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On order-preserving isomorphism of $C_0(X)$

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RIASSUNTO – *È ben noto che se esiste un isomorfismo lineare di $C_0(X)$ in $C_0(Y)$ che conserva l'ordine, allora esiste, per ogni numero ordinale α, una costante positiva K tale che* $\operatorname{card} X^{(\alpha)} \leq K \operatorname{card} Y^{(\alpha)}$. *Pertanto non esiste alcun isomorfismo lineare di c in c_0 che conservi l'ordine.*

ABSTRACT – *It is shown that if there is an order-preserving linear isomorphism of $C_0(X)$ into $C_0(Y)$ then there is a positive constant K such that* $\operatorname{card} X^{(\alpha)} \leq K \operatorname{card} Y^{(\alpha)}$ *for every ordinal number α. Consequently there is no order-preserving linear isomorphism of c into c_0.*

KEY WORDS – *Order-preserving - Isomorphism - Continuous functions - Derived sets - Cardinality.*

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1 – Introduction

Let X be a locally compact space[1]. We shall denote by $C_0(X)$ the Banach space of all continuous complex-valued functions that vanish at infinity on X, provided with the supremum norm. If X is compact we shall write $C(X)$ instead of $C_0(X)$. $C_0^r(X)$ (resp. $C^r(X)$ when X is compact) will denote the real elements of $C_0(X)$ (resp. $C(X)$).

Let T be a linear isomorphism (i.e. T and T^{-1} are continuous) of $C_0(X)$ onto $C_0(Y)$. The spaces X and Y need, of course, not be home-

[1] Throughout this paper all topological spaces are assumed to be Hausdorff.

omorphic, unless $\|T\|\ \|T^{-1}\| < 2$ (see [1]). However, they share some common properties, for one, they should have the same cardinality (see [2]). Although, we also know of some other properties which are preserved under surjective isomorphisms of $C_0(X)$ (see [4]), the problem is basically open.

For into isomorphisms our expectations should naturally be more modest. In [6], Holsztyński proves that if T is an isometry from $C^r(X)$ into $C^r(Y)$ then X is the continuous image of a closed subset of Y. The same result was obtained previously by GĘBA and SEMADENI [5] with the additional hypothesis that T and T^{-1} are order-preserving. In [3], the author was able to generalize this result to the locally compact case as follows: if T is a linear isomorphism of $C_0(X)$ into $C_0(Y)$ such that $\|T\|\ \|T^{-1}\| < \frac{3}{2}$ then for each ordinal number α, $X^{(\alpha)}$ is the continuous image of a subset of Y^α, where for a topological space Z, $Z^{(\alpha)}$ denotes the α-th derived set of Z defined by transfinite induction: $Z^{(0)} = Z$, $Z^{(1)}$ is the set of all non-isolated points of Z, and for any $\alpha > 1$, $Z^{(\alpha)} = (Z^{(\beta)})^{(1)}$ if $\alpha = \beta + 1$, and $Z^{(\alpha)} = \bigcap_{\beta<\alpha} Z^{(\beta)}$ otherwise.

In this article we will drop the norm condition $\|T\|\ \|T^{-1}\| < \frac{3}{2}$ and assume, instead, that T is order-preserving, and prove that for every ordinal number α, $\operatorname{card} X^{(\alpha)} \leq K \operatorname{card} Y^{(\alpha)}$, where $K = \inf \|\phi\|\ \|\phi^{-1}\|$, and where ϕ runs through the set of all order-preserving linear isomorphisms of $C_0^r(X)$ into $C_0^r(Y)$.

Let us recall that a linear map T from $C_0(X)$ into $C_0(Y)$ is said to be order-preserving if $Tf \geq 0$ for all $f \in C_0(X)$, $f \geq 0$. It should be noted that the inverse of an order-preserving linear isomorphism need not be order-preserving. For example, if we let X be the discrete space of positive integers and Y be its one-point compactification with ∞ as the point at infinity, and define $T(f)(k) = f(k+1) + \frac{1}{2}f(1)$ for all $k \geq 1$ and $T(f)(\infty) = \frac{1}{2}f(1)$, $f \in C_0(X)$, then clearly, T, (from $c_0 = C_0(X)$ onto $c = C(Y)$) is order-preserving but T^{-1} is not. For another simple example, let $X = Y = [0,1]$ and define $T(f)(t) = f(t) + \int_X f\, d\lambda$, $t \in [0,1]$, $f \in C(X)$, where λ is the Lebesgue measure on $[0,1]$.

Let T be an isomorphism of $C_0(X)$ into $C_0(Y)$. T^* will denote its adjoint, and for each $y \in Y$, δ_y will denote the evaluation map on $T(C_0(X))$ at y. We may and will identify bounded functionals on $C_0(X)$ with finite regular Borel measures on X.

2 – The results

THEOREM. *Let X, Y be locally compact spaces. If there exists an order-preserving linear isomorphism of $C_0^r(X)$ into $C_0(Y)$ then* $\operatorname{card} X^{(\alpha)} \le K \operatorname{card} Y^{(\alpha)}$ *for every ordinal number α.*

We need the following lemma.

LEMMA. *Let X, Y be locally compact spaces, and let T be a norm-increasing, order-preserving linear isomorphism of $C_0(X)$ into $C_0(Y)$ with $\|T^{-1}\| = 1$. For each x in X let $\{U_{xi}: i \in I_x\}$ denote the family of all open neighbourhoods of x in X. For each x in X and $i \in I_x$ let $f_{xi} \in C_0^r(X)$ be such that $1 = \|f_{xi}\| = f_{xi}(x) \ge f_{xi}(z) \ge 0$ for all $z \in U_{xi}$ and $f_{xi}(z) = 0$ for all $z \in X \setminus U_{xi}$. For each $0 < x < 1$, $x \in X$ and $i \in I_x$ let $K_{xi}(\varepsilon) = \{y \in Y: T(f_{xi})(y) \ge \varepsilon\}$ and $Y_x(\varepsilon) = \{y \in Y: T^*\delta_y(\{x\}) \ge \varepsilon\}$. Then,*

(a) *$Y_x(\varepsilon) = \cap\{K_{xi}(\varepsilon): i \in I_x\}$ for all $x \in X$ and $0 < \varepsilon < 1$,*
(b) *$Y_x(\varepsilon) \cap Y^{(\beta)} \neq \emptyset$ for every ordinal number β, every $x \in X^{(\beta)}$ and $0 < \varepsilon < 1$.*

PROOF. Fix $x \in X$ and $0 < \varepsilon < 1$. It is easy to see that for each $y \in Y$, $T^*\delta_y(\{x\}) = \lim_i T(f_{xi})(y)$, and therefore $\{K_{xi}(\varepsilon): i \in I_{xi}\} \subset Y_x(\varepsilon)$.

Let $y \in Y_x(\varepsilon)$ and write $T^*\delta_y = a\mu_x + \mu$ where $a = T^*\delta_y(\{x\})$, μ_x is the unit point mass at x and $\mu = T^*\delta_y - a\mu_x$. Since T is order-preserving, $T^*\delta_y$ is a positive measure. Thus, $\mu \ge 0$. Therefore, for each $i \in I_x$ we have $T(f_{xi})(y) = T^*\delta_y(f_{xi}) = af_{xi}(x) + \int_X f_{xi} d\mu \ge a \ge \varepsilon$ from which it follows that $Y_x(\varepsilon) \subset \cap\{K_{xi}(\varepsilon): i \in I_x\}$, and this completes the proof of (a).

Since for any finite subset $\{i_1, \dots, i_n\}$ of I_n,

$$\bigcap_{k=1}^{n} K_{xi_k}(\varepsilon) \supset \{y \in Y: T(f_{xi_1} f_{xi_2} \cdot \ldots \cdot f_{xi_n})(y) \ge \varepsilon\}$$

and this latter set is nonvoid, it follows that $\{K_{xi}(\varepsilon): i \in I_x\}$ is a family of compact subset of Y with the finite intersection property. Therefore, $Y_x(\varepsilon) = \cap\{K_{xi}(\varepsilon): i \in I_x\}$ is nonvoid.

The preceding argument shows that (b) is true for $\beta = 0$. Now let $\beta > 0$ and assume that (b) holds for every ordinal numbers $\delta < \beta$.

Suppose that β is a limit ordinal. For each x in X and $0 < \varepsilon < 1$, by part (a), $Y_x(\varepsilon)$ is a compact subset of Y. Now fix $x \in X^{(\beta)}$ and $0 < \varepsilon < 1$. Then, by the induction hypothesis, $\{Y_x(\varepsilon) \cap Y^{(\delta)} : \delta < \beta\}$ is a family of compact subsets of Y with the finite intersection property, and hence, $Y_x(\varepsilon) \cap Y^{(\beta)} = \bigcap_{\delta<\beta} Y_x(\varepsilon) \cap Y^{(\delta)}$ is nonempty.

Now let us assume that $\beta = \gamma + 1$. Suppose that there exist $x \in X^{(\beta)}$ and $0 < \varepsilon < 1$ such that $Y_x(\varepsilon) \cap Y^{(\beta)}$ is empty. Then it follows that there exists a finite subset $\{i_1, i_2, \ldots, i_n\}$ of I_x such that $Y^{(\beta)} \cap K_{xi_1}(\varepsilon) \cap \ldots \cap K_{xi_n}(\varepsilon)$ is void. Let $f = f_{xi_1} \cdot f_{xi_2} \cdot \ldots \cdot f_{xi_n}$ and $K_f(\varepsilon) = \{y \in Y : T(f)(y) \geq \varepsilon\}$. Then $K_f(\varepsilon) \cap Y^{(\beta)} = \emptyset$.

Let us choose a sequence $\{x_1, x_2, \ldots\}$ of distinct points in $X^{(\gamma)}$ with $f(x_n) \geq \sqrt{\varepsilon}$ for all n, and a sequence $g_1, g_2, \ldots$ in $C_0^r(X)$ such that $1 = \|g_n\| = g_n(x_n) \geq g_n(z) \geq 0$ for all $z \in X$ and $g_n g_m = 0$ for all $n \neq m$. For each n, let $F_n = \{y \in Y : T(g_n f)(y) \geq \varepsilon\}$. Since $0 \leq g_n f \leq f$ and T is order-preserving, we have

$$F_n \subset K_f(\varepsilon) \qquad \text{for all } n. \tag{1}$$

Now fix n, let $y \in Y_{x_n}(\sqrt{\varepsilon})$ and write $T^*\delta_y = a\mu_x + \mu$ where $a = T^*\delta_y(\{x\})$. Then $a \geq \sqrt{\varepsilon}$ and therefore $T(g_n f)(y) = af(x_n) + \int_X g_n f \, d\mu \geq \varepsilon$ which shows that $y \in F_n$. Thus,

$$F_n \cap Y^{(\gamma)} \neq \emptyset \qquad \text{for all } n. \tag{2}$$

Let $n_1, n_2, \ldots, n_k$ be distinct integers, and let $g = \sum_{j=1}^{k} g_{n_j} f$. Then, clearly $\|g\| \leq 1$, and $\|T\| \geq |T(g)(y)| = \sum_{j=1}^{k} T(g_{n_j} f)(y) \geq k\varepsilon$ for every $y \in F_{n_1} \cap F_{n_2} \cap \ldots \cap F_{n_k}$. Hence we have:

(3) If $k > \|T\|/\varepsilon$, then $F_{n_1} \cap F_{n_2} \cap \ldots \cap F_{n_k} = \emptyset$ for any k distinct integers $n_1, n_2, \ldots, n_k$.

From (1)—(3) it follows that $K_f(\varepsilon) \cap Y^{(\gamma)}$ is an infinite set which in turn implies that $K_f(\varepsilon) \cap Y^{(\beta)} \supset (K_f(\varepsilon) \cap Y^{(\gamma)})^{(1)} \neq \emptyset$ (by compactness of the set $K_f(\varepsilon) \cap Y^{(\gamma)}$), which is a contradiction. This completes the proof of our lemma.

PROOF OF THE THEOREM. Let T be an order-preserving linear isomorphism of $C_0^r(X)$ into $C_0^r(Y)$. We may assume that T is norm-increasing (i.e. $\|Tf\| \geq \|f\|$ for all $f \in C_0(X)$) and that $\|T^{-1}\| = 1$ (for otherwise we may replace T by $\|T^{-1}\|T$ which has, as easily shown, these properties).

Let α be an arbitrary ordinal number. If $Y^{(\alpha)}$ is empty then, by the lemma, $X^{(\alpha)}$ must be empty. We may, therefore, assume that $Y^{(\alpha)} \neq \emptyset$. Let us fix $0 < \varepsilon < 1$ and for each $y \in Y^{(\alpha)}$ define $X_y = \{x \in X^{(\alpha)} \colon T^{*}\delta_y(\{x\}) \geq \varepsilon\}$. Then by part (b) of the lemma we have $\operatorname{card} X_y \leq \|T\|/\varepsilon$ and $X^{(\alpha)} \subset \bigcup\{X_y \colon y \in Y^{(\alpha)}\}$. Thus,

$$\operatorname{card} X^{(\alpha)} \leq \sum_{y \in Y^{(\alpha)}} \operatorname{card} X_y \leq \frac{1}{\varepsilon}\|T\| \operatorname{card} Y^{(\alpha)},$$

and since $0<\varepsilon<1$ is arbitrary we conclude that $\operatorname{card} X^{(\alpha)} \leq \|T\| \operatorname{card} Y^{(\alpha)}$, which completes the proof.

COROLLARY.

i) *There is no order-preserving linear isomorphism from c into c_0.*

ii) *If Y is dispersed (i.e. $Y^{(\alpha)} = \emptyset$ for some α) but X is not, then there is no order-preserving linear isomorphism of $C_0(X)$ into $C_0(Y)$.*

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