems or how to blow bubbles*, Arch. Rat. Mech. Anal. 89, (1985), 21-56.

[7] H. BREZIS - L. NIREMBERG: *Positive solutions of nonlinear elliptic equations involving critical Sobolev Exponents*, Comm. Pure App. Math. XXXVI (1983), 437-477.

[8] A CAPOZZI - D. FORTUNATO - G. PALMIERI: *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Ins. H. Poincaré Anal. Nonlinéaire t. 2 (1985), 463-470.

[9] G. CERAMI - D. FORTUNATO - M. STRUWE: *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Ins. H. Poincaré - Anal. Nonlinéaire t. I (1984), 341- 350.

[10] G. CERAMI - S. SOLIMINI - M. STRUWE: *Some existence results for superlinear elliptic boundary value problems involving critical exponents*, J. of Funct. Anal. 69, (1986), 289-306.

[11] H. EGNELL - F. PACELLA - M. TRICARICO: *Some remarks on Sobolev inequalities*, Nonlinear Analysis T.M.A. 9, (1989), 763-773.

[12] B. GIDAS: *Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic problems*, Nonlinear Part. Diff. Eq. in Engineering and Applied Sciences. Ed. R. Sternberg. A. Kalinowski - J. Papadakis (New York - Dekker 1980).

[13] M. GROSSI: *On some semilinear elliptic equations with critical nonlinearities and mixed boundary conditions*, Rend. Mat. Roma (199).

[14] M. GROSSI - F. PACELLA: *Positive solutions of nonlinear elliptic equations with critical Sobolev exponent and mixed boundary conditions*, Proceed. of Royal Soc. of Edimb. 116 a (1990), 23-45.

[15] H. HOFER: *Variational and topological methods in partially ordered Hilbert spaces*, Math. Ann. 261, (1982), 493-514.

[16] P.L. LIONS: *The concentration compactness principle in the calculus of variations. The limit case (part 1 and part 2)*, Riv. Mat. Iberoamericana 1, (1985), 145-201 and 45-121.

[17] P.L. LIONS - F. PACELLA: *Isoperimetric inequalities for convex cones*, Proceed Amer. Math. Society. 109, (1990), 477-485.

[18] P.L. LIONS - F. PACELLA - M. TRICARICO: *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. 37, (1988), 301-324.

[19] C. MIRANDA: *Un'osservazione sul teorema di Brouwer*, Boll. Un. Mat. Ital. Serie II n°3, (1941), 5-7.

[20] M. STRUWE: *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. 187, (1984), 511-517.

[21] G. TALENTI: *Best constants in Sobolev inequality*, Ann. Mat. Pura e Appl. 110, (1976), 353-372.

Lavoro pervenuto alla redazione il 5 marzo 1991
ed accettato per la pubblicazione il 10 aprile 1991
su parere favorevole di U. Mosco e di F. Pacella

INDIRIZZO DELL'AUTORE:

Maria Vittoria Marchi - Dipartimento di Matematica - Università degli Studi di Roma "La Sapienza" - P.le A. Moro, 2 - 00185 Roma - Italia