Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 285-297

New results on univalence of Gelfond-Leontev derivatives of functions defined by gap power series

J. PATEL

RIASSUNTO – *Sia $f(z)$ una funzione analitica definita mediante una serie di potenze con raggio di convergenza R e lacune non superiori a $(k-1)$, essendo k un intero positivo ≥ 1. Sia $D^n f$ l'n-esima iterata della derivata secondo Gelfond-Leontev di f. Nel presente lavoro vengono dimostrate delle relazioni che collegano R con il raggio di univalenza ρ_n di $D^n f$ e con i parametri di crescenza γ e δ della stessa f.*

ABSTRACT – *Let $f(z)$ be an analytic function defined by a power series of radius convergence R and gaps no greater than $(k-1)$, k being a positive integer ≥ 1. Let $D^n f$ be the n-th iterate of the Gelfond-Leontev derivative of f. In the present paper, we find relations connecting R, the radii of univalence ρ_n of $D^n f$ and the growth numbers γ and δ of an entire function f.*

KEY WORDS – *Univalent - Entire - Gelfond-Leontev derivative.*

A.M.S. CLASSIFICATION: 30C45 - 30C50

1 – Introduction

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^{\lambda_j} \tag{1.1}$$

where $a_j \neq 0$ and $\{\lambda_j\}_{j=0}^{\infty}$ is a strictly increasing sequence of positive integers, be analytic in $|z| < R$, $0 < R \leq \infty$. For a strictly increasing sequence $\{d_j\}_{j=1}^{\infty}$ of positive numbers, the Gelfond-Leontev derivative of f, given by (1.1), is defined as [1]

$$Df(z) = \sum_{j=1}^{\infty} d_{\lambda_j} a_j z^{\lambda_j - 1} \tag{1.2}$$

For $n = 1, 2, \dots$, the nth iterate $D^n f$ of Df is given by

$$D^n f(z) = \sum_{j=n}^{\infty} d_{\lambda_j} d_{\lambda_j - 1} \dots d_{\lambda_j - n+1} a_j z^{\lambda_j - n}$$

We note that for $d_j = j, j = 1, 2, 3, \dots, Df$ is the usual derivative of f. We shall assume throughout in the sequel that $d_j \to \infty$ as $j \to \infty$ and

$$\{d_{j+1}/d_j\}_{j=1}^{\infty} \quad \text{decreases to } 1 \text{ as } j \to \infty. \tag{1.3}$$

A function f of the form (1.1) is said to be defined by a power series with gaps no greater $(k-1)$, k being a positive integer ≥ 1, if $\limsup_{j\to\infty}(\lambda_{j+1} - \lambda_j) = k$. If $k = \infty$, we say that f is defined by a power series with unbounded gaps. Define,

$$\overline{R} = \limsup_{j\to\infty} \{|a_j/a_{j+1}|\}^{1/(\lambda_{j+1}-\lambda_j)}$$

$$\underline{R} = \liminf_{j\to\infty} \{|a_j/a_{j+1}|\}^{1/(\lambda_{j+1}-\lambda_j)}$$

If R is the radius of convergence of the power series given by (1.1), then $\underline{R} \leq R \leq \overline{R}$[7, p.422]. If f is an entire function, i. e., $R = \infty$, we set $\mu(r) = \max_{j\geq 0}\{|a_j|r^{\lambda_j}\}$, $\nu(r) = \max\{\lambda_j : \mu(r) = |a_j|r^{\lambda_j}\}$ and define the growth number γ and δ of f as

$$\gamma = \limsup_{j\to\infty}\left\{\frac{\nu(r)}{r}\right\} \tag{1.4}$$

$$\delta = \liminf_{j\to\infty}\left\{\frac{\nu(r)}{r}\right\}$$

In [4.], it is proved that if $\{|a_j/a_{j+1}|^{1/(\lambda_{j+1}-\lambda_j)}\}_{j=1}^{\infty}$ is an eventually non-decreasing function of j, then

$$\tag{1.5} \begin{aligned} \gamma &= \limsup_{j\to\infty}\left\{\frac{\lambda_j}{|a_j/a_{j+1}|^{1/(\lambda_{j+1}-\lambda_j)}}\right\} \\ \delta &= \liminf_{j\to\infty}\left\{\frac{\lambda_j}{|a_j/a_{j+1}|^{1/(\lambda_{j+1}-\lambda_j)}}\right\} \end{aligned}$$

Let $h(z) = z + \sum_{j=2}^{\infty} h_j z^j$. It is known [2] that $h(z)$ is univalent in the disc of radius ρ centred at origin if

$$\tag{1.6} \sum_{j=2}^{\infty} j|h_j|\rho^{j-1} \leq 1\,.$$

The radius of univalence of $D^n f$ is defined to be the largest positive number ρ_n with the property that $D^n f$ is analytic and univalent in the disc $|z| < \rho_n$.

It this paper, we establish relations of the radii of univalence ρ_n of $D^n f$ with $\underline{R}$ and $\overline{R}$ of the function f defined by (1.1) with gaps no greater than $(k-1)$. We further find relation between ρ_n and the growth numbers γ and δ of an entire functiom f. The results obtained here generalize the corresponding results of Shah and Trimble [6].

For ease of notations we shall sometimes write $a_n = a(n)$, $\rho_n = \rho(n)$ and $d_n = d(n)$.

2 – Statements and Proofs of the results

THEOREM 1. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^{\lambda_j}, a_j \neq 0, \{\lambda_j\}_{j=0}^{\infty}$ is a strictly increasing sequence of positive integers, be analytic in $|z| < R$, $0 < R \leq \infty$, and have gaps no greater than $(k-1)$, k being a positive integer ≥ 1. Let ρ_n be the radius of univalence of $D^n f$ about the origin. Then there is a strictly increasing sequence $\{y_k\}_{k=1}^{\infty}$ of positive numbers given by*

$$\tag{2.1} \sum_{j=k}^{\infty}(j+1)\frac{y_k^j}{d_{j+1}\cdot d_j \dots d_2} = 1$$

such that

$$(2.2)\qquad y_k\underline{R} \le \limsup_{n\to\infty} d_n\rho_n \le \begin{cases} \alpha\overline{R} & ,k<k_0 \\ \left(2\frac{d_{k+1}d_k\dots d_2}{k+1}\right)^{1/k}\overline{R} & ,k\ge k_0 \end{cases}$$

where $\alpha = \max\left\{ (2\frac{d_{k+1}d_k\dots d_2}{k+1})^{1/k} : 1 \le k \le 5\right\}$ *and*

$$k_0 = \min\left\{k : \alpha \le \left(2\frac{d_{k+1}\dots d_2}{k+1}\right)^{1/k}\right\}$$

We need the following lemmas.

LEMMA 1. [3]. *Let* $g(z) = z + \sum_{n=0}^{\infty} a_{p+n}z^{p+n}$ *be analytic and univalent in the unit disc* $U = \{z : |z| < 1\}$. *Then*

$$(2.3)\qquad |a_p| \le \frac{2}{p-1}\,.$$

LEMMA 2. *Let* k *be any positive integer* ≥ 1. *If* $\{\lambda_m\}_{m=1}^{\infty}$ *is a strictly increasing sequence of positive integers, then there is a unique, strictly increasing sequence* $\{y_{m,k}\}_{m=1}^{\infty}$ *of positive numbers with the following properties.*

$$(2.4)\qquad \sum_{j=0}^{\infty}(k+j+1)\frac{d(\lambda_m+k+j)\dots d(\lambda_m+1)}{d(k+j+1)\dots d(2)}\left(\frac{y_{m,k}}{d(\lambda_m)}\right)^{k+j} = 1$$

$$(2.5)\qquad \lim_{m\to\infty} y_{m,k} = y_k$$

Further, $\{y_k\}_{k=1}^{\infty}$ *is a strictly increasing sequence of positive number such that*

$$(2.6)\qquad \sum_{j=k}^{\infty}(j+1)\frac{y_k^j}{d_{j+1}d_j\dots d_2} = 1$$

and

$$(2.7)\qquad \lim_{k\to\infty}\frac{y_k}{(d_{k+1}\dots d_2)^{1/k}} = 1$$

PROOF. Define $\Phi_{m,k}$ on $(0, d(\lambda_m))$ by

$$\Phi_{m,k}(x)=\sum_{j=0}^{\infty}(k+j+1)\frac{d(\lambda_m+k+j)\dots d(\lambda_m+1)}{d(k+j+1)\dots d(2)}\left(\frac{x}{d(\lambda_m)}\right)^{k+j}. \tag{2.8}$$

Clearly, $\Phi_{m,k}$ is a strictly increasing function of x. Moreover, $\Phi_{m,k}(0) = 0$ and $\Phi_{m,k}(x) > 2\frac{d(\lambda_m+1)}{d(2)}\left(\frac{x}{d(\lambda_m)}\right)^k$. Letting $t = d(\lambda_m)\left(\frac{d(2)}{d(\lambda_m+1)}\right)^{1/k}$, we have $\Phi_{m,k}(t) > 2$ and thus by Intermediate Value Theorem there is a unique positive number $y_{m,k}$ in $(0, d(\lambda_m))$ such that $\Phi_{m,k}(y_{m,k}) = 1$.

We claim that $\{y_{m,k}\}_{m=1}^{\infty}$ is an increasing function of m. Since, for a fixed m, $\Phi_{m,k}$ is a strictly increasing function of x, it is enough to show that for a fixed value of x, $\Phi_{m,k}$ is a strictly decreasing function of m. From (1.3), $\{d(\lambda_{m+1}+j)/d(\lambda_m+j)\}$ is a decreasing function of j and hence

$$\frac{d(\lambda_{m+1}+k+j)\dots d(\lambda_{m+1}+1)}{d(\lambda_m+k+j)\dots d(\lambda_m+1)} < \left(\frac{d(\lambda_{m+1}+1)}{d(\lambda_m+1)}\right)^{k+j}$$

That is,

$$(k+j+1)\frac{d(\lambda_{m+1}+k+j)\dots d(\lambda_{m+1}+1)}{d(\lambda_m+k+j)\dots d(\lambda_m+1)}\left(\frac{x}{d(\lambda_{m+1})}\right)^{k+j} <$$

$$< (k+j+1)\frac{d(\lambda_m+k+j)\dots d(\lambda_m+1)}{d(\lambda_m+k+j)\dots d(2)}\left(\frac{x}{d(\lambda_m)}\right)^{k+j}$$

which shows that $\Phi_{m,k}$ is a decreasing function of m.

Next, we prove that $\{y_{m,k}\}_{m=1}^{\infty}$ is bounded above. From (2.4), we have for all m,

$$(k+1)\frac{d(\lambda_m+k)\dots d(\lambda_m+1)}{d(k+1)\dots d(2)}\left(\frac{y_{m,k}}{d(\lambda_m)}\right)^k < 1$$

or,

$$\begin{aligned} y_{m,k} &< \left(\frac{d_{k+1}\dots d_2}{k+1}\right)^{1/k}\frac{d(\lambda_m)}{(d(\lambda_m+k)\dots d(\lambda_m+1))^{1/k}} < \\ &< \left(\frac{d_{k+1}\cdot d_k\dots d_2}{k+1}\right)^{1/k} \end{aligned} \tag{2.9}$$

Since the right side of the above inequality is independent of m, the sequence $\{y_{m,k}\}$ is bounded above. And, since this sequence $\{y_{m,k}\}$ increases with m, it converges to a finite limit. Let $\lim_{m\to\infty} y_{m,k} = y_k$ (say). This proves (2.5).

To prove (2.6), we note that our hypothesis (1.3) gives, $d(\lambda_m + 1)/d(\lambda_m)$ decreases to 1 as $m \to \infty$, from which it follows that the radius of convergence of the series in (2.8) is atleast $d(\lambda_m)$. Let K be any compact subset of $\{x : 0 < x < \sup_m d(\lambda_m)\}$. Then, we can find an integer N so large that K is contained in $\{x : 0 < x < d(\lambda_N)\}$. Further, $\{\Phi_{m,k}\}$ being decreasing function of m, we have for $m \geq N$

$$\Phi_{m,k}(x) \leq \Phi_{N,k}(x) \quad \text{for all} \quad x \in K\,.$$

This gives $\{\Phi_{m,k}\}$ is locally uniformly bounded. Also, it is easily seen that $\{\Phi_{m,k}\}$ is equicontinuous on $(0, d(\lambda_N))$. Hence, the sequence $\{\Phi_{m,k}(x)\}$ converges uniformly on compact subsets of $\{x : 0 < x < d(\lambda_m)\}$. But, since $\lim_{m\to\infty} y_{m,k} = y_k$ and

$$\lim_{m\to\infty} \frac{d(\lambda_m + k + j)\ldots d(\lambda_m + 1)}{d(\lambda_m)^{k+j}} = 1, j = 1, 2, \ldots$$

it follows that y_k satisfies (2.6).

To see that $\{y_k\}_{k=1}^{\infty}$ is strictly increasing, define ψ_k on $(0, \infty)$ by

$$\psi_k(y) = \sum_{j=0}^{\infty} (k + j + 1) \frac{y^{k+1}}{d(k + j + 1)\ldots d(2)}$$

Then $\psi_k(y_k) = 1$. For $j = 0, 1, 2, \ldots$ and $k = 1, 2, 3, \ldots$, we have

$$\text{(2.10)} \qquad \frac{k + j + 2}{(k + j + 1)(k + 2)^{1/(k+1)}} \leq 1$$

and from (2.9)

$$y_k = \lim_{m\to\infty} y_{m,k} \leq \left(\frac{d_{k+1} \cdot d_k \ldots d_2}{k + 1}\right)^{1/k} < \frac{d_{k+1}}{(k + 1)^{1/k}}$$

The above inequality followed by (2.10) gives

$$\psi_k(y_k) = 1 = \sum_{j=0}^{\infty}(k+j+2)\frac{y_{k+1}^{k+j+1}}{d(k+j+2)\dots d(2)} <$$

$$< \sum_{j=0}^{\infty}\frac{(k+j+2)}{(k+2)^{1/(k+1)}}\frac{y_{k+1}^{k+j}}{d(k+j+1)\dots d(2)} <$$

$$< \sum_{j=0}^{\infty}(k+j+1)\frac{y_{k+1}^{k+j}}{d(k+j+1)\dots d(2)} = \psi_k(y_{k+1})$$

and ψ_k being strictly increasing function of y, we must have $y_k < y_{k+1}$, showing that the sequence $\{y_k\}_{k=1}^{\infty}$ is strictly increasing.

It remains to see that $\{y_k\}$ satisfies (2.7). From (2.6), we get

$$1 = \frac{y_k^k}{\left(\frac{d_{k+1}\dots d_2}{k+1}\right)}\sum_{j=0}^{\infty}(k+j+1)\frac{d_{k+1}\dots d_2}{(k+1)d(k+j+1)\dots d(2)}y_k^j$$

$$< \frac{y_k^k}{\left(\frac{d_{k+1}\dots d_2}{k+1}\right)}\left[1 + \left(\frac{k+2}{k+1}\right)\frac{y_k}{d_{k+2}} + \left(\frac{k+2}{k+1}\right)^2\left(\frac{y_k}{d_{k+2}}\right)^2 + \dots\right]$$

$$= \frac{y_k^k}{\left(\frac{d_{k+1}\dots d_2}{k+1}\right)}\left[1 - \left(\frac{k+2}{k+1}\right)\frac{y_k}{d_{k+2}}\right]^{-1}.$$

Thus,

$$1 - \frac{(k+2)y_k}{(k+1)d_{k+2}} < \frac{y_k^k}{\left(\frac{d_{k+1}\dots d_2}{k+1}\right)} < 1. \tag{2.11}$$

On taking limit superior, the second inequality in (2.11) yields

$$\limsup_{k\to\infty}\frac{y_k}{(d_{k+1}\dots d_2)^{1/k}} \le 1. \tag{2.12}$$

Now,

$$\frac{(k+2)y_k}{(k+1)d_{k+2}} < \frac{(k+2)y_k}{(k+1)(d_{k+1}\dots d_2)^{1/k}}.$$

Using the above inequality followed by the first inequality in (2.11), we get

$$\liminf_{k\to\infty} \frac{y_k}{(d_{k+1}\dots d_2)^{1/k}} \geq 1.$$

Combining this with (2.12), we immediately get (2.7). This completes the proof of Lemma 2.

LEMMA 3. *Let k be any positive integer ≥ 1. Let $\{d_m\}_{m=1}^{\infty}$ be a strictly increasing sequence of positive numbers defined as in (1.1) and $\{\lambda_m\}_{m=1}^{\infty}$ be a strictly increasing sequence of positive integers as in Lemma 2. If $\{y_{m,k}\}_{m=1}^{\infty}$ is defined by (2.4), then*

$$\frac{y_{m,k}}{d(\lambda_m)} < \frac{(k+1)d(k+2)}{(k+2)d(\lambda_m+k+1)} \tag{2.13}$$

PROOF. From (2.4), we get

$$(k+2)\frac{d(\lambda_m+k+1)\dots d(\lambda_m+1)}{d(k+2)\dots d(2)}\left(\frac{y_{m,k}}{d(\lambda_m)}\right)^{k+1} < 1$$

By (1.3), it easily follows that $\{d(\lambda_m+j+1)/d(\lambda_m+j)\}_{m=1}^{\infty}$ is a decreasing function of m and thus

$$(k+2)\left(\frac{d(\lambda_m+k+1)}{d(k+2)}\cdot\frac{y_{m,k}}{d(\lambda_m)}\right)^{k+1} < 1$$

or,

$$\frac{y_{m,k}}{d(\lambda_m)} < \frac{d(k+2)}{d(\lambda_m+k+1)(k+2)^{1/(k+1)}}$$

which gives (2.13), because $(k+2)^k \leq (k+1)^{k+1}$ for $k = 1,2,3,\dots$.

PROOF OF THEOREM 1. For $m \geq 1$, let

$$F_m(z) = \frac{D^{\lambda_m-1}f(\rho(\lambda_m-1)z) - D^{\lambda_m-1}f(o)}{(\lambda_m-1)D^{\lambda_m}f(o)} =$$

$$= z + \frac{d(\lambda_{m+1})\dots d(\lambda_m+1)}{d(\lambda_{m+1}-\lambda_m+1)\dots d(2)}\frac{a(m)}{a(m+1)}\cdot$$

$$\cdot\,\rho(\lambda_m-1)^{(\lambda_{m+1}-\lambda_m)}z^{(\lambda_{m+1}-\lambda_m+1)} + \dots$$

Then F_m is univalent in the unit disc U and Lemma 1 gives

$$\frac{d(\lambda_{m+1})\ldots d(\lambda_m+1)}{d(\lambda_{m+1}-\lambda_m+1)\ldots d(2)}\left|\frac{a(m)}{a(m+1)}\right|\rho(\lambda_m-1)^{(\lambda_{m+1}-\lambda_m)}\le\frac{2}{\lambda_{m+1}-\lambda_m}.$$

Since $d(\lambda_m+1)<(d(\lambda_m+1)\ldots d(\lambda_{m+1}))^{1/(\lambda_{m+1}-\lambda_m)}$, it follows that

$$(2.14)\qquad d(\lambda_m+1)\rho(\lambda_m-1)< \\ <\left[2\frac{d(\lambda_{m+1}-\lambda_m+1)\ldots d(2)}{\lambda_{m+1}-\lambda_m}\left|\frac{a(m+1)}{a(m)}\right|\right]^{1/(\lambda_{m+1}-\lambda_m)}$$

Therefore,

$$(2.15)\qquad \limsup_{n\to\infty} d_n\rho_n\le \\ \le\overline{R}\limsup_{m\to\infty}\left[2\frac{d(\lambda_{m+1}-\lambda_m+1)\ldots d(2)}{\lambda_{m+1}-\lambda_m}\right]^{1/(\lambda_{m+1}-\lambda_m)}$$

Let $b_n=\left(2\frac{d(n+1)\ldots d(2)}{n}\right)^{1/n}$. We note that $\{b_n\}_{n=5}^{\infty}$ is a strictly increasing sequence of n so that

$$\limsup_{n\to\infty} d_n\rho_n\le\begin{cases}\alpha\overline{R} & ,k<k_0\\ \left(2\frac{d_{k+1}\cdots d_2}{k+1}\right)^{1/k}\overline{R} & ,k\ge k_0\end{cases}$$

where $\alpha=\max\left\{\left(\frac{d_{k+1}\cdots d_2}{k+1}\right)^{1/k}:1\le k\le 5\right\}$ and

$$k_0=\min\left\{k:\alpha\le\left(2\frac{d_{k+1}\ldots d_2}{k+1}\right)^{1/k}\right\}$$

This proves the right side inequality of (2.2).

To prove the left side inequality of (2.2), assume that $\underline{R}>0$ and let $0<r<\underline{R}$. Then, for sufficiently large m,

$$r\le\left|\frac{a(m)}{a(m+1)}\right|^{1/(\lambda_{m+1}-\lambda_m)}$$

That is, for sufficiently large m,

$$|a(m+1)|r^{\lambda_{m+1}} \leq |a(m)|r^{\lambda_m}$$

Now, an inductive argument on m in the above inequality gives for sufficiently large m and for $j \geq m$

$$|a(j)|r^{\lambda_j} \leq |a(m)|r^{\lambda_m} \tag{2.16}$$

Choose m such that (2.16) holds and $\lambda_{m+1} - \lambda_m = k$. Let $y_{m,k}$ be defined as in (2.4). Set,

$$H_m(z) = D^{\lambda_m - 1} f\left(\frac{r y_{m,k}}{d(\lambda_m)}\right) =$$

$$= D^{\lambda_m - 1} f(0) + \sum_{j=0}^{\infty} \frac{d(\lambda_m + j) \dots d(2)}{d(\lambda_{m+j} - \lambda_m + 1) \dots d(2)} \cdot$$

$$\cdot\, a(m+j)\left(\frac{r y_{m,k}}{d(\lambda_m)}\right)^{\lambda_{m+j} - \lambda_m + 1} z^{\lambda_{m+j} - \lambda_m + 1} =$$

$$= D^{\lambda_m - 1} f(0) + d(\lambda_m) \dots d(2) a(m) \left(\frac{r y_{m,k}}{d(\lambda_m)}\right) z + \dots$$

we first show that H_m is univalent in U. In view of (2.16), we need to prove

$$\sum_{j=0}^{\infty} (\lambda_{m+j} - \lambda_m + 1) \frac{d(\lambda_{m+j+1}) \dots d(2)}{d(\lambda_{m+j+1} - \lambda_m + 1) \dots d(2)} \cdot$$

$$\cdot\, |a(m+j)| \left(\frac{r y_{m,k}}{d(\lambda_m)}\right)^{\lambda_{m+j} - \lambda_m + 1} \leq$$

$$\leq \frac{r y_{m,k}}{d(\lambda_m)} |a(m)| d(\lambda_m) \dots d(2)$$

Since by (1.3), $\{d(\lambda_{m+j+1} + 1)/d(\lambda_{m+j+1} - \lambda_m + 2)\}_{j=0}^{\infty}$ is a decreasing function of j, we get

$$\frac{d(\lambda_{m+j+1} + 1)}{d(\lambda_{m+j+1} - \lambda_m + 2)} \leq \frac{d(\lambda_{m+1} + 1)}{d(\lambda_{m+1} - \lambda_m + 2)} = \frac{d(\lambda_{m+1} + 1)}{d(k+2)}$$

Further, for $j = 0, 1, 2, \ldots$ and $k = 1, 2, 3, \ldots$

$$\frac{\lambda_{m+j+1} - \lambda_m + 2}{\lambda_{m+j+1} - \lambda_m + 1} \leq \frac{k+2}{k+1}$$

Using Lemma 3 and the fact that $(\lambda_{m+1} - \lambda_m) = k$, we have

$$\frac{\lambda_{m+j+1} - \lambda_m + 2}{\lambda_{m+j+1} - \lambda_m + 1} \cdot \frac{d(\lambda_{m+j+1} + 1)}{d(\lambda_{m+j+1} - \lambda_m + 2)} \left(\frac{y_{m.k}}{d(\lambda_m)} \right) \leq$$

$$\leq \frac{k+2}{k+1} \frac{d(\lambda_{m+1} + 1)}{d(k+2)} \cdot \frac{y_{m.k}}{d(\lambda_m)} <$$

$$< 1.$$

This gives for $j = 0, 1, 2, \ldots$

$$(\lambda_{m+j+1} - \lambda_m + 2) \frac{d(\lambda_{m+j+1} + 1)}{d(\lambda_{m+j+1} - \lambda_m + 2)} \cdot \left(\frac{y_{m,k}}{d(\lambda_m)} \right)^{\lambda_{m+j+1} - \lambda_m + 1} \leq$$

$$\leq (\lambda_{m+j+1} - \lambda_m + 1) \frac{d(\lambda_{m+j+1}) \ldots d(2)}{d(\lambda_{m+j+1} - \lambda_m + 1)} \left(\frac{y_{m,k}}{d(\lambda_m)} \right)^{\lambda_{m+j+1} - \lambda_m}$$

Now, in view of (2.16), the above inequality gives

$$\sum_{j=0}^{\infty} (\lambda_{m+j+1} - \lambda_m + 1) \frac{d(\lambda_{m+j+1}) \ldots d(2)}{d(\lambda_{m+j+1} - \lambda_m + 1) \ldots d(2)} \cdot$$

$$\cdot |a(m+j)| \left(\frac{r y_{m,k}}{d(\lambda_m} \right)^{\lambda_{m+j+1} - \lambda_m + 1} \leq$$

$$\leq \frac{r y_{m,k}}{d(\lambda_m)} |a(m)| d(\lambda_m) \ldots d(2) \sum_{j=0}^{\infty} (k + j + 1) \cdot$$

$$\cdot \frac{d(\lambda_m + k + j) \ldots d(\lambda_m + 1)}{d(k + j + 1) \ldots d(2)} \left(\frac{y_{m,k}}{d(\lambda_m)} \right)^{k+j} =$$

$$= \frac{r y_{m,k}}{d(\lambda_m)} |a(m)| d(\lambda_m) \ldots d(2), \qquad \text{by}(2.4).$$

Thus $H_m(z)$ is univalent in U and hence

$$ry_{m,k} \le d(\lambda_m)\rho(\lambda_m - 1). \tag{2.17}$$

Since $\lim_{m\to\infty} y_{m,k} = y_k$ and $0 < r < \underline{R}$, on proceedings to limit as $m \to \infty$, we get the left side inequality in (2.2). This completes the proof of Theorem 1.

THEOREM 2. *Let f, k, k_0, α and $\{y_k\}_{k=1}^{\infty}$ be as in Theorem 1. Suppose $\{|a_j/a_{j+1}|^{1/(\lambda_{j+1}-\lambda_j)}\}_{j=0}^{\infty}$ is eventually a non-decreasing function tending to ∞. Then f is entire. Further, if $k < \infty$, then*

$$\frac{y_k}{\gamma} \le \limsup_{n\to\infty}\left(\frac{d_n\rho_n}{n}\right) \le \begin{cases} \dfrac{\alpha}{\delta} & , \quad k < k_0 \\ \dfrac{\left(2\dfrac{d_{k+1}\dots d_2}{k+1}\right)^{1/k}}{\delta} & , \quad k \ge k_0 \end{cases} \tag{2.18}$$

PROOF. Since $\underline{R} \le R \le \overline{R}$ and $\{|a_j/a_{j+1}|^{1/(\lambda_{j+1}-\lambda_j)}\}_{j=0}^{\infty}$ increases to ∞, we have $\overline{R} = \infty$ and hence f is an entire function. Suppose $k < \infty$. Then (2.14) gives for all $m \ge 1$,

$$\frac{d(\lambda_m+1)\rho(\lambda_m-1)}{\lambda_m+1} < \left[2\frac{d(\lambda_{m+1}-\lambda_m+1)\dots d(2)}{\lambda_{m+1}-\lambda_m}\right]^{1/(\lambda_{m+1}-\lambda_m)} \cdot \frac{1}{(\lambda_m+1)/|a(m)/a(m+1)|^{1/(\lambda_{m+1}-\lambda_m)}}$$

The right side inequality in (2.18) follows by using (1.5) and by following the same lines of proof that established the right side of (2.2). To prove the left side inequality in (2.18), we let $\lambda_{m+1} - \lambda_m = k$ and $r = |a(m)/a(m+1)|^{1/(\lambda_{m+1}-\lambda_m)}$. From (2.17), we have

$$\frac{|a(m)/a(m+1)|^{1/(\lambda_{m+1}-\lambda_m)}y_{m,k}}{\lambda_m} \le \frac{d(\lambda_m+1)\rho(\lambda_m-1)}{\lambda_m}$$

Again, using (1.5) and the technique that proved the left side of (2.2), we deduce the left side of (2.18). This proves Theorem 2.

Remark. For $d_n = n$ the results of Shah and Trimble [6] follow from the results found in this paper.

Acknowledgements

The author is grateful for the referee's valuable suggestions.

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Lavoro pervenuto alla redazione il 1 Dicembre 1989
ed accettato per la pubblicazione il 28 Novembre 1991
su parere favorevole di M. Nacinovich e di P.E. Ricci

INDIRIZZO DELL'AUTORE:

J. Patel - Department of Mathematics Utkal University - 751004 - Vani Vihar, Bhubaneswar - India