Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 305-327

Locally conformal cosymplectic manifolds foliated by generalized Hopf manifolds

J.C. MARRERO(*)

RIASSUNTO – *Si studia una classe particolare di varietà cosimplettiche localmente conformi che sono fogliettate da varietà generalizzate di Hopf e, come principale risultato, si dimostra che lo spazio di rivestimento universale di tale varietà è il prodotto di una varietà c-Sasakiana con lo spazio iperbolico di dimensione 2.*

ABSTRACT – *In this paper, we study a particular class of locally conformal cosymplectic manifolds which are foliated by generalized Hopf manifolds and, as main result, we prove that the universal covering space of such manifolds is the product of a c-Sasakian manifold with the 2-dimensional hyperbolic space.*

KEY WORDS · *Cosymplectic manifolds - Locally conformal cosymplectic manifolds - Generalized Hopf manifolds - Sasakian manifolds.*

A.M.S. CLASSIFICATION: 53C15 - 53C25

– Introduction

An almost Hermitian manifold M^{2n} is called locally conformal Kähler if its metric is conformally related to a Kähler metric in some neighbourhood of every point of M^{2n}. Such manifolds have been studied by various authors (see, for instance, [12], [19], [20], [22], [13], [8], ...). Examples of locally conformal Kähler manifolds are provided by the generalized Hopf manifolds which are locally conformal Kähler manifolds with Lee form parallel (see [20] and [22]). The main non-Kähler example of such

(*)Supported by the "Consejería de Educación del Gobierno de Canarias".

manifolds is the Hopf manifold (see [19]), which is defined as the quotient $H = \frac{C^n - \{0\}}{\Delta_\lambda}$, where Δ_λ is a cyclic group of transformations (it is known that H is diffeomorphic with $S^1 \times S^{2n-1}$, see [11]). Other example of non-Kähler compact generalized Hopf manifold is the nilmanifold $N(r,1) \times S^1$, where $N(r,1) = \Gamma(r,1) \setminus H(r,1)$ is a compact quotient of the generalized Heisenberg group $H(r,1)$ by a discreet subgroup $\Gamma(r,1)$ (see [3]).

On the other hand, if M^{2n+1} is a differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g), a conformal change of the metric g leads to a metric which is no more compatible with the almost contact structure (φ, ξ, η). This can be corrected by a convenient change of ξ and η which implies rather strong restrictions. Such a definition is given by I. VAISMAN in [21]. Using this definition for the conformal change of an almost contact metric structure Vaisman introduces, in [21], a class of almost contact metric manifolds, called locally conformal cosymplectic manifolds. An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be locally conformal cosymplectic if the structure (φ, ξ, η, g) is conformally related to a cosymplectic structure in some neighbourhood of every point of M, or equivalently if $N_\varphi = 0$ and there is a closed 1-form ω on M, which we call Lee form of M, such that $d\Phi = -2\Phi \wedge \omega$ and $d\eta = \eta \wedge \omega$, where N_φ and Φ are the Nijenhuis torsion of φ and the fundamental 2-form of M, respectively. Recently, in [5], [6] and [7], we have continued the study of the locally conformal cosymplectic manifolds, and we have obtained some interesting examples of locally conformal cosymplectic structures on the real Hopf manifolds ([23]) and on a compact quotient of a certain solvable non-nilpotent three-dimensional Lie group.

In this paper, we study a particular class of locally conformal cosymplectic manifolds which we call PC-manifolds. A PC-manifold is a locally conformal cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ with Lee form $\omega \neq 0$ at every point and such that $\omega(\xi) = 0$ and the leaves of the foliation $\eta = 0$ with the induced almost Hermitian structure are generalized Hopf manifolds. In section 1, we give some results on locally conformal cosymplectic and c-Sasakian manifolds. In section 2, we define and characterize the PC-manifolds (see proposition 2.3) and we obtain some properties of these manifolds (see proposition 2.4). We also prove that a compact manifold can not be a PC-manifold. In section 3, we study the Riemann curvature tensor R of a PC-manifold $(M, \varphi, \xi, \eta, g)$. We determine the vector fields

$R(X,Y)\xi$, $R(X,Y)U$ and $R(X,Y)V$, for all X,Y vector fields on M, in terms of $\eta, u, v = -u \circ \varphi, \xi, U$ and $V = \varphi U$, where u and U are the unit Lee form and the unit Lee vector field respectively of M (see propositions 3.1 and 3.2). In particular, we obtain explicit formulas for the sectional curvature of a plane section containing ξ, U or V and for the Ricci curvature in the direction of these vectors (see corollary 3.1). Finally, in section 4, by using the results of the above sections, we prove that the universal covering space $\overline{M}$ of a PC-manifold (M,φ,ξ,η,g) is the product of a c-Sasakian manifold $(N,\varphi_N,\xi_N,\eta_N,g_N)$ with the 2-dimensional hyperbolic space and we describe the induced PC-structure $(\bar\varphi,\bar\xi,\bar\eta,\bar g)$ on $\overline{M}$. Moreover, if N is of constant φ_N-sectional curvature, then we determine, up to almost contact isometries, the almost contact metric manifold $(\overline{M},\bar\varphi,\bar\xi,\bar\eta,\bar g)$ (see theorem 4.2).

1 – Preliminaries

All the manifolds considered in this paper are assumed to be connected and of class C^∞.

Let V be an almost Hermitian manifold with metric g and almost complex structure J. Denote by $\mathfrak{X}(V)$ the Lie algebra of C^∞ vector fields on V. The Kähler form Ω is given by $\Omega(X,Y) = g(X,JY)$ and the Lee form is the 1-form θ defined by $\theta(X) = 1/(n-1)\delta\Omega(JX)$, where δ denotes the coderivate, $\dim V = 2n$ and $X,Y \in \mathfrak{X}(V)$.

Recall that V is said to be Kähler if $d\Omega = 0$ and $N_J = 0$ and locally conformal Kähler (l.c.K.) if $d\Omega = \theta \wedge \Omega$ and $N_J = 0$, N_J being the Nijenhuis tensor of J. Among the locally conformal Kähler manifolds, those such that the Lee form θ is parallel are called generalized Hopf manifolds.

On the other hand, let M be an almost contact metric manifold with metric g and almost contact structure (φ,ξ,η). Then, we have

$$\varphi^2 = -I + \eta\otimes\xi, \qquad \eta(\xi) = 1$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for X, $Y \in \mathfrak{X}(M)$, where I denotes the identity transformation. Using the above relations we deduce that

$$\varphi(\xi)=0\,,\quad \eta\circ\varphi=0\,,\quad \eta(X)=g(X,\xi)\,,$$

for all $X\in\mathfrak{X}(M)$. The fundamental 2-form Φ of the almost contact metric manifold (M,φ,ξ,η,g) is defined by $\Phi(X,Y)=g(X,\varphi Y)$ and the Lee form by $\omega(X)=(\nabla_\xi\Phi)(\xi,\varphi X)+\frac{\delta\eta}{2n}\eta(X)$, where ∇ is the Riemannian connection of g and $\dim M=2n+1$.

An almost contact metric structure (φ,ξ,η,g) on M is said to be:

Normal if $N_\varphi+2d\eta\otimes\xi=0$, N_φ being the Nijenhuis tensor of φ; Cosymplectic if it is normal and $d\eta=0$, $d\Phi=0$; Locally conformal cosymplectic (l.c.C.) if every point $x\in M$ has an open neighbourhood U such that the structure $(\varphi,e^{-\sigma}\xi,e^{\sigma}\eta,e^{2\sigma}g)$ is cosymplectic on U, where $\sigma\colon U\longrightarrow\mathbb{R}$ is a real differentiable function on U (see [16], [1], [5] and [6]).

Let (M,φ,ξ,η,g) be an almost contact metric manifold with Lee form ω and ∇ the Riemannian connection of g. Consider

$$\overline{\nabla}_XY=\nabla_XY+\omega(X)Y+\omega(Y)X-g(X,Y)B \tag{1.1}$$

for all X, $Y\in\mathfrak{X}(M)$, where B is the Lee vector field on M given by $\omega(X)=g(X,B)$. $\overline{\nabla}$ is a torsionless linear connection on M. Moreover, if (M,φ,ξ,η,g) is l.c.C. then $\overline{\nabla}$ is the Riemannian connection of the local metrics $e^{2\sigma}g$ (see [6]). In fact, in [6], the authors prove

THEOREM 1.1. *Are equivalent:*

i) (M,φ,ξ,η,g) *is a* l.c.C. *manifold.*

ii) *The Lee form* ω *is closed and*

$$\overline{\nabla}_X\varphi=0 \tag{1.2}$$

for all $X\in\mathfrak{X}(M)$.

iii) *The Lee form* ω *is closed and*

$$(\nabla_X\varphi)Y=\omega(Y)\varphi X-\omega(\varphi Y)X+\Phi(X,Y)B-g(X,Y)\varphi B \tag{1.3}$$

for all X, $Y\in\mathfrak{X}(M)$.

iv) *The Lee form* ω *is closed and*

$$d\Phi=-2\Phi\wedge\omega\,,\qquad d\eta=\eta\wedge\omega\,,\qquad N_\varphi=0\,. \tag{1.4}$$

We remark that if $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold with Lee form ω and σ is a real differentiable function on an open U such that the structure $(\varphi, e^{-\sigma}\xi, e^{\sigma}\eta, e^{2\sigma}g)$ is cosymplectic on U, then $\omega = d\sigma$ on U.

Let $(M, \varphi, \xi, \eta, g)$ be a l.c.C. manifold with Lee vector field B and Lee form $\omega \neq 0$ at every point. Then, through of this paper, we shall use the following notation,

$$(1.5) \qquad c = \|\omega\|, \quad u = \omega/c, \quad U = B/c, \quad v = -u \circ \varphi, \quad V = \varphi U.$$

From (1.5) we obtain that $u(V) = v(U) = 0$.

A l.c.C. manifold $(M, \varphi, \xi, \eta, g)$ with Lee form ω is of class $C_4 \oplus C_{12}$ if $\omega(\xi) = 0$ (see [5]). Consequently, if B is the Lee vector field of M then $\eta(B) = 0$.

If $(M, \varphi, \xi, \eta, g)$ is a l.c.C. manifold of class $C_4 \oplus C_{12}$ and ω and B are the Lee form and the Lee vector field of M respectively then, using (1.3) and the fact that $\varphi(\xi) = 0$, we deduce

$$(1.6) \qquad \nabla\xi = \eta \otimes B$$

$$(1.7) \qquad \nabla\eta = \eta \otimes \omega.$$

Moreover, if $\omega \neq 0$ at every point, we have

$$(1.8) \qquad \Phi = 2v \wedge u + \psi$$

where Φ is the fundamental 2-form of the structure (φ, ξ, η, g) and ψ is a 2-form of rank $2n-2$ such that

$$(1.9) \qquad \psi(X, U) = \psi(X, V) = \psi(X, \xi) = 0, \qquad u \wedge v \wedge \psi^{n-1} \neq 0$$

being $\dim M = 2n + 1$.

On the other hand, an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be c-Sasakian ($c \in \mathbb{R}, \quad c \neq 0$) if it is normal and $d\eta = c\Phi$, where Φ is the fundamental 2-form (see [10]). The structure (φ, ξ, η, g) is said to be Sasakian if it is 1-Sasakian.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and x a point of M. A plane section Π in the tangent space to M at x, T_xM, is called a φ-section if there exists a unit vector X in T_xM orthogonal to ξ such that $\{X, \varphi X\}$ is an orthonormal basis of Π. Then the sectional curvature $K_{X\varphi X} = g(R(X, \varphi X)\varphi X, X)$ is called a φ-sectional curvature.

A c-Sasakian manifold is said to be a c-Sasakian space form if M has constant φ-sectional curvature. Examples of Sasakian space forms are provided on the manifolds S^{2n+1}, $\mathbb{R}^{2n+1}$ and $\mathbb{R} \times CD^n$. In fact, the unit sphere S^{2n+1} has a Sasakian structure of constant φ-sectional curvature k, for all $k > -3$ (see [17] and [18]); the real $(2n+1)$-dimensional number space $\mathbb{R}^{2n+1}$ is a Sasakian space form with $k = -3$ [14]; and the product manifold $\mathbb{R} \times CD^n$, where CD^n is a simply connected bounded complex domain in $\mathbb{C}^n$ with negative constant holomorphic sectional curvature, has a Sasakian structure of constant φ-sectional curvature k, for all $k < -3$ [18].

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold with constant φ-sectional curvature k. Put

$$\varphi' = \varphi, \quad \xi' = c\xi, \quad \eta' = 1/c\, \eta, \quad g' = 1/c^2\, g$$

where $c \in \mathbb{R}$, $c \neq 0$. Then, $(M, \varphi', \xi', \eta', g')$ is a c-Sasakian space form of constant φ-sectional curvature kc^2. We denote by $M(c, kc^2)$, the c-Sasakian manifold with this structure.

In [18], TANNO proves that if $(M, \varphi, \xi, \eta, g)$ and $(M', \varphi', \xi', \eta', g')$ are $(2n+1)$-dimensional complete simply connected Sasakian manifolds of constant φ-sectional curvature k, then, M is almost contact isometric to M', i.e. there exists an isometry F of M into M' such that $F_* \circ \varphi = \varphi' \circ F_*$ and $F_*\xi = \xi'$. Therefore, by using this result, we deduce

THEOREM 1.2. *Let M be a $(2n+1)$-dimensional complete simply connected c-Sasakian manifold with constant φ-sectional curvature k.*

i) *If $k > -3c^2$, then M is almost contact isometric to $S^{2n+1}(c, k)$.*

ii) *If $k = -3c^2$, then M is almost contact isometric to*

$$\mathbb{R}^{2n+1}(c, -3/c^2) = \mathbb{R}^{2n+1}(c).$$

iii) *If $k < -3c^2$, then M is almost contact isometric to*

$$(\mathbb{R} \times CD^n)(c, k).$$

2 – PC- manifolds

Let (M,φ,ξ,η,g) be a l.c.C. manifold with Lee form ω.

Let $i\colon F \longrightarrow M$ be the immersion of a generic leaf of the foliation G given by $\eta = 0$. Then, F carries an induced natural almost Hermitian structure (J,h), which is l.c.K. with Lee form $\omega_F = -2i^*\omega$ (see [6]). If we denote by ∇ and ∇^F the Riemann connections of the metrics g and h on M and F respectively, we have

PROPOSITION 2.1.

$$\left(\nabla^F_X\omega_F\right)Y = -2\left[(\nabla_X\omega)Y + (\omega(\xi))^2 g(X,Y)\right],$$

for all $X,Y \in \mathfrak{X}(F)$.

PROOF. If X, $Y \in \mathfrak{X}(F)$, then we deduce

$$(\nabla^F_X\omega_F)Y = X(\omega_F(Y)) - \omega_F(\nabla^F_X Y)\,.$$

Using this relation and the equation of Gauss, we obtain

$$(\nabla^F_X\omega_F)Y = -2\Big((\nabla_X\omega)Y - \omega(\xi)g(\nabla_X\xi,Y)\Big)\,.$$

Now, from (1.3) and since $\varphi(\xi) = 0$ and $\eta(X) = 0$, we deduce that

$$\nabla_X\xi = -\omega(\xi)X\,,$$

which ends the proof of our assertion. □

Using the above result, we have

COROLLARY 2.1. *Let* (M,φ,ξ,η,g) *be a* l.c.C. *manifold. Then, the leaves of* G *carry an induced* l.c.K. *structure with parallel Lee form if and only if*

$$(\nabla_X\omega)Y = -\omega(\xi)^2 g(X,Y)\,,$$

for all X, $Y \in \mathfrak{X}(M)$ *such that* $\eta(X) = \eta(Y) = 0$.

Next, we shall suppose that (φ, ξ, η, g) is, moreover, of class $C_4 \oplus C_{12}$. Let c be as in (1.5). We deduce

PROPOSITION 2.2. *If $(M, \varphi, \xi, \eta, g)$ is a* l.c.C. *manifold of class $C_4 \oplus C_{12}$, then,*

$$(\nabla_\xi \omega)X = (\nabla_X \omega)\xi = -c^2\eta(X), \tag{2.1}$$

for all $X \in \mathfrak{X}(M)$.

PROOF. Since ω is a closed 1-form, we obtain the first relation of (2.1). The second relation follows from (1.6) and using that $\omega(\xi) = 0$. □

Now, by using (2.1), corollary 2.1 and the relation $\omega(\xi) = 0$, we have

COROLLARY 2.2. *Let $(M\varphi, \xi, \eta, g)$ be a* l.c.C. *manifold of class $C_4 \oplus C_{12}$. Then, the leaves of G carry an induced* l.c.K. *structure with parallel Lee form if and only if*

$$\nabla\omega = -c^2\eta \otimes \eta .$$

The corollary 2.2 suggests us to give the following definition.

DEFINITION 2.1. *A* l.c.C. *manifold of class $C_4 \oplus C_{12}$ with Lee form $\omega \neq 0$ at every point is called a* PC*-manifold if*

$$\nabla\omega = -c^2\eta \otimes \eta$$

where $c = \|\omega\|$.

If $(M, \varphi, \xi, \eta, g)$ is a PC-manifold then M is said to have a PC-structure (φ, ξ, η, g).

Let $(M, \varphi, \xi, \eta, g)$ be a l.c.C. manifold of class $C_4 \oplus C_{12}$ with Lee form $\omega \neq 0$ at every point and consider c, u, U, v, V as in (1.5). Denote by L the Lie derivate and by ψ the 2-form on M given by $\psi = \Phi - 2v \wedge u$, (see (1.8)). Then, using (1.3) and the expression $\nabla_X V = (\nabla_X \varphi)U + \varphi(\nabla_X U)$, we obtain

PROPOSITION 2.3. *If (M,φ,ξ,η,g) is a l.c.C. manifold of class $C_4 \oplus C_{12}$ with Lee form $\omega \neq 0$ at every point, then, (M,φ,ξ,η,g) is a PC-manifold if and only if $c =$ constant and one of the following relations holds*

$$\nabla u = -c\eta\otimes\eta, \quad \nabla U = -c\eta\otimes\xi, \quad \nabla v = -c\psi, \quad \nabla V = c(\varphi + v\otimes U - u\otimes V).$$

Next, we deduce another results for a PC-manifold.

PROPOSITION 2.4. *Let (M,φ,ξ,η,g) be a PC-manifold. Then, V is a Killing vector field for the metric g. Moreover, the following relations hold*

$$[U,V]=0\,,\quad [\xi,V]=0\,,\quad [U,\xi]=c\xi\,. \tag{2.2}$$

$$L_U\varphi = 0\,,\quad L_\xi\varphi = cv\otimes\xi\,,\quad L_U v = 0\,,\quad L_\xi v = 0\,. \tag{2.3}$$

$$dv = -c\psi\,. \tag{2.4}$$

PROOF. Since ∇ is a torsionless linear connection on M, from (1.6) and proposition 2.3, we obtain (2.2).

Let X be a vector field on M. From proposition 2.3 we deduce that

$$(L_U\varphi)X = (\nabla_U\varphi)X$$

and therefore, by using (1.3), $L_U\varphi = 0$.

The second relation of (2.3) follows from (1.3), (1.6) and the formula

$$(L_\xi\varphi)X = (\nabla_\xi\varphi)X + \varphi(\nabla_X\xi) - \nabla_{\varphi X}\xi\,.$$

Now, we shall prove (2.4). Let $X, Y \in \mathfrak{X}(M)$, then

$$2dv(X,Y) = -X(u(\varphi Y)) + Y(u(\varphi X)) + u(\varphi[X,Y])\,. \tag{2.5}$$

Replacing the two first terms according to the formula

$$X(u(Y)) = u(\nabla_X Y) - c\eta(X)\eta(Y)$$

obtained from proposition 2.3, (2.5) gives

$$2dv(X,Y) = u((\nabla_Y \varphi)X - (\nabla_X \varphi)Y) .$$

Then, by using (1.3) and (1.8), we get just the desired relation.

On the other hand, by the classical formula of the Levi-Civita connection [11] we have that,

$$(L_V g)(X,Y) = 2g(\nabla_X V, Y) - 2dv(X,Y)$$

and thus, from (2.4) and proposition 2.3, we deduce that V is a Killing vector field.

Finally, using (1.9), (2.4) and the relations

$$L_U v = d(i_U v) + i_U(dv) , \qquad L_\xi v = d(i_\xi v) + i_\xi(dv)$$

we obtain that $L_U v = L_\xi v = 0$. □

In [5] (see also [6]) we have obtained an example of locally conformal cosymplectic structure on the real Hopf manifold $\mathbb{R}H^{2n+1}$ ([23]) and on a certain quotient $M^3(k)$ of a solvable non-nilpotent three-dimensional Lie group. $M^3(k)$ and $\mathbb{R}H^{2n+1}$ are compact manifolds which can have no cosymplectic structures. Moreover, the locally conformal cosymplectic structures on $M^3(k)$ and $\mathbb{R}H^{2n+1}$ obtained in [5] are not P$\mathcal{C}$-structures. In fact, using the proposition 2.4, we deduce the following result.

COROLLARY 2.3. *A compact manifold can not admit a* P$\mathcal{C}$*-structure.*

PROOF. Let $(M, \varphi, \xi, \eta, g)$ be a compact P$\mathcal{C}$-manifold with $\dim M = 2n+1$ and let Φ be the fundamental 2-form of the structure (φ, ξ, η, g). Then, from (1.8) and (1.9), we obtain that

$$2n(\eta \wedge u \wedge v \wedge \psi^{n-1}) = \eta \wedge \Phi^n . \tag{2.6}$$

Therefore, if γ is the $(2n+1)$-form on M given by

$$\gamma = \eta \wedge u \wedge v \wedge \psi^{n-1} ,$$

using (2.6), we deduce that $\gamma \neq 0$ at every point, i.e. γ is a volume element.

On the other hand, from (1.4) and (2.4), we obtain that

$$\gamma = d(1/c(\eta \wedge v \wedge \psi^{n-1}))$$

which, in view of Stoke's theorem, is a contradiction. □

3 – The curvature tensor on a P$\mathcal{C}$-manifold

In this section, we shall study the Riemann curvature tensor of a P$\mathcal{C}$-manifold.

Let $(M, \varphi, \xi, \eta, g)$ be a P$\mathcal{C}$-manifold with $\dim M = 2n+1$ and let c, u, U, v, V, be as in (1.5). Then, if R is the Riemann curvature tensor of M, we have

PROPOSITION 3.1.

$$R(X,Y)\xi = 2c^2[(\eta \wedge u)(X,Y)]U\,, \tag{3.1}$$

$$R(X,\xi)Y = c^2(u(X)u(Y)\xi - u(X)\eta(Y)U)\,, \tag{3.2}$$

$$R(X,Y)U = -2c^2[(\eta \wedge u)(X,Y)]\xi\,, \tag{3.3}$$

$$R(X,U)Y = c^2(\eta(X)\eta(Y)U - u(Y)\eta(X)\xi)\,, \tag{3.4}$$

for all $X, Y \in \mathfrak{X}(M)$.

PROOF. From (1.6) and proposition 2.3 we deduce that

$$R(X,Y)\xi = c[2d\eta(X,Y)U + \eta(Y)\nabla_X U - \eta(X)\nabla_Y U] = 2c\, d\eta(X,Y)U$$

$$R(X,Y)U = -c[2d\eta(X,Y)\xi + \eta(Y)\nabla_X \xi - \eta(X)\nabla_Y \xi] = -2c\, d\eta(X,Y)\xi$$

for all X, $Y \in \mathfrak{X}(M)$.

Thus, using (1.4), we obtain (3.1) and (3.3).

(3.2) and (3.4) follow from (3.1) and (3.3) respectively and using the relation

$$g(R(X,Y)Z,W) = -g(R(Z,W)Y,X) \tag{3.5}$$

for all X, Y, Z, $W \in \mathfrak{X}(M)$. □

Also, we deduce

PROPOSITION 3.2.

$$R(X,Y)V = c^2\Big(2(v\wedge u)(X,Y)U + 2(v\wedge\eta)(X,Y)\xi + \\ - v(X)Y + v(Y)X\Big), \tag{3.6}$$

$$R(X,V)Y = c^2\Big(v(Y)X - u(X)v(Y)U + (u(X)u(Y) + \\ + \eta(X)\eta(Y) - g(X,Y))V - \eta(X)v(Y)\xi\Big), \tag{3.7}$$

for all $X, Y \in \mathfrak{X}(M)$.

PROOF. By using proposition 2.3 we obtain that

$$R(X,Y)V = c\Big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + 2dv(X,Y)U + v(Y)\nabla_X U + \\ - u(Y)\nabla_X V - v(X)\nabla_Y U + u(X)\nabla_Y V\Big) = \\ = c\Big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + 2dv(X,Y)U + cu(X)\varphi Y - cu(Y)\varphi X + \\ + c(v(X)\eta(Y) - v(Y)\eta(X))\xi + c(u(X)v(Y) - u(Y)v(X))U\Big).$$

Thus, from (1.3), (1.8) and (2.4), we deduce (3.6).
(3.7) follows from (3.5) and (3.6). □

Let x be a point of M. Denote by K_{XY} and by $\rho(X,X)$ the sectional curvature for the plane section in T_xM with orthonormal basis $\{X,Y\}$ and the Ricci curvature in the direction X respectively. Then, by using (3.2), (3.4) and (3.7), we obtain

COROLLARY 3.1.

$$K_{X\xi} = -c^2u(X)^2, K_{XU} = -c^2\eta(X)^2, K_{XV} = c^2(1 - u(X)^2 - \eta(X)^2).$$

$$\rho(\xi,\xi) = -c^2, \qquad \rho(U,U) = -c^2, \qquad \rho(V,V) = 2(n-1)c^2.$$

If for all $X \in \mathfrak{X}(M)$ we denote by X' the vector field on M defined by $X' = X - u(X)U - \eta(X)\xi$, then, from proposition 3.1, we have

COROLLARY 3.2.

$$R(X,Y)Z = R(X',Y')Z' + 2c^2(\eta \wedge u)(X,Y)[\eta(Z)U - u(Z)\xi],$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Next, we obtain the relation between R and the curvature tensor $\overline{R}$ of the connection $\overline{\nabla}$ given in (1.1).

PROPOSITION 3.3.

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z - c^2\Big\{&(\eta(X)\eta(Z) + u(X)u(Z))Y - (\eta(Y)\eta(Z) + \\ &+ u(Y)u(Z))X - g(Y,Z)(\eta(X)\xi + u(X)U) + \\ &+ g(X,Z)(\eta(Y)\xi + u(Y)U) + g(Y,Z)X + \\ &- g(X,Z)Y\Big\},\end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

PROOF. This follows from proposition 2.3 and using a well-known formula (see [9], pag. 115). □

Now, from corollary 3.2 and proposition 3.3, we deduce

COROLLARY 3.3.

$$\overline{R}(X,Y)Z = R(X',Y')Z' - c^2\{g(Y',Z')X' - g(X',Z')Y'\}, \tag{3.8}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

4 – The universal covering space of a P$\mathcal{C}$-manifold

Let $(M, \varphi, \xi, \eta, g)$ be a P$\mathcal{C}$-manifold and let c, u, U, v, V be as in (1.5).

Denote by F the foliation given by $\eta = 0$, $u = 0$. F defines on M a foliation of codimension two, which we call the canonical foliation of M. Using (1.6), (1.7), (2.2), proposition 2.3 and corollary 3.1, we deduce

THEOREM 4.1. *The canonical foliation F of a P$\mathcal{C}$-manifold M is totally geodesic with integrable normal bundle. Moreover, if $F^{\perp}$ is the foliation determined by the normal bundle of F, then $F^{\perp}$ also is totally geodesic and its leaves are of constant sectional curvature $-c^2$.*

Let $i\colon F \longrightarrow M$ be the inmersion of a generic leaf of the canonical foliation F. We define an almost contact metric structure $(\varphi_F, \xi_F, \eta_F, g_F)$ on F by

$$(4.1)\quad \varphi_F X = \varphi X + (i^* v)(X) U_{|F'} \quad \xi_F = -V_{|F'} \quad \eta_F = -(i^* v) \quad g_F = i^* g\,.$$

Then, we have

PROPOSITION 4.1. *The almost contact metric structure $(\varphi_F, \xi_F, \eta_F, g_F)$ on F is c-Sasakian.*

PROOF. If Φ is the fundamental 2-form of the P$\mathcal{C}$-structure (φ, ξ, η, g) then, using (4.1), we obtain that fundamental 2-form Φ_F of the structure $(\varphi_F, \xi_F, \eta_F, g_F)$ is $\Phi_F = i^*\psi = i^*\Phi - 2(i^*v) \wedge (i^*u)$. Thus, from (2.4), we deduce that

$$d\eta_F = c\Phi_F\,.$$

On the other hand, if N_φ and N_{φ_F} are the Nijenhuis tensors of φ and φ_F respectively then, from (1.4), (2.4) and using first and third relation of (2.3) and the formula

$$\begin{aligned} N_{\varphi_F}(X,Y) + 2d\eta_F(X,Y)\xi_F = N_\varphi(X,Y) &+ v(X)\Big\{(L_U\varphi)Y + ((L_U v)Y)U\Big\} + \\ &- v(Y)\Big\{(L_U\varphi)X + ((L_U v)X)U\Big\} + \\ &+ 2(dv(\varphi X, Y) + dv(X, \varphi Y))U \end{aligned}$$

we see that the structure $(\varphi_F, \xi_F, \eta_F)$ is normal.

Consequently, $(\varphi_F, \xi_F, \eta_F, g_F)$ is a c-Sasakian structure on F. □

Next, we shall suppose that F with the c-Sasakian structure $(\varphi_F, \xi_F, \eta_F, g_F)$ is of constant φ_F-sectional curvature k. Then, from (4.1) and using a theorem of OGUIE [15] and the fact that the foliation F is totally geodesic, we have that

$$
\begin{aligned}
R(X,Y)Z &= \frac{1}{4}(k+3c^2)(g(Y,Z)X - g(X,Z)Y)+\\
&+\frac{1}{4}(k-c^2)\Big\{v(X)v(Z)Y - v(Y)v(Z)X+\\
&+(g(X,Z)v(Y) - g(Y,Z)v(X))V+\\
&+g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y + 2g(X,\varphi Y)\varphi Z+\\
&+(v(X)g(\varphi Y,Z) - v(Y)g(\varphi X,Z)+\\
&+2v(Z)g(X,\varphi Y))U\Big\}
\end{aligned}
\tag{4.2}
$$

for all $X, Y, Z \in \mathfrak{X}(F)$, where R is the Riemann curvature tensor of M.

Now, we give the following definition.

DEFINITION 4.1. *A* P*C-manifold is called a* P$C(k)$*-manifold* ($k \in \mathbb{R}$) *if every leaf F of the canonical foliation F is of constant φ_F-sectional curvature k, where $(\varphi_F, \xi_F, \eta_F, g_F)$ is the induced c-Sasakian structure on F given by* (4.1).

If $(M, \varphi, \xi, \eta, g)$ is a P$C(k)$-manifold then M is said to have a P$C(k)$-structure (φ, ξ, η, g).

A PC-manifold M is a P$C(k)$-manifold if and only if the relation (4.2) is satisfied for $X, Y, Z \in \mathfrak{X}(F)$, i.e. for $X, Y, Z \in \mathfrak{X}(M)$ such that $\eta(X) = \eta(Y) = \eta(Z) = u(X) = u(Y) = u(Z) = 0$.

Let $(M, \varphi, \xi, \eta, g)$ be a l.c.C. manifold with Lee form $\omega \neq 0$ at every point and let c, u, U, v, V be as in (1.5). Denote by R the Riemann curvature tensor of M and by $\overline{R}$ the curvature tensor of the connection $\overline{\nabla}$ on M given by (1.1).

From (4.2) and using corollaries 3.2 and 3.3, we obtain

COROLLARY 4.1. *If $(M, \varphi, \xi, \eta, g)$ is a* P*C-manifold then, the following conditions are equivalent:*

i) (M,φ,ξ,η,g) *is a* P*C(k)-manifold.*

ii) *For all* $X,Y,Z\in\mathfrak{X}(M)$

$$\begin{aligned}R(X,Y)Z=&\frac{1}{4}(k+3c^2)(g(Y',Z')X'-g(X',Z')Y')+\\&+\frac{1}{4}(k-c^2)\Big\{v(X)v(Z)Y'-v(Y)v(Z)X'+\\&+(g(X',Z')v(Y)-g(Y',Z')v(X))V+\\&+g(\varphi Y',Z')\varphi X'-g(\varphi X',Z')\varphi Y'+2g(X',\varphi Y')\varphi Z'+\\&+(v(X)g(\varphi Y',Z')-v(Y)g(\varphi X',Z')+\\&+2v(Z)g(X',\varphi Y'))U\Big\}+\\&+2c^2(\eta\wedge u)(X,Y)(\eta(Z)U-u(Z)\xi),\end{aligned}\tag{4.3}$$

where X',Y' *and* Z' *are the orthogonal projections of* X,Y *and* Z *respectively onto the tangent planes of the leaves of the canonical foliation.*

iii) *For all* $X,Y,Z\in\mathfrak{X}(M)$

$$\begin{aligned}\overline{R}(X,Y)Z=&\frac{1}{4}(k-c^2)\Big\{g(Y',Z')X'-g(X',Z')Y'+\\&+v(X)v(Z)Y'-v(Y)v(Z)X'+\\&+(g(X',Z')v(Y)-g(Y',Z')v(X))V+\\&+g(\varphi Y',Z')\varphi X'-g(\varphi X',Z')\varphi Y'+2g(X',\varphi Y')\varphi Z'+\\&+(v(X)g(\varphi Y',Z')-v(Y)g(\varphi X',Z')+\\&+2v(Z)g(X',\varphi Y'))U\Big\}\end{aligned}\tag{4.4}$$

where X',Y' *and* Z' *are the orthogonal projections of* X,Y *and* Z *respectively onto the tangent planes of the leaves of the canonical foliation.*

Let (M,φ,ξ,η,g) be a l.c.C. manifold. Then, every point $x \in M$ has an open neighbourhood U such that the structure $(\varphi, e^{-\sigma}\xi, e^{\sigma}\eta, e^{2\sigma}g)$ is cosymplectic on U and $\overline{R}$ is the curvature tensor of the local metric $e^{2\sigma}g$, where $\sigma\colon U \longrightarrow \mathbb{R}$ is a real differentiable function on U (see §1). Moreover, using (3.8), (4.3) and (4.4) we deduce

COROLLARY 4.2. *If (M,φ,ξ,η,g) is a* P*C-manifold with Lee form ω and $c = \|\omega\|$, then the following conditions are equivalent:*

i) *(M,φ,ξ,η,g) is a* P*C*(c^2)*-manifold.*

ii) *The leaves of the canonical foliation are of constant sectional curvature c^2.*

iii) *The local metrics $e^{2\sigma}g$ are flat, i.e., $\overline{R} = 0$.*

Next, we give some results about the universal covering space of a PC-manifold. First, we introduce some definitions and we prove some previous results.

Let N, k be a $(2n-1)$-dimensional manifold and a real number respectively and let H_c^2 be the 2-dimensional hyperbolic space, i.e., H_c^2 is the space of 2-tuples of real numbers (s,t) with the Riemannian metric given by

$$d\tau^2 = ds^2 + e^{-2cs}dt^2$$

where c is a positive constant. Then,

DEFINITION 4.2. *A distinguished* P*C*(c) *(respectively* P*C*(c,k)*)-structure on $M = N \times H_c^2$ is a* P*C(respectively* P*C*(k)*)-structure (φ,ξ,η,g) on M, such that:*

a) *The metric g is of the form*

$$g = d\sigma^2 + ds^2 + e^{-2cs}dt^2 = d\sigma^2 + d\tau^2\,,$$

where $d\sigma^2$ is a Riemann metric on N and,

b) *$\xi = e^{cs}\partial/\partial t$ and the unit Lee vector field is $U = \partial/\partial s$.*

PROPOSITION 4.2. *If (φ,ξ,η,g) is a distinguished* P*C*(c)*-structure on $M = N \times H_c^2$, then the manifold N carries an induced c-Sasakian structure $(\varphi_N,\xi_N,\eta_N,g_N)$. Moreover, if (φ,ξ,η,g) is a distinguished* P*C*(c,k)*-structure on M, then N is of constant φ_N-sectional curvature k.*

PROOF. From definition 4.2, we have that

$$g = d\sigma^2 + ds^2 + e^{-2cs}dt^2\,, \quad U = \partial/\partial s\,, \quad \xi = e^{cs}\partial/\partial t \tag{4.5}$$

where $d\sigma^2$ is a Riemann metric on N. Thus, if ω and u are the Lee 1-form and the unit Lee 1-form on M respectively, then we obtain that

$$\omega = c\,ds\,, \quad u = ds\,, \quad \eta = e^{-cs}dt\,. \tag{4.6}$$

By using first and second relation of (2.2) and third and fourth relation of (2.3) we deduce that $\xi_N = -\varphi U = -V$ and $\eta_N = u\circ\varphi = -v$ define a vector field and a 1-form respectively on N.

Let X be a vector field on N. Then, $X = \overline{X} + v(X)V$ with $v(\overline{X}) = 0$. Define $\varphi_N X = \varphi\overline{X}$.

From first and second relation of (2.3) we obtain that φ_N defines a (1,1)-tensor field on N.

Now, it is easy to check that $(\varphi_N, \xi_N, \eta_N, g_N = d\sigma^2)$ is an almost contact metric structure on N.

On the other hand, from (4.6), we deduce that the leaves of the canonical foliation of M are $N \times \{(s_0, t_0)\}$, with $(s_0,t_0) \in \mathbb{R}^2$. Thus, by proposition 4.1, we get a c-Sasakian structure on each $N \times \{(s_0,t_0)\}$, $(s_0,t_0) \in \mathbb{R}^2$. In fact, if $(s_0,t_0) \in \mathbb{R}^2$ then, it is not difficult to check that the application $i_{(s_0,t_0)}$ of $N \times \{(s_0,t_0)\}$ into N given by $i_{(s_0,t_0)}(x, s_0, t_0) = x$ is an almost contact isometry. This, in view of proposition 4.1 and definition 4.1, ends the proof of proposition. □

Reciprocally, let $(N, \varphi_N, \xi_N, \eta_N, g_N = d\sigma^2)$ be an almost contact metric manifold with $\dim N = 2n-1$.

Then, we define an almost contact metric structure (φ, ξ, η, g) on $M = N \times H_c^2$ by

$$\varphi\Big(X, a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\Big) = \Big(\varphi_N X - a\xi_N, \eta_N(X)\frac{\partial}{\partial s}, 0\frac{\partial}{\partial t}\Big)$$

$$\xi = \Big(0, 0\frac{\partial}{\partial s}, e^{cs}\frac{\partial}{\partial t}\Big)$$

$$\eta\Big(X, a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\Big) = e^{-cs}b$$

$$g\Big(\Big(X, a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\Big), \Big(X', a'\frac{\partial}{\partial s}, b'\frac{\partial}{\partial t}\Big)\Big) = d\sigma^2(X, X') + aa' + e^{-2cs}bb'$$

for all $X, X' \in \mathfrak{X}(N)$ and a, a', b, b' differentiable functions on M, where (s,t) are the coordinates on H_c^2.

Denote by L the Lie derivate on N, by ∇ the Riemann connection of the metric g, by N_φ and N_{φ_N} the Nijenhuis tensors of φ and φ_N respectively and by Φ and Φ_N the fundamental 2-forms of the structures in M and N respectively. Then, by a direct computation, we have

LEMMA 4.1.

$$\begin{aligned} &N_\varphi\Big(\Big(X, a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\Big), \Big(X', a'\frac{\partial}{\partial s}, b'\frac{\partial}{\partial t}\Big)\Big) = \\ (4.7)\quad &= \Big(N_{\varphi_N}(X,X') + 2d\eta_N(X,X')\xi_N + a'\Big(L_{\xi_N}\varphi_N\Big)X + \\ &- a\Big(L_{\xi_N}\varphi_N\Big)X', \; 2\Big(d\eta_N(\varphi_N X, X') - d\eta_N(\varphi_N X', X) + \\ &+ a' d\eta_N(\xi_N, X) - a\, d\eta_N(\xi_N, X')\Big)\frac{\partial}{\partial s}, \; 0\frac{\partial}{\partial t}\Big), \end{aligned}$$

$$(4.8)\quad \Phi = \Pi^*\Phi_N + 2ds\wedge\Pi^*\eta_N, \quad d\Phi = d\Big(\Pi^*\Phi_N\Big) - 2ds\wedge d\Big(\Pi^*\eta_N\Big),$$

$$(4.9)\quad d\eta = \eta\wedge(c\,ds),$$

$$(4.10)\quad \nabla_{(X, a\partial/\partial s, b\partial/\partial t)}\Big(0, \frac{\partial}{\partial s}, o\frac{\partial}{\partial t}\Big) = \Big(0, 0\frac{\partial}{\partial s}, -cb\frac{\partial}{\partial t}\Big),$$

for all $X, X' \in \mathfrak{X}(N)$ and a, a', b, b' differentiable functions on M, where $\Pi\colon M \longrightarrow N$ is the projection onto the first factor.

Now, by using lemma 4.1, we deduce

COROLLARY 4.3. *If the structure $(\varphi_N, \xi_N, \eta_N, d\sigma^2)$ on N is c-Sasakian, then (φ, ξ, η, g) is a distinguished $PC(c)$-structure on M. Moreover, if N is of constant φ_N-sectional curvature k, then (φ, ξ, η, g) is a distinguished $PC(c,k)$-structure on M.*

PROOF. If the structure $(\varphi_N, \xi_N, \eta_N, d\sigma^2)$ is c-Sasakian then it is normal, $d\eta_N = c\Phi_N$ and $L_{\xi_N}\varphi_N = 0$ (see [10]). Thus, from (1.4) and from lemma 4.1 and proposition 2.3, we obtain that (φ, ξ, η, g) is a distinguished P$C(c)$-structure on M. Therefore, $(N, \varphi_N, \xi_N, \eta_N, d\sigma^2)$ is almost contact isometric to the leaves of the canonical foliation of M with the induced c-Sasakian structure (see proof of proposition 4.2). This proves the rest of corollary. □

Next, we give the announced results on the universal covering space of a PC-manifold.

THEOREM 4.2. *The universal covering space of a $(2n+1)$-dimensional complete* PC*-manifold M with Lee form ω, is a product space $\overline{M} = N \times H_c^2$, where N is the universal covering space of an arbitrary leaf of the canonical foliation of M, $c = \|\omega\|$ and H_c^2 is the 2-dimensional hyperbolic space. The lift of the* PC*-structure to $\overline{M}$ gives a distinguished* P$C(c)$*-structure on $\overline{M}$. Moreover, if the structure of M is a* P$C(k)$*-structure, then, considering the induced c-Sasakian structure on N, we have:*

i) *If $k > -3c^2$, then N is almost contact isometric to $S^{2n-1}(c,k)$;*

ii) *If $k = -3c^2$, then N is almost contact isometric to $\mathbb{R}^{2n-1}(c)$;*

iii) *If $k < -3c^2$, then N is almost contact isometric to $(\mathbb{R} \times CD^{n-1})$ (c,k).*

PROOF. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional complete PC-manifold and ω, u the Lee form and the unit Lee form respectively of M.

Denote by $\bar{g}$ the induced metric on $\overline{M}$. Then, using theorem 4.1 and theorem A of [2], we deduce that $(\overline{M}, \bar{g})$ is the Riemannian product $N \times H_c^2$, where N is the universal covering space of an arbitrary leaf of the canonical foliation F, $c = \|\omega\|$ and H_c^2 is the hyperbolic 2-dimensional space. Moreover, if $F^\perp$ is the foliation determined by the normal bundle of F then, the lift of the foliations F and $F^\perp$ to $\overline{M}$ are the foliations with leaves of the form $N \times \{q\}(q \in H_c^2)$ and $\{p\} \times H_c^2(p \in N)$ respectively.

Now, let $\bar{\eta}$ and $\bar{u}$ be the lift of η, u respectively to $\overline{M}$. Then, it is clear, from (1.4) and from the fact that $\bar{u}$ is a closed 1-form, that $\{\bar{\eta}, \bar{u}\}$ is a global basis of 1-forms on H_c^2. The dual basis of vector fields on H_c^2 is given by $\{\bar{\xi}, \overline{U}\}$, being $\bar{\xi}$ and $\overline{U}$ the lift of ξ and U respectively to $\overline{M}$.

Thus, using the following lemma 4.2, we obtain that

$$\overline{U} = \frac{\partial}{\partial s}, \qquad \bar{\xi} = e^{cs}\frac{\partial}{\partial t}$$

where (s,t) are the coordinates on H_c^2.

Consequently, the lift of the PC-structure (φ, ξ, η, g) to $\overline{M}$ is a distinguished $PC(c)$-structure on $\overline{M}$.

If (φ, ξ, η, g) is a $PC(k)$-structure on M, then the lift of this $PC(k)$-structure to $\overline{M}$ gives a distinguished $PC(c,k)$-structure on $\overline{M}$ and therefore, since N is a simply connected complete manifold, the rest of theorem follows using proposition 4.2 and theorem 1.2. □

LEMMA 4.2. *Let M be a 2-dimensional complete, simply connected, Riemannian manifold of constant negative curvature $-c^2 (c \neq 0)$ and let U, ξ be vector fields on M such that $\{U, \xi\}$ form an orthonormal basis for M and $[U, \xi] = c\xi$. Then, there is an isometry F of M to the 2-dimensional hyperbolic space H_c^2, satisfying*

$$F_*U = \frac{\partial}{\partial s}, \qquad F_*\xi = e^{cs}\frac{\partial}{\partial t}$$

where (s,t) are the coordinates on H_c^2.

PROOF. Let x be a point of M. We consider the linear isometry L of T_xM onto $T_{(0,0)}(H_c^2)$ given by

$$L(U_x) = \frac{\partial}{\partial s}_{|(0,0)}, \qquad L(\xi_x) = \frac{\partial}{\partial t}_{|(0,0)}.$$

Then, there is an isometry F of M onto H_c^2 such that the differential of F at x is L (see, for instance, [11]) and thus, using the relation $[U, \xi] = c\xi$, we prove that

$$F_*U = \frac{\partial}{\partial s}, \qquad F_*\xi = e^{cs}\frac{\partial}{\partial t}.$$

□

Acknowledgments

The author is grateful to D. Chinea and to the referee for helpful suggestions and remarks concerning this paper.

REFERENCES

[1] D.E. BLAIR: *The theory of quasi-Sasakian structures*, J. Diff. Geom. 1 (1967), 331-345.

[2] R.A. BLUMENTHAL- J.J. HEBDA: *De Rham decomposition theorems for foliated manifolds*, Ann. Inst. Fourier, Grenoble, 33 (1983), 183-198.

[3] L.A. CORDERO - M. FERNANDEZ - M. DE LEON: *Compact locally conformal Kähler nilmanifolds*, Geometriae Dedicata 21 (1986), 187-192.

[4] Y. CARRIERE - E. GHYS: *Feuilletages totalement géodésiques*, Ann. Acad. Brasil Ciênc. 53 (1981), 427-432.

[5] D. CHINEA - J.C. MARRERO: *Conformal changes of almost contact metric structures*, preprint.

[6] D. CHINEA - J.C. MARRERO: *Conformal changes of almost cosymplectic manifolds*, preprint.

[7] D. CHINEA - J.C. MARRERO: *On Invariant Submanifolds of Locally Conformal Almost Cosymplectic Manifolds*, Bollettino U.M.I. (7) 4-A (1990), 357-364.

[8] B.Y. CHEN - P. PICCINNI: *The canonical foliations of a locally conformal Kähler manifold*, Ann. di Mat. Pura ed appl. 141 (4) (1985), 289-305.

[9] S.I. GOLDBERG: *Curvature and homology*, Academic Press, New York, 1962.

[10] D. JANSSENS - L. VANHECKE: *Almost contact structures and curvature tensors*, Kodai Math. J. 4 (1981), 1-27.

[11] S. KOBAYASHI - K. NOMIZU: *Foundations of Differential Geometry*, Vol. I and II Intersc. Publ., New York (1969).

[12] P. LIBERMANN: *Sur les structures presque complexes et autres structures infinitesimales regulieres*, Bull. Soc. Mat. France 83 (1955), 195-224.

[13] K. MATSUMOTO: *On locally conformal Kähler space forms*, Internat. J. Math. Sci. 8 (1) (1985), 69-74.

[14] M. OKUMURA: *On infinitesimal conformal and projective transformations of normal contact spaces*, Tôhoku Math. J. 14 (1962), 398-412.

[15] K. OGIUE: *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kodai Math. Sem. Rep. 16 (1964), 223-232.

[16] S. SASAKI - Y. HATAKEYAMA: *On the differentiable manifolds with certain structures which are closely related to almost contact structure II*, Tôhoku Math. J. 13 (1961), 281-294.

[17] S. TANNO: *The topology of contact Riemannian manifolds*, Illinois J. Math. 12 (1968), 700-717.

[18] S. TANNO: *Sasakian manifolds with constant φ-holomorphic sectional curvature*, Tôhoku Math. J. 21 (1969), 501-507.

[19] I. VAISMAN: *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976), 338-351.

[20] I. VAISMAN: *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. Roma 12 (1979), 263-284.

[21] I. VAISMAN: *Conformal changes of almost contact metric structures*, Lecture Notes in Math. 792 (1979), Springer.

[22] I. VAISMAN: *Generalized Hopf manifolds*, Geometriae Dedicata 13 (1982), 231-255.

[23] I. VAISMAN - C. REISCHER: *Local similarity manifolds*, Ann. di Mat. Pura ed Appl. 135 (IV) (1983), 279-292.

Lavoro pervenuto alla redazione il 26 giugno 1990
ed accettato per la pubblicazione il 16 ottobre 1991
su parere favorevole di S. Marchiafava e di M. Bruni

INDIRIZZO DELL'AUTORE:

J.C. Marrero - Dpto. de Matemática Fundamental - Universidad de La Laguna - La Laguna, Tenerife - Canary Island - Spain