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On the maximum size of a maximal partial plane

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RIASSUNTO – *Un piano parziale di ordine n è costituito da un insieme di* n^2+n+1 *punti e da una famiglia di sottoinsiemi di cardinalità* $n+1$, *detti rette, tali che due rette qualsivoglia abbiano al più un punto in comune. Per* $n \geq 15$ *si dimostra che un piano parziale di ordine n o è immergibile in un piano proiettivo di ordine n, oppure ha al più* n^2+1 *rette e questa diseguaglianza è ottimale nel caso in cui n sia l'ordine di un piano proiettivo.*

ABSTRACT – *A partial plane of order n consists of a set of* n^2+n+1 *points and a family of subsets of size* $n+1$, *called lines, such that any two lines meet in at most one point. For* $n \geq 15$, *it is proved that a partial plane of order n can either be extended to a projective plane of order n or has at most* n^2+1 *lines, and this inequality is sharp whenever n is the order of a projective plane.*

KEY WORDS – *Partial plane - Projective plane.*

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1 – Introduction

A *partial plane* Γ of order n consists of a set of n^2+n+1 *points* and a family B of b sets of $n+1$ points, called *lines*, such that any two lines have at most one point in common. The partial plane Γ is called *maximal* if there is no partial plane containing Γ properly. The number of lines is denoted by b.

In this paper the possible number of lines in a maximal partial plane of order n is investigated. Z. FÜREDI and L. SPISSICH [3] showed that a maximal partial plane has at least $(3n + 4)/2$ lines if n is even and at least $(3n + 3)/2$ lines if n is odd and this bound is sharp. On the other hand, any line of a partial plane of order n covers $(n + 1)n$ of the $(n^2 + n + 1)(n^2 + n)$ pairs of points so that the number of lines is at most $n^2 + n + 1$ (here equality holds only if and only if any two points are joined by a line, which implies that every point lies on $n + 1$ lines and in turn that any two lines meet so that the partial plane is a projective plane). Using a theorem of S. VANSTONE [6], it follows that any partial plane with mutually intersecting lines can be extended to a projective plane provided that it has at least n^2 lines. Hence if Γ is a maximal partial plane with $n^2 \leq b < n^2 + n + 1$ then Γ has a pair of disjoint lines. D.R. STINSON [5] showed that every such partial plane has $b \leq n^2 + n/2$ lines, and Z. FÜREDI and J. KAHN [2] proved $b \leq n^2 + n/3$. In this paper, we improve these bounds.

THEOREM. *If Γ is a partial plane of order $n \geq 15$ with two disjoint lines then $b \leq n^2 + 1$. Furthermore, this bound is sharp whenever n is the order of a projective plane.*

The following construction due to D.R. STINSON [5] shows that this bound is sharp whenever n is the order of a projective plane.

Let p and q be two points of a projective plane $\mathbf{P}$ of order n, and let L be a line other than pq passing through p. Remove the line L and all lines other than pq passing through q to obtain a partial plane Γ'. Then the set S consisting of q and the n points other than p of L meets every line of Γ' in at most one point. Hence we obtain a partial plane Γ if we adjoin S as a new line to Γ'. In Γ the line S is disjoint to all lines other than pq through p. Furthermore Γ has $n^2 + 1$ lines.

Using the result of S. Vanstone mentioned above, we obtain

COROLLARY. *If n is not the order of a projective plane then every partial plane has at most $n^2 + 1$ lines.*

It should be mentioned that the examples of a partial plane with $n^2 + 1$ lines and two disjoint lines have been constructed starting with a projective plane of order n. This might indicate that the bound in the corollary is far from being best possible.

2 – Proof of the Theorem

The theorem will be proved indirectly. Suppose for the rest of the paper that Γ is a partial plane of order n with at least n^2+2 lines and a pair of disjoint lines. We may assume that Γ is a counter-example to the theorem with the maximal possible number of lines (where n is fixed).

We use the following notation. The number of lines is denoted by $b = n^2+n+1-\alpha$ for some integer α, which satisfies $\alpha \leq n-1$. The set of lines is denoted by B. The *degree* r_p of a point p is the number of lines through p and $d_p\colon = n+1-r_p$ is called the *deficiency* of p. By $p^\perp$ we denote the set consisting of p and all points q which are not joined to p. Since p is joined to $r_p \cdot n$ points, we have $|p^\perp| = 1 + d_p \cdot n$. If p is a point of degree n, then the set $p^\perp$ is called a *projective set*. Finally, a line is called *bad* if it meets some projective set in at least two points.

We shall show that distinct projective sets have a unique point in common, and that there exist more projective sets than bad lines. Then we remove the bad lines and adjoin the projective sets as new lines to obtain a new partial plane Γ', which has more lines than Γ. The maximality of Γ implies then that Γ' can be extended to a projective plane, and this information will be sufficient to obtain a contradiction.

LEMMA 1. *If p is a point outside a line L then $|L \cap p^\perp| \geq d_p$.*

PROOF. This is clear, since p is joined to at most r_p points of L.

LEMMA 2. *If two distinct points p and q are joined then $|p^\perp \cap q^\perp| \leq d_p \cdot d_q$.*

PROOF. By Lemma 1, every line other than pq through p meets $q^\perp$ in at least d_q points. Hence p is joined to at least $1+(r_p-1)d_q$ of the $1+d_q \cdot n$ points of $q^\perp$, which implies that $|p^\perp \cap q^\perp| \leq d_q \cdot n - (r_p-1)d_q = d_p \cdot d_q$.

LEMMA 3. *If a point p of degree n is not joined to a point q then $|p^\perp \cap q^\perp| \geq n+2-d_q$.*

PROOF. Set $x = |p^\perp \cap q^\perp|$. If $p^\perp \subseteq q^\perp$, then $x = |p^\perp| = n+1 \geq n+2-d_q$ as $d_q \geq 1$. We may therefore assume that $p^\perp$ is not contained in $q^\perp$. Choose a point s of maximum degree of $p^\perp - q^\perp$, set $d = d_q$ and $e = d_s$. The lines $\neq sq$ through s join s to at least $(r_s - 1)d$ points of $q^\perp$ (Lemma 1) and therefore to at least $(r_s-1)\cdot d - |q^\perp - p^\perp| = (n-e)d - |q^\perp| + |q^\perp \cap p^\perp| = (n-e)d-(1+dn)+x = x-1-de$ points of $p^\perp \cap q^\perp$. Since q is joined to $|p^\perp - q^\perp| = n+1-x$ points of $p^\perp$ and since every line L meets $p^\perp$ in at least one point (Lemma 1), it follows that $\Sigma_L(|L \cap p^\perp| - 1) \geq x-1-de+n+1-x = n-de$. Now count incident point line pairs (y, L) with y in $p^\perp$ to obtain

$$\sum_{y\in p^\perp} r_y = \sum_{L\in B} |L\cap p^\perp| = b+\sum_{L\in B}(|L\cap p^\perp|-1) \geq b+n-de \geq n^2+2+n-de\,.$$

On the other hand (notice that every point of $p^\perp - \{p\}$ has degree at most n, since it is not joined to p, while the $n+1-x$ points of $p^\perp - q^\perp$ have degree at most $n+1-e$)

$$\sum_{y\in p^\perp} r_y \leq r_q + |(p^\perp \cap q^\perp) - \{q\}|\cdot n + |p^\perp - q^\perp|\cdot(n+1-e)$$

$$= (n+1-d)+(x-1)n+(n+1-x)(n+1-e)$$

$$= n^2+n+1-d-(n+1-x)(e-1)\,.$$

Hence $1+(n+1-x)(e-1) \leq d(e-1)$, in particular $e \neq 1$ and therefore $n+1-x < d$, i. e. $x > n+1-d$.

LEMMA 4. *If points p and q of degree n are not joined then $p^\perp = q^\perp$.*

PROOF. This is immediate from the preceding lemma.

LEMMA 5. *Distinct projective sets have a unique point in common.*

PROOF. Let $p^\perp$ and $q^\perp$ be distinct projective sets. Then p and q are joined (Lemma 4) and $p^\perp$ and $q^\perp$ have at most one point in common (Lemma 2). Assume by way of contradiction that $p^\perp$ and $q^\perp$ are disjoint, and denote by x resp. y the number of n-points in $p^\perp$ resp. $q^\perp$. We may assume that $x \geq y$.

Suppose that s is a point of degree n of $p^\perp$. Since $s^\perp = p^\perp$ (Lemma 4), the point s is joined to every point of $q^\perp$. Hence s lies on a line which has at least two points in $q^\perp$. If s and s' are distinct points of degree n of $p^\perp$ then $s^\perp = p^\perp = s'^\perp$ implies that s and s' are not on a common line. It follows that there are at least x different lines which have two points in $q^\perp$. Counting incident point line pairs (z, Z) with z in $q^\perp$ and using Lemma 1, we obtain

$$\sum_{z \in q^\perp} r_z = \sum_{Z \in B} |Z \cap q^\perp| \geq 2x + (b - x) \geq n^2 + 2 + x\,.$$

On the other hand

$$\sum_{z \in q^\perp} r_z \leq yn + (n + 1 - y)(n - 1) = n^2 - 1 + y\,.$$

Hence $x < y$, a contradiction.

LEMMA 6. *Every point of degree $\leq n$ lies in a projective set.*

PROOF. Let p be a point of degree $n+1-d < n+1$. Then every point of $p^\perp - \{p\}$ has degree at most n, since a point of degree $n+1$ is joined to every other point. Furthermore, every line not passing through p meets $p^\perp - \{p\}$ in at least d points (Lemma 1). Counting incident point-line pairs (q, L) with $q \in p^\perp - \{p\}$, we obtain

$$\sum_{q \in p^\perp - \{p\}} r_q \geq (b - r_p)d \geq (b - n)d \geq (n^2 - n + 2)d > dn(n - 1)\,.$$

Since $|p^\perp - \{p\}| = dn$, it follows that some point of $p^\perp - \{p\}$ has degree at least n.

LEMMA 7. *The sum $\sum d_p$ of the deficiencies d_p of all points p equals $\alpha(n + 1)$.*

PROOF. Since there are $n^2 + n + 1$ points, this follows from $\sum r_p = b(n + 1)$ and $b = n^2 + n + 1 - \alpha$.

LEMMA 8. *There are at least $\alpha/2$ different projective sets.*

PROOF. In view of $\alpha \leq n-1$, the preceding lemma shows that it exists a point p of degree $n+1$. Let L be a line through p, let $p_1, \dots, p_s$ be the points of L which have not degree $n+1$, and denote the deficiency of p_j by d_j. Since p has degree $n+1$, it is joined to every other point, which implies that the lines through p meet every other line. Hence, the line L meets every other line, which implies that $\sum d_j = \alpha$. The set $p_j^\perp$ has $1 + d_j n$ points and distinct sets $p_j^\perp$ and $p_k^\perp$ have at most $d_j d_k$ points in common (Lemma 2). Since every point contained in one of the sets $p_j^\perp$ has degree at most n, it follows that the number of points of degree $\leq n$ is at least

$$\sum_{j=1}^{s}(1+d_j n)-\sum_{j<k} d_j d_k = s+\alpha n-\frac{1}{2}\left(\alpha^2-\sum_{j=1}^{s} d_j^2\right) > 1+\alpha n-\frac{1}{2}\alpha^2 > 1+\frac{1}{2}\alpha n .$$

Suppose there are fewer than $\frac{1}{2}\alpha$ distinct projective sets, say $q_1^\perp, \dots, q_m^\perp$ with $m < \frac{1}{2}\alpha$. By Lemma 6, the union of the projective sets is the set of points of degree at most n. Now (see Lemma 5)

$$\Big|\bigcup_{j=1}^{m} q_j^\perp\Big| \leq |q_1^\perp| + |q_2^\perp - q_1^\perp| + \dots + |q_m^\perp - q_1^\perp|$$

$$= n + 1 + (m-1)n = 1 + mn < 1 + \frac{1}{2}\alpha n ,$$

But every point of degree $\leq n$ lies in one of the projective sets $q_j^\perp$, a contradiction.

LEMMA 9. *It exists a bad line.*

PROOF. Since Γ is a counterexample to our theorem with the maximum number of lines, every projective set S contains two points which are joined (if not we could adjoin S as a new line), i.e. every projective set meets some bad line in two points. Hence if $\alpha > 0$ then there is a bad line (Lemma 8). But we have already noticed in the introduction that every partial plane has at most $n^2 + n + 1$ lines with equality if and only if it is a projective plane. Since Γ has disjoint lines, it is certainly not a projective plane, so $\alpha = n^2 + n + 1 - b > 0$.

LEMMA 10. *Every point of a bad line has degree at most* $n-1$.

PROOF. Suppose L is a bad line which intersects the projective set $p^\perp$ in two points. By Lemma 4, the points of $L \cap p^\perp$ have degree at most $n-1$. Now let q be a point of $L - p^\perp$. Then q has not degree $n+1$, because every line through a point of degree $n+1$ intersects $p^\perp$ (Lemma 1) in a (necessarily) unique point.

Assume that q has degree n. Since every line through q meets $p^\perp$ and since L meets $p^\perp$ in two points, it follows that q is joined to each of the $n+1$ points of $p^\perp$. But this implies that the projective sets $p^\perp$ and $q^\perp$ are disjoint, which contradicts Lemma 5. □

LEMMA 11. *It two points* x *and* y *of a projective set* $p^\perp$ *are joined then* $(d_x+1)(d_y+1) \geq n+4$.

PROOF. By Lemma 3 we have $|p^\perp \cap x^\perp| \geq n+2-d_x$ and $|p^\perp \cap y^\perp| \geq n+2-d_y$ and Lemma 2 shows that $|x^\perp \cap y^\perp| \leq d_x d_y$. It follows that

$$2n+4-d_x-d_y \leq |p^\perp \cap x^\perp| + |p^\perp \cap y^\perp| \leq |p^\perp| + |p^\perp \cap x^\perp \cap y^\perp| \leq n+1+d_x d_y .$$

Consequently $(d_x+1)(d_y+1) \geq n+4$.

LEMMA 12. *If* $n \geq 3$, *then there are at most* $n-2$ *bad lines.*

PROOF. Let M be any set of $n+1-\beta$ bad lines with some integer $\beta \geq 0$. Since distinct lines meet in at most one point, the lines of M cover at least $\frac{1}{2}(n+2)(n+1) - \frac{1}{2}(\beta+1)\beta$ points. Since every point of a bad line has deficiency at least 2 (Lemma 10), Lemma 7 implies that $(n+2)(n+1) - (\beta+1)\beta \leq \alpha(n+1)$. In view of $\alpha \leq n-1$, it follows that $\beta(\beta+1) \geq 3(n+1)$ so that $\beta \geq 3$. □

LEMMA 13. *If* $n \geq 13$ *then there are more projective sets than bad lines.*

PROOF. Let $n+1-\beta$ be the number of bad lines and assume that there are at most $n+1-\beta$ projective sets. Let S be the set of points which have degree at most $n-3$, and set $s=|S|$.

It L is a bad line then every point of L lies in a projective set (Lemma 6 and Lemma 10). In view of $n \geq 13$, Lemma 11 shows that a projective set contains at most one point of $L-S$. Since there are at most $n+1-\beta$ projective sets, it follows that every bad line contains at least β points of S. Let w be the number of points covered by all bad lines. As in the proof of Lemma 12 we have $2w \geq (n+2)(n+1)-(\beta+1)\beta$. Since points of bad lines have deficiency at least 2 (Lemma 10) and since the points of S have deficiency at least 4, we have $2w+2s=(w-s)\cdot 2+s\cdot 4 \leq \alpha(n+1)$ by Lemma 7. Hence $2s \leq \beta(\beta+1)-(n+2-\alpha)(n+1)$.

Consider first the case that $n+1-\beta \geq \beta$ so that there are at least β bad lines. Then $s \geq \frac{1}{2}(\beta+1)\beta$, since a bad line contains at least β points of S. But we have just shown that $2s < \beta(\beta+1)$, a contradiction.

Now suppose that $\beta \geq n+1-\beta$. Since each of the $n+1-\beta$ bad lines contains at least β points of S, if follows that $s \geq \beta+(\beta-1)+\ldots+[\beta-(n-\beta)]$ so that $2s \geq (\beta+1)\beta-(2\beta-n)(2\beta-n-1)$. Together with the upper bound for $2s$ we obtain $(n+2-\alpha)(n+1) \leq (2\beta-n)(2\beta-n-1)$ or $2(n+1)+2\beta(2n+1-2\beta) \leq \alpha(n+1)$. In view of $2\beta \geq n+1$, it follows that $2(n+1)+(n+1)(2n+1-2\beta) \leq \alpha(n+1)$ which implies that $1+2(n+1-\beta) \leq \alpha$ or $n+1-\beta < \frac{1}{2}\alpha$. Since there are at most $n+1-\beta$ projective sets, this contradicts Lemma 8. □

Let Γ' be the partial plane obtained from Γ by removing the bad lines. Then each line of Γ' meets each projective set in at most one point. Lemma 5 shows that we obtain again a partial plane Γ'' if we add the projective sets as new lines to Γ'. By the preceding lemma, Γ'' has more lines than Γ. The choice of Γ implies now that Γ'' can be extended to a projective plane $\mathbf{P}$ of order n. Since Γ has at least n^2+2 lines, it follows that Γ' can be extended to the same projective plane by adjoining at most $n-1+\delta$ lines where δ is the number of bad lines of Γ. By Lemma 12, we have $\delta \leq n-2$. Since the points of degree $n+1$ of Γ have still degree $n+1$ in Γ' (Lemma 10), the partial plane Γ' has at least $n^2+n+1-\alpha(n+1) \geq n+2$ points of degree $n+1$ (Lemma 7). Let $S_1,\ldots,S_\delta$ be the bad lines of Γ and denote by D the set of lines of $\mathbf{P}$ which are not lines of Γ'. Since D contains the projective sets of Γ, each line S_j meets some line of D in at least two points. Therefore each bad line S_j of Γ is a set of $n+1$ non-collinear points of the projective plane $\mathbf{P}$. The following proposition shows that we have derived a situation which is not possible and this completes the proof of the theorem.

PROPOSITION 14. *Suppose* $\mathbf{P}$ *is a projective plane of order* $n \geq 15$ *and let* δ *be a positive integer. Suppose furthermore that there exists a set* D *consisting of (at most)* $n-1+\delta$ *lines and sets* $S_1, \ldots, S_\delta$ *of points satisfying the following properties.*

(1) *Each set* S_j *consists of* $n+1$ *non-collinear points.*

(2) *Any two points of* S_j *are joined by a line of* D.

(3) *Distinct sets* S_j *share at most one point.*

(4) *At least* $n+2$ *points do not lie on any line of* D.

Then $\delta \geq n-1$.

PROOF. Assume by way of contradiction that $\delta \leq n-2$. For every point p, we denote by d_p the number of lines of D passing through p, and we call d_p the deficiency of p.

Since the points of S_1 are non-collinear, the lines (of D) which have at least two points in S_1 induce a linear space $\mathbf{L}$ on the points of S_1. Since every linear space has at least as many lines as points (see [1]), it follows that D has at least $n+1$ lines. Hence $\delta \geq 2$.

We define a function $f(x) := (x+1)^2(n-x)$ for real values of x. Since f has two extrema one of which for $x=-1$, it follows that $f(x_1) \leq f(x) \leq f(x_2)$ for all real x, x_1, x_2 with $0 \leq x_1 \leq x \leq x_2$. Denote by $k+1$ the maximum line degree of $\mathbf{L}$. Then $k \leq n-1$ and $\mathbf{L}$ has at least $1 + (f(k)/n)$ lines (see [4]). Since the lines of $\mathbf{L}$ are induced by the lines of D, it follows that $f(k) \leq (|D|-1) \cdot n = (n-2+\delta)n$. Since $\delta \leq n-2$, $f(\sqrt{(2n)}) \geq 2n^2$ and $f(n-2) \geq 2n^2$, it follows from the above properties of f that $k = n-1$ or $k < \sqrt{(2n)}$. However $k = n-1$ is not possible. (Assume to the contrary that there is a line L in D which meets S_1 in n points. Then the point of $S_1 - L$ is joined to the n points of $L \cap S_1$ by lines of D, which implies that the lines of D cover at least n^2+1 points. This contradicts condition (4)). Hence $k < \sqrt{(2n)}$. If $\delta \leq 3$ then $f(k) \leq (n+1)n$, and the same argument shows in this case that $k < \sqrt{n}$.

Let p be a point of $\mathbf{L}$. Since p is joined to every other point of $\mathbf{L}$ by a line of degree at most $k+1$, it follows that p has degree at least n/k

in $\mathbf{L}$. Hence every point of $\mathbf{L}$ lies on at least n/k lines of D, i.e. every point of S_1 has deficiency $d_p \geq n/k$. The same argument shows that every point contained in one of the sets S_j has deficiency at least n/k. Let S be the union of the sets $S_1, \ldots, S_\delta$. Since any two distinct sets S_j share at most one point, we have $|S| \geq (n+1) + n + \ldots + (n+2-\delta) = \frac{1}{2}(n+2)(n+1) - \frac{1}{2}(n+2-\delta)(n+1-\delta) = \frac{1}{2}\delta(2n+3-\delta)$. Counting in two ways triples (p, L_1, L_2) consisting of a point p and distinct lines L_1, L_2 of D which intersect in p, we obtain $\Sigma d_p(d_p - 1) = |D|(|D| - 1) \leq (n-1+\delta)(n-2+\delta)$. Since the points of S have deficiency at least n/k it follows that

$$(1) \qquad \delta(2n+3-\delta)n(n-k) \leq 2(n-1+\delta)(n-2+\delta)k^2 .$$

First suppose that $\delta \in \{2, 3\}$. Then $k < \sqrt{n}$ and (1) yields

$$(2) \qquad \delta(2n+3-\delta)(n-\sqrt{n}) < 2(n-1+\delta)(n-2+\delta) .$$

In view of $n \geq 15$ and $\delta \in \{2, 3\}$, this is a contradiction.
Hence $4 \leq \delta \leq n-2$. Now $k < \sqrt{(2n)}$ and (1) yields

$$(3) \qquad g(\delta) := \frac{\delta(2n+3-\delta)}{(n-1+\delta)(n-2+\delta)} < \frac{2k^2}{n(n-k)} .$$

Restrict g to values $\delta \geq 0$. Then the function $g'(\delta)(n-1+\delta)^2(n-2+\delta)^2$ is a polynomial of degree 2 in δ (where g' is the derivation of g). Hence g has at most two extrema. Since $g(0) = g(2n-3) = 0$ and since $g(\delta)$ is positive for $0 < \delta < 2n-3$, it follows that g has a unique extremum for $0 < \delta < 2n-3$. Hence $\min\{g(\delta_1), g(\delta_2)\} \leq g(\delta)$ for all real $\delta, \delta_1, \delta_2$ with $0 \leq \delta_1 \leq \delta \leq \delta_2$. Since $4 \leq \delta \leq n-2$, it follows that (3) is satisfied for $\delta = 4$ or $\delta = n-2$. In view of $k < \sqrt{(2n)}$, the right hand side of (3) is smaller than $4/(n - \sqrt{(2n)})$. It follows easily in the case $\delta = 4$ as well as in the case $\delta = n-2$ that $n \leq 18$. Now $k < \sqrt{(2n)}$ implies that $k \leq 5$. But then (3) is also not satisfied for $n \in \{15, 16, 17, 18\}$. This contradiction completes the proof of the proposition. □

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