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Approximation numbers of continuous linear mappings and compact operators on non-archimedean spaces

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RIASSUNTO – *Si studiano operatori compatti e semi-compatti tra spazi localmente convessi su un campo non archimedeo e si esamina la connessione tra operatori compatti e i loro "numeri di approssimazione", precedentemente introdotti.*

ABSTRACT – *Compact and semi-compact operators between locally convex spaces over a non-Archimedean valued field are investigated and the connection between compact operators and their approximation numbers is examined.*

KEY WORDS - *Approximation numbers - Compact operator - t-orthogonal sequence - Pseudo-reflexive space.*

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– Introduction

Throughout this paper, $\mathbb{K}$ will be a complete non-archimedean non-trivially valued field. If the valuation is discrete, we will denote by π an element of $\mathbb{K}$ such that $|\pi| < 1$ is the generator of the value group of $\mathbb{K}$. For a subset S of a vector space E over $\mathbb{K}$, we will denote by $\mathrm{co}(S)$ the absolutely convex hull of S. For $A \subset E$, the linear subspace spanned by A will be denoted by [A].

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Let now E be a locally convex space over $\mathbb{K}$. If B is a bounded absolutely convex subset of E, we will denote by E_B the vector space [B] equipped with the Minkowski functional p_B of B (i.e. for $x \in$[B], $p_B(x) = \inf\{|\lambda| : x \in \lambda B\}$) . B is said to be completing if the normed space E_B is complete. For V, W convex neighborhoods of zero, we write $V \prec W$ if W absorbs V. In this case, for each non-negative integer n, we define $\delta_n(V,W) = \inf\{|\lambda| : \lambda \in \mathbb{K}, V \subset F + \lambda W, F$ linear subspace of E, $\dim F \leq n\}$. The diametral dimension space $\Delta(E)$ of E is the collection of all sequences $(\delta_n)_{n \geq 0}$ of non-negative real numbers such that, for each convex neighborhood W of zero in E, there exists another one $V \prec W$ such that $\delta_n(V,W) \leq \delta_n$ for all n.

We will denote by $\widehat{E}$ the completion of a separated locally convex space E over $\mathbb{K}$. If E, F are locally convex space over $\mathbb{K}$, $L(E,F)$ is the space of all linear continuous maps from E nto F. By $\widehat{T} : \widehat{E} \to \widehat{F}$ we mean the extension of $T \in L(E,F)$ to the completions of E and F.

If E is a non-archimedean normed space over $\mathbb{K}$, B_E will denote the unit ball in E (i.e. $B_E = \{x \in E : \|x\| \leq 1\}$). Also for $r > 0$, we denote by $B(0,r)$ the set $\{x \in E : \|x\| \leq r\}$.

For all further basic notions and notations we refer to [10].

1 – Approximation numbers of linear mappings

It this section E, F are going to be non-archimedean normed spaces over $\mathbb{K}$. For each non-negative integer n, we will denote by $\mathcal{A}_n(E,F)$ the subset of $L(E,F)$ consisting of those $T \in L(E,F)$ for which $\dim T(E) \leq n$.

1.1. Recall, for each non-negative integer n, $T \in L(E,F)$ and a bounded subset B of E, the following definitions:

(a) $\alpha_n(T) := \inf\{\|T - A| : A \in \mathcal{A}_n(E,F)\}$.

(b) $u_n(T) := \inf\{\|T|D\| : D$ is a linear subspace of E of codimension $\leq n\}$.

(c) $\delta_n^*(B) := \inf\{r > 0 : B \subset G + B(0,r)$, G linear subspace of E, $\dim G \leq n\}$.

The numbers $\alpha_n(T)$ were introduced by the first author in [6] where they are called approximation numbers. The numbers $u_n(T)$ were introduced by A.C.M. Van Rooij in [10, p. 144]. As concerning to $\delta_n^*(B)$, the

Kolmogorov diameters of B, they were introduced by authors in [7]. A closely related diameter, $\delta_n(B)$, had been previously introduced by the first author in [6].

The aim of this section is to compare $\alpha_n(T)$, $u_n(T)$ as well as $\delta_n^*(T(B_E))$.

The following lemma is in fact a part of the proof of theorem 4.7 in [8]. For the sake of completeness we include a sketch of the proof.

LEMMA 1.2. *Let D be a linear subspace of a normed space E with finite codimension. Then, for each $\varepsilon > 0$ there exists a continuous linear projection P from E onto D such that $\|P\| \leq 1+\varepsilon$.*

PROOF. Let $\varepsilon > 0$ and choose $t \in (0,1)$ such that $\frac{1}{1+\varepsilon} < t$. Let $\{Q(e_1 \ldots, Q(e_n)\}$ be a t-orthogonal basis of E/D where $Q : E \to E/D$ denote the quotient map. For each i $\in \{1, \ldots, n\}$, choose $x_i \in Q(e_i)$ such that $\|x_i\| \leq t(1+\varepsilon)\|Q(e_i)\|$. Then the linear map $H : E/D \to E$ defined by $Q(e_i) \mapsto x_i$ is injective and $\|H\| \leq 1+\varepsilon$. Finally $P = I - HQ$ satisfies the required conditions.

THEOREM 1.3. $\alpha_n(T) = u_n(T)$.

PROOF. Let $A \in \mathcal{A}_n(E,F)$. Then Ker A is a linear subspace of E of codimension $\leq n$. Also $\|T|\operatorname{Ker} A\| = \|(T-A)|\operatorname{Ker} A\| \leq \|T - A\|$. So, $u_n(T) \leq \alpha_n(T)$.

Conversely, let D be a linear subspace of E of codimension $\leq n$ and let $\varepsilon > 0$. By 1.2 there exists a projection P from E onto D such that $\|P\| \leq 1+\varepsilon$. Set $A := T(I-P)$. Then $A \in \mathcal{A}_n(E,F)$ and $\|T - A\| \leq (1+\varepsilon)\|T|D\|$. So, $\alpha_n(T) \leq u_n(T)$.

THEOREM 1.4. *Assume F to be pseudoreflexive. Then,*

(a) $\alpha_n(T) \leq \delta_n^*(T(B_E))$ *if the valuation of* $\mathbb{K}$ *is dense.*

(b) $|\pi|\alpha_n(T) \leq \delta_n^*(T(B_E))$ *if the valutation of* $\mathbb{K}$ *is discrete.*

PROOF. Let $T(B_E) \subset G + B(0,r)$, where G is a subspace of F with $\dim G \leq n$. Since F is pseudoreflexive, there exists for any given $\varepsilon > 0$ a projection P of F onto G with $\|P\| \leq 1+\varepsilon$ ([8], theorem 2.1). Let $A = PT \in \mathcal{A}_n(E,F)$ and let $\nu \in \mathbb{K}, |\nu| > 1$. Given $x \neq 0$ in E, there exists an integer k such that $|\nu|^{k-1} < \|x\| \leq |\nu|^k$. Now $\nu^{-k}Tx = y + w$, where $y \in G$ and $\|w\| \leq r$. Since $Py = y$, we have $\nu^{-k}Ax = y + Pw$ and so

$$|\nu|^{-k}\|Tx - Ax\| = \|w - Pw\| \leq (1+\varepsilon)r\,.$$

Thus

$$\|Tx - Ax\| \leq |\nu|(1+\varepsilon)r\|x\|\,,$$

which implies that

$$\alpha_n(T) \leq \|T - A\| \leq |\nu|(1+\varepsilon)r$$

and so $\alpha_n(T) \leq |\nu|r$ since $\varepsilon > 0$ was arbitrary. Now (a) and (b) follow.

LEMMA 1.5. $\delta_n^*(T(B_E)) \leq \alpha_n(T)$.

PROOF. It takes only the obvious changes with respect to the proof of the proposition 3.7 in [6].

COROLLARY 1.6. *Assume F to be pseudoreflexive. Then,*
(a) $\alpha_n(T) = \delta_n^*(T(B_E))$ *if the valuation of* $\mathbb{K}$ *is dense.*
(b) $|\pi|\alpha_n(T) \leq \delta_n^*(T(B_E)) \leq \alpha_n(T)$ *if the valuation of* $\mathbb{K}$ *is discrete.*

REMARKS 1.7. The hypothesis of pseudoreflexivity on F can not be dropped in general as the following example shows.

EXAMPLE ([8]). Let F be a Banach space for which $F' = \{0\}$ (e. g. ℓ^∞/c_0 over a non-spherically complete ground field). Let $\beta \in \mathbb{K} - \{0\}$, $e \in F - \{0\}$ and $\nu \in \mathbb{K}$ such that $0 < |\nu| < \|e\|$. Let E be the space F endowed with the Minkowski functional $p_A(x) = \inf\{|\lambda| : x \in \lambda A\}$ where $A = (\nu/B)B_F + \mathrm{co}\{e\}$. The identity map $T : E \to F$ is continuous (in fact, it is a homeomorphism) and for all $n \geq 1$, $\alpha_n(T) = \|T\| > |\nu|$ because $\mathcal{A}_n(E,F) = \{0\}$ whereas $\delta_n^*(T(B_E)) = \delta_n^*(\{x \in E : p_A(x) \leq 1\}) \leq \delta_n^*(yA) \leq |y\nu/B|$ for all $y \in \mathbb{K}, |y| > 1$. It follows that $|\beta|\delta_n^*(T(B_E)) < \alpha_n(T)$.

1.8. From the above theorem it follows that $\lim \delta_n^*(T(B_E)) = \lim u_n(T)$ if the valuation on $\mathbb{K}$ is dense and $\lim \delta_n^*(T(B_E)) \leq \lim u_n(T) \leq |\pi|^{-1} \lim \delta_n^*(T(B_E))$ otherwise. So, we have abtained with a different proof lemma 4.8 of [8].

LEMMA 1.9. *Assume F to be complete. Let M be a dense subspace of E, G a dense subspace of F and $T \in L(M,G)$. If $\widehat{T} \in L(E,F)$ is the extension of T, then $\alpha_n(T) = \alpha_n(\widehat{T})$ for all n.*

PROOF. Since G is dense in F, we have $\alpha_n(T) = \alpha_n^F(T)$ by [6, proposition 2.4], where $\alpha_n^F(T)$ denotes the nth approximation number of T considered as a map fram M to F. Hence we may assume that $F = G$. Given $A \in \mathcal{A}_n(M,F)$, the space $A(M)$ is closed in F and so $\widehat{A}(E) = A(M)$.Thus

$$\alpha_n(\widehat{T}) \leq \|\widehat{T} - \widehat{A}\| = \|T - A\|,$$

which proves that $\alpha_n(\widehat{T}) \leq \alpha_n(T)$. On the other hand, if $B \in \mathcal{A}_n(E,F)$ and if $A = B|M$, then

$$\alpha_n(T) \leq \|T - A\| = \|\widehat{T} - B\|,$$

and so $\alpha_n(T) \leq \alpha_n(\widehat{T})$

PROPOSITION 1.10. *Let $T \in L(E,F)$. If there exists $S \in L(F,E)$ such that $TS = I_F$, where I_F is the identity map on F, then for $n < \dim F$ we have $\alpha_n(T)\|S\| \geq 1$.*

PROOF. If $\widehat{T} \in L(\widehat{E},\widehat{F})$ and $\widehat{S} \in L(\widehat{F},\widehat{E})$ are the extensions of T and S respectively, then $\widehat{T}\widehat{S} = I_{\widehat{F}}$, $\alpha_n(\widehat{T}) = \alpha_n(T)$ and $\|\widehat{S}\| = \|S\|$. Hence, we may assume that both E and F are complete. Suppose now that $\alpha_n(T)\|S\| < 1$. Then, there exists $A \in \mathcal{A}_n(E,F)$ such that $\|T - A\|\|S\| < 1$. Thus $\|(T-A)S\| < 1$ and so $AS = I_F - (T-A)S$ is invertible, which is a contradiction since the range of AS is a proper subspace of F.

COROLLARY 1.11.

(1) *If there exists a linear homeomorphism T from E onto F, then $\alpha_n(T)\|T^{-1}\| \leq 1$ for each $n < \dim F$.*
(2) *For $n < \dim E$, we have $\alpha_n(I_E) = 1$ while for $n \geq \dim E$ we have $\alpha_n(I_E) = 0$.*

PROOF.
(1) If follows from the preceding proposition.
(2) Clearly $\alpha_n(I_E) = 0$ if $n \geq \dim E$. For $n < \dim E$, we have

$$1 = \|I_E\| \geq \alpha_n(I_E) = \alpha_n(I_E)\|I_E^{-1}\| \geq 1.$$

COROLLARY 1.12. *Let $T \in L(E,F)$. Then, $T \in \mathcal{A}_n(E,F)$ if and only if $\alpha_n(T) = 0$.*

PROOF. Clearly $\alpha_n(T) = 0$ if $T \in \mathcal{A}_n(E,F)$. Conversely, if $\alpha_n(T) = 0$, then

$$\delta_n^*(T(B_E)) \leq \alpha_n(T) = 0,$$

which implies that $\dim T(E) \leq n$ by [7, Corollary 2.7].

2 – Compact operators in normed spaces

2.1. Recall that, if E, F are locally convex spaces over $\mathbb{K}$, then a linear map $T : E \to F$ is called (see [3, 2.1] and [4, 2.1 and 2.10]):

(a) Compact if there exists a neighborhood V of zero in E such that $T(V)$ is compactoid and $\overline{T(V)}$ is complete. The set of all compact operators from E to F will be denoted by $C(E,F)$.

(b) Semi-compact if there exists a compactoid, completing subset D of F such that $T^{-1}(D)$ is a neighborhood of zero. The set of all semi-compact operators from E to F will be denoted by $SC(E,F)$.

(c) Compactoid if there exists a neighborhood of zero in E such that $T(V)$ is compactoid. The set of all compactoid operators from E to F will be denoted by $CO(E,F)$.

2.2. It is obvious that $C(E,F) \subset SC(E,F) \subset CO(E,F)$. Also, if F is a complete space then $C(E,F) = SC(E,F) = CO(E,F)$. However, in general the above inclusions are strict even when E and F are normed spaces ([3]). N. de Grande-de Kimpe and the second author have recently developed ([4]) a Fredholm theory for semi-compact operators. Notice that the composition of a compact operator with a continuous one does not need to be compact in general ([4], 2.10).

The next theorem is basically a slight generalization of [10, theorem 4.40] to the context of normed (not necessarily complete) spaces.

THEOREM 2.3. *Let E, F be normed spaces over $\mathbb{K}$ and let $T \in L(E,F)$. Then the following are equivalent,*

(1) *T is compactoid.*
(2) $\lim \alpha_n(T) = 0$.
(3) $\lim u_n(T) = 0$.
(4) *For each $t \in (0,1)$ there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F, with $\|g_n\| \leq 1$ and y_n converging to zero such that*

$$Tx = \sum_1^{\infty} g_n(x) y_n \quad (x \in E).$$

(5) *For each $t \in (0,1)$ there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F such that $\|g_n\|\|y_n\|$ tends to zero and*

$$Tx = \sum_1^{\infty} g_n(x) y_n \quad (x \in E).$$

(6) *There exists a sequence (h_n) in E' with $\lim \|h_n\| = 0$, such that*

$$\|Tx\| \leq \sup_n |h_n(x)| \quad (x \in E).$$

(7) *There exists $S \in C(E, c_0)$ such that $\|Tx\| \leq \|Sx\|$ for all $x \in E$.*

PROOF. (1) $\Leftrightarrow$ (2). First assume that T is compactoid. Then, the extension $\widehat{T} \in L(\widehat{E}, \widehat{F})$ is also compactoid. Given $\varepsilon > 0$, there exists n and $A \in \mathcal{A}_n(\widehat{E}, \widehat{F})$ such that $\|\widehat{T} - A\| < \varepsilon$ (by [10, Theorem 4.39]). If now $m \geq n$, then

$$\alpha_m(T) = \alpha_m(\widehat{T}) \leq \alpha_n(\widehat{T}) \leq \|\widehat{T} - A\| < \varepsilon,$$

and so $\lim \alpha_n(T) = 0$. The converse was proved in [6, Proposition 2.5].

(2) $\Leftrightarrow$ (3). If follows from 1.3.

(1) $\Rightarrow$ (4). Since $T \in CO(E,F)$, its extension $\widehat{T} \in L(\widehat{E}, \widehat{F})$ is compact. Then $T(B_E)$ is a compactoid subset subset of the Banach space of

countable type $G = cl_{\widehat{F}}(D)$ where $D = \widehat{T}(\widehat{E})$. So, given $t \in (0,1)$ and $\beta \in \mathbb{K}, 0 < |\beta| < 1$, there exists by [10, Lemma 4.36 (A)] a t-orthogonal sequence (y_n) in $\beta^{-1}T(B_E)$ (and hence in F) such that $\lim y_n = 0$ and $T(B_E) \subset cl_{\widehat{F}} \operatorname{co}\{y_1, \ldots, y_n, \ldots\}$. Proceeding as in [10, Theorem 4.40, $(\alpha) \Rightarrow (\delta)$] one can easily prove that there exists a sequence (g_n) in E' such that $\|g_n\| \leq 1$ for all n and

$$Tx = \sum_{1}^{\infty} g_n(x) y_n \quad (x \in E).$$

(4) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (6). Let $\lambda \in \mathbb{K}$ with $0 < |\lambda| < 1$. For each n with $y_n \neq 0$ choose $\nu_n \in \mathbb{K}$, $|\lambda| < \|\nu_n y_n\| \leq 1$. Set $h_n = 0$ if $y_n = 0$ and $h_n = \nu_n^{-1} g_n$ otherwise. Now $\lim h_n = 0$ and

$$\|Tx\| \leq \sup_n |h_n(x)| \quad (x \in E).$$

(6)$\Rightarrow$ (7). It suffices to take

$$S : E \to c_0, Sx = (h_n(x)).$$

(7) $\Rightarrow$ (1). Let $\widehat{S} \in C(\widehat{E}, c_0)$ and $\widehat{T} \in L(\widehat{E}, \widehat{F})$ be the extensions of S and T. Then $\|\widehat{T}x\| \leq \|\widehat{S}x\|$ for all $x \in \widehat{E}$. Let $(x_i)_{i \in I}$ be a bounded net converging weakly to 0 in $\widehat{E}$. By compactness of $\widehat{S}$, $(\widehat{S}(x_i))_{i \in I}$ (and hence $(\widehat{T}(x_i))_{i \in I}$) converges to 0 in the norm topology. By [9, Theorem 1.2], it follows that $\widehat{T} \in C(\widehat{E}, \widehat{F})$ and hence $T \in C0(E, F)$.

REMARKS 2.4.

a) The preceding theorem holds also for seminormed spaces E, F.

b) In the above theorem the assumption of t-orthogonality made in (4) and (5) can be dropped. On the other hand it is possible to assume in (4) and (5) that the sequence (y_n) lies in $\beta^{-1}T(B_E)$ for any choice of $\beta \in \mathbb{K}$ with $|\beta| \in (0,1)$.

c) If every one-dimensional subspace of $\widehat{F}(F?)$ is orthocomplemented, then the above theorem holds for $t = 1$. Also, if the valuation of $\mathbb{K}$ is discrete one can assume in (4) and (5) that the sequence (y_n) is orthogonal and lies in $T(B_E)$ (see [10, Lemma 4.36 (B), (C)]).

COROLLARY 2.5. *Let E, F be normed spaces over $\mathbb{K}$ and let $T \in L(E,F)$. Then the following are equivalent,*

(a) *T is compact.*

(b) *For each $t \in (0,1)$ there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F, with $\|g_n\| \leq 1$ and y_n converging to zero such that*

$$Tx = \sum_{l}^{\infty} g_n(x) y_n \quad (x \in E)$$

and $\overline{\mathrm{co}}\{y_n : n \in \mathbb{N}\}$ is complete.

PROOF.

$(a) \Rightarrow (b)$. It is left to prove that $cl_{\widehat{F}} \mathrm{co}\{y_n : n \in N\} \subset F$. Since T is compact, the $\widehat{F}$-closure of $\widehat{T}(B_{\widehat{E}})$ lies in F ([3], Theorem 2.11). Then,

$$cl_{\widehat{F}} \mathrm{co}\{y_n : n \in N\} \subset \beta^{-1} cl_{\widehat{F}} T(B_E) \subset F$$

where β is as in 2.4 (a).

$(b) \Rightarrow (a)$. We know that T is compactoid. Also, if $x \in B_E$, then $Tx \in \overline{\mathrm{co}}\{y_n : n \in \mathbb{N}\}$. Since this subset is complete, the $\widehat{F}$-closure of $\widehat{T}(B_{\widehat{E}})$ lies in $\overline{\mathrm{co}}\{y_n : n \in \mathbb{N}\}$ (and hence in F). By ([3]), Theorem 2.11), T is compact.

COROLLARY 2.6. *Let E, F be normed spaces over $\mathbb{K}$ and let $T \in L(E,F)$.*

(a) *If T is semi-compact then for each $t \in (0,1)$ there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F, with $\|g_n\| \leq 1$ and y_n converging to zero such that*

$$Tx = \sum_{l}^{\infty} g_n(x) y_n \quad (x \in E)$$

and $\mathrm{co}\{y_n : n \in \mathbb{N}\}$ is contained in a completing subset of F.

(b) *If for each $t \in (0,1)$ there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in F, with $\|g_n\| \leq 1$ and y_n converging to zero such that*

$$Tx = \sum_{l}^{\infty} g_n(x) y_n \quad (x \in E)$$

and $\overline{\text{co}}\{y_n : n \in \mathbb{N}\}$ *is completing, then* T *is semi-compact.*

PROOF.

(a) Let $\beta \in \mathbb{K} - \{0\}$ with $|\beta| < 1$. By 2.3 there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in $\beta^{-1}T(B_E)$, with $\|g_n\| \le 1$ and y_n converging to zero such that

$$Tx = \sum_{I}^{\infty} g_n(x) y_n \quad (x \in E).$$

Also, there exists a completing compactoid D in F such that $T(B_E) \subset D$. Hence $\text{co}\{y_n : n \in \mathbb{N}\}$ is contained in the completing subset $\beta^{-1}D$.

(b) By hypothesis $T(B_E) \subset \overline{\text{co}}\{y_n : n \in \mathbb{N}\}$ is completing and compactoid, then T is semi-compact.

COROLLARY 2.7. *Let* E, F *be normed spaces and let* $T, T_1 \in L(E,F)$. *If* T *is compactoid and* $\|T_1 x\| \le \|Tx\|$ *for all* $x \in E$, *then* T_1 *is also compactoid.*

3 – Compact operators in locally convex spaces

In this paragraph E, F are locally convex spaces over $\mathbb{K}$. If p is a continuous seminorm on E, then for $f \in (E,p)'$ we define $\|f\|_p$ by

$$\|f\|_p := \inf\{M \ge 0 : |f(x)| \le Mp(x), \forall x \in E\}.$$

THEOREM 3.1. *Let* E, F *be locally convex over* $\mathbb{K}$. *For a* $T \in L(E,F)$ *the following are equivalent:*

(1) T *is compactoid.*

(2) *There exists a continuous seminorm* p *on* E *such that, for each continuous seminorm* q *on* F, *there is a sequence* (f_n) *in* $(E,p)'$ *such that* $\lim \|f_n\|_p = 0$ *and* $q(Tx) \le \sup_n |f_n(x)|$ *for all* $x \in E$.

PROOF.

(1) $\Rightarrow$ (2). Since T is compactoid, there exists a continuous seminorm p on E such that $T(V_p)$ is compactoid in F where

$$V_p = \{x \in E : p(x) \leq 1\}.$$

Since $T(V_p)$ is bounded, it follows that $q(Tx) = 0$ for each $x \in \operatorname{Ker} p$ and each continuous seminorm q on F. Set $E_p := E/\operatorname{Ker} p$ and $F_q := F/\operatorname{Ker} q$ normed by $\|[x]_p\| = p(x)$ and $\|[y]_q\| = q(y)$, respectively. Let

$$\pi_p : E \to E_p \quad , \quad \pi_q : F \to F_q$$

be the canonical surjections and let

$$\Psi = \Psi_{q,p} : E_p \to F_q, [x]_p \to [Tx]_q.$$

The set $\pi_p(V_p)$ is the closed unit ball in E_p. Since $\Psi(\pi_p(V_p)) = \pi_q(T(V_p))$, the mapping Ψ is compactoid. In view of Theorem 2.3, there exists a sequence (g_n) in E_p', with $\lim \|g_n\| = 0$, such that

$$\|\Psi(z)\| \leq \sup_n |g_n(z)| \quad (z \in E_p).$$

If $f_n = g_n \pi_p$, then $f_n \in (E,p)'$ and $\|f_n\|_p = \|g_n\|$. Moreover,

$$q(Tx) \leq \sup_n |f_n(x)| \quad (x \in E).$$

(2) $\Rightarrow$ (1). Let p be as in (2) and let $V = V_p$. In order to show that the set $A = T(V)$ is compactoid in F, it suffices to prove that $\pi_q(A)$ is compactoid in F_q for each continuous seminorm q on F. So, let q be given. Our assumption (2) implies that $q(Tx) = 0$ if $p(x) = 0$. Consider the mapping

$$\Psi = \Psi_{q,p} : E_p \to F_q, [x]_p \mapsto [Tx]_q.$$

Let (f_n) be a sequence in $(E,p)'$, with $\lim \|f_n\|_p = 0$, such that $q(Tx) \leq \sup_n |f_n(x)|$. If

$$g_n : E_p \to \mathbb{K}, [\mathrm{x}]_{\mathrm{p}} \mapsto \mathrm{f_n(x)},$$

and $\overline{\text{co}}\{y_n : n \in \mathbb{N}\}$ *is completing, then* T *is semi-compact.*

PROOF.

(a) Let $\beta \in \mathbb{K} - \{0\}$ with $|\beta| < 1$. By 2.3 there exists a sequence (g_n) in E' and a t-orthogonal sequence (y_n) in $\beta^{-1}T(B_E)$, with $\|g_n\| \leq 1$ and y_n converging to zero such that

$$Tx = \sum_{1}^{\infty} g_n(x) y_n \quad (x \in E).$$

Also, there exists a completing compactoid D in F such that $T(B_E) \subset D$. Hence $\text{co}\{y_n : n \in \mathbb{N}\}$ is contained in the completing subset $\beta^{-1}D$.

(b) By hypothesis $T(B_E) \subset \overline{\text{co}}\{y_n : n \in \mathbb{N}\}$ is completing and compactoid, then T is semi-compact.

COROLLARY 2.7. *Let* E, F *be normed spaces and let* $T, T_1 \in L(E, F)$. *If* T *is compactoid and* $\|T_1 x\| \leq \|Tx\|$ *for all* $x \in E$, *then* T_1 *is also compactoid.*

3 – Compact operators in locally convex spaces

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THEOREM 3.1. *Let* E, F *be locally convex over* $\mathbb{K}$. *For a* $T \in L(E, F)$ *the following are equivalent:*

(1) T *is compactoid.*

(2) *There exists a continuous seminorm* p *on* E *such that, for each continuous seminorm* q *on* F, *there is a sequence* (f_n) *in* $(E, p)'$ *such that* $\lim \|f_n\|_p = 0$ *and* $q(Tx) \leq \sup_n |f_n(x)|$ *for all* $x \in E$.

PROOF.

(1) $\Rightarrow$ (2). Since T is compactoid, there exists a continuous seminorm p on E such that $T(V_p)$ is compactoid in F where

$$V_p = \{x \in E : p(x) \leq 1\}.$$

Since $T(V_p)$ is bounded, it follows that $q(Tx) = 0$ for each $x \in \operatorname{Ker} p$ and each continuous seminorm q on F. Set $E_p := E/\operatorname{Ker} p$ and $F_q := F/\operatorname{Ker} q$ normed by $\|[x]_p\| = p(x)$ and $\|[y]_q\| = q(y)$, respectively. Let

$$\pi_p : E \to E_p \quad , \quad \pi_q : F \to F_q$$

be the canonical surjections and let

$$\Psi = \Psi_{q,p} : E_p \to F_q, [x]_p \to [Tx]_q.$$

The set $\pi_p(V_p)$ is the closed unit ball in E_p. Since $\Psi(\pi_p(V_p)) = \pi_q(T(V_p))$, the mapping Ψ is compactoid. In view of Theorem 2.3, there exists a sequence (g_n) in E_p', with $\lim \|g_n\| = 0$, such that

$$\|\Psi(z)\| \leq \sup_n |g_n(z)| \quad (z \in E_p).$$

If $f_n = g_n \pi_p$, then $f_n \in (E,p)'$ and $\|f_n\|_p = \|g_n\|$. Moreover,

$$q(Tx) \leq \sup_n |f_n(x)| \quad (x \in E).$$

(2) $\Rightarrow$ (1). Let p be as in (2) and let $V = V_p$. In order to show that the set $A = T(V)$ is compactoid in F, it suffices to prove that $\pi_q(A)$ is compactoid in F_q for each continuous seminorm q on F. So, let q be given. Our assumption (2) implies that $q(Tx) = 0$ if $p(x) = 0$. Consider the mapping

$$\Psi = \Psi_{q,p} : E_p \to F_q, [x]_p \mapsto [Tx]_q.$$

Let (f_n) be a sequence in $(E,p)'$, with $\lim \|f_n\|_p = 0$, such that $q(Tx) \leq \sup_n |f_n(x)|$. If

$$g_n : E_p \to \mathbb{K}, [\mathrm{x}]_\mathrm{p} \mapsto \mathrm{f}_\mathrm{n}(\mathrm{x}),$$

then g_n is well defined and $\lim \|g_n\| = \lim \|f_n\|_p = 0$. Moreover

$$\|\Psi([x]_p)\| = q(Tx) \leq \sup_n |f_n(x)| = \sup_n |g_n([x]_p)| .$$

By Theorem 2.3, it follows that Ψ is compactoid and so $\pi_q(A) = \Psi(\pi_p(V_p))$ is compactoid. This completes the proof.

As an application, we next give a different proof of a well known result of N. de Grande-de Kimpe [2, Theorem 4.5 and Corollary 4.6 i)]. Notice that we have removed the hypothesis of polarity on the space E.

THEOREM 3.2. *A locally convex space E is nuclear if and only if for each continuous seminorm p on E there exist an equicontinuous sequence (g_n) in E' and an element (λ_n) of c_0 such that $p(x) \leq \sup_n |\lambda_n g_n(x)|$ for all $x \in E$.*

PROOF. First suppose E is nuclear. Given a continuous seminorm p, there exists another continuous seminorm $q \geq p$ such that the canonical mapping $\varphi_{p,q} : E_q \to E_p$ is compactoid. The map

$$T = \varphi_{p,q}\pi_q : E \to E_p$$

is compactoid and thus (by Theorem 3.1) there exists an equicontinuous sequence (g_n) in E' and $(\lambda_n) \in c_0$ such that

$$p(x) \leq \sup |\lambda_n g_n(x)| \quad (x \in E) .$$

Conversely assume that the condition is satisfied. Given a continuous seminorm p, let (g_n) be an equicontinuous sequence in E' and $(\lambda_n) \in c_0$ such that $p(x) \leq \sup_n |\lambda_n g_n(x)|$. By Theorem 3.1, the canonical surjection $\pi_p : E \to E_p$ is compactoid. Let $q \geq p$ be a continuous seminorm on E such that $\pi_p(V_q)$ is compactoid in E_p. Since $\pi_q(V_q)$ is the closed unit ball in E_q and since $\varphi_{p,q}(\pi_q(V_q)) = \pi_p(V_q)$, it follows that $\varphi_{p,q}$ is compactoid, which proves that E is nuclear.

THEOREM 3.3. *Let E be locally convex. If there exists $(\delta_n) \in \Delta(E)$ with $\inf \delta_n = 0$, then E is nuclear.*

PROOF. Let W be a convex neighborhood of zero in E and let V be another one with $V \prec W$ and $\delta_n(V,W) \leq \delta_n$ for all n. Let p,q be the Minkowski functionals of W and V respectively. We will show that the canonical mapping

$$\varphi = \varphi_{p,q} : E_q \to E_p$$

is compactoid. In fact, the set $A = \pi_q(V)$ is a neighborhood of zero in E_q and $\varphi(A) = \pi_p(V)$. By [6, Lemma 4.2], we have

$$\delta_n(\varphi(A)) = \delta_n(V,W) \leq \delta_n.$$

Since $\delta_n(V,W) \leq \delta_k(V,W)$ for $k \leq n$, we have $\lim \delta_n(\varphi(A)) = 0$ and so $\varphi(A)$ is compactoid by [6, Theorem 3.3], which completes the proof.

THEOREM 3.4. *Let E, F be locally convex spaces, where T is metrizable, and let $T \in L(E,F)$. Then, T is compactoid if and only if there are normed spaces X, Y and $T_1 \in L(E,X)$, $\overline{T} \in CO(X,Y)$, $T_2 \in L(Y,F)$ such that $T - T_2\overline{T}T_1$.*

PROOF. The sufficiency is clear. In order to prove the necessity, let p be a continuous seminorm on E such that $T(V_p)$ is compactoid in F, where $V_p := \{x \in E : p(x) \leq 1\}$. Since $A = T(V_p)$ is bounded, there exists a bounded absolutely convex subset B of F containing A such that F and F_B induce the same topology on A (by [1], Lemma 11]). Since A is absolutely convex and compactoid in F, it is also compactoid in F_B by [5, Theorem 4.3]. Now, take $X = E_p$, $Y = F_B$, $T_1 = \pi_p$, $T_2 : F_B \to F$ the injection map and

$$\overline{T} : E_p \to F_B, [x]_p \mapsto Tx.$$

REMARK 3.5 Under the hypothesis of 3.4, T is compactoid if and only if there are a normed space X and $T_1 \in L(E,X)$, $T_2 \in CO(X,F)$ such that $T = T_2T_1$.

Notice that the following related result has been recently proved in [4, Prop. 10] under no additional hypothesis over F : T is semicompact if and only if there are a Banach space X and $T_1 \in L(E,X)$, $T_2 \in SC(X,F)$ such that $T = T_2T_1$. Another related result is proposition 12 in [1].

THEOREM 3.6. *Let $T \in L(E,F)$ where E, F are locally convex spaces and F is metrizable. Then, T is compactoid if and only if there exist an equicontinuous sequence (f_n) in E', an element (λ_n) of c_0 and a bounded absolutely convex subset B of F such that $T(E) \subset F_B$ and*

$$p_B(Tx) \leq \sup_n |\lambda_n f_n(x)| \quad (x \in E)$$

where p_B is the Minkowski functional of B in F_B.

PROOF. First assume T to be compactoid. Let p be a continuous seminorm on E such that $T(V_p)$ is compactoid. As in the proof of the preceding theorem, there exists a bounded absolutely convex set $B \supset T(V_p)$ such that $T(V_p)$ is compactoid in F_B and so $T : E \to F_B$ is compactoid. Now the conclusion follows from Theorem 2.3.

Conversely, assume that T satisfies the condition in the statement of the theorem. By Theorem 3.1 the mapping

$$T_1 : E \longrightarrow F_B \quad , \quad T_1 x = Tx$$

is compactoid and so $T = T_2 T_1$ is compactoid, where $T_2 : F_B \to F$ is the injection map.

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