Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 357-364

Approximation of continuous linear functionals in real normed spaces

S.S. DRAGOMIR

RIASSUNTO – *Vengono dati alcuni teoremi di approssimazione per i funzionali lineari continui su spazi reali normati in termini di derivate della norma.*

ABSTRACT – *Some approximation theorems for the continuous linear functionals on real normed spaces in terms of norm derivatives are given*

KEY WORDS – *Continuous linear functionals - Norm derivatives - Reflexivity - James' theorem - Bishop-Phelps' theorem.*

A.M.S. CLASSIFICATION: 46B20

1 – Introduction

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives (see [2] or [5]):

$$(x,y)_{i(s)} := \lim_{t\to 0-(+)} (\|y+tx\|^2 - \|y\|^2)/2t \qquad \text{for all} \quad x,y \quad \text{in} \quad X\,.$$

For the sake of completeness we list some usual properties of these mappings that will be used in the sequel [2]:

(i) $(x,x)_p = \|x\|^2$ for all x in X;

(ii) $(-x,y)_s = (x,-y)_s = -(x,y)_i$ if x,y are in X;

(iii) $(\alpha x, \beta y)_p = \alpha\beta(x,y)_p$ for all x, y in X and $\alpha\beta \geq 0$;

(iv) $(\alpha x + y, x)_p = \alpha(x,x)_p + (y,x)_p$ if x, y belong to X and α is in $\mathbb{R}$;

(v) the element x in X is Birkhoff orthogonal over y in X, i.e., $\|x + ty\| \geq \|x\|$ for all t in $\mathbb{R}$ iff $(y,x)_i \leq 0 \leq (y,x)_s$;

(vi) $(x + y, z)_p \leq \|x\| \, \|z\| + (y,z)_p$ for all x, y, z in X;

(vii) the space $(X, \|\cdot\|)$ is smooth iff $(y,x)_i = (y,x)_s$ for all x, y in X or iff $(\, , \,)_p$ is linear in the first variable;

where $p = s$ or $p = i$.

For other properties of $(\, , \,)_p$ in connection to best approximation element or continuous linear functionals see [2] where further references are given.

2 – A characterization of reflexivity

To recall some well-known theorems of reflexivity due to R.C. James we need the following concept: the nonzero element $u \in X$ is a maximal element for the functional $f \in X^*$ if $f(u) = \|f\| \, \|u\|$ [6, p. 35].

THEOREM 1. [3]. *Let X be a Banach space. Then X is reflexive iff every nonzero continuous linear functional on E has at least one maximal element in X.*

Another famous result of R.C. James is the following.

THEOREM 2. [4]. *Let X be a Banach space. X is reflexive iff for every closed and homogeneous hyperplane H in X (i.e., H contains the null element) there exists a point $u \in X \setminus \{0\}$ such that $u \perp_B H$.*

The following characterization of reflexivity in terms of norm derivatives also holds.

THEOREM 3. *Let X be a Banach space. X is reflexive if and only if for every continuous linear functional f on X there exists an element u in X such that the next evaluation holds:*

$$(x,u)_i \leq f(x) \leq (x,u)_s \qquad \textit{for all } \; x \; \textit{ in } \; X \tag{1}$$

and $\|f\| = \|u\|$.

PROOF. Let H be a closed and homogeneous hyperplane in X and $f: X \to \mathbb{R}$ be a continuous linear functional on X such that $H = \mathrm{Ker}(f)$. Then from (1) follows that $u \perp_B H$ and by Theorem 2 we conclude that X is reflexive.

Now, assume that X is reflexive and let f be a nonzero continuous linear functional on it. Since $\mathrm{Ker}(f)$ is a closed and homogeneous hyperplane in X then there exists, by Theorem 2, a nonzero element w_0 in X so that:

$$(x, w_0)_i \leq 0 \leq (x, w_0)_s \qquad \text{for all} \quad x \in \mathrm{Ker}(f). \tag{2}$$

Because $f(x)w_0 - f(w_0)x \in \mathrm{Ker}(f)$ for all x in X, from (2) we derive that:

$$\left(f(x)w_0 - f(w_0)x, w_0\right)_i \leq 0 \leq \left(f(x)w_0 - f(w_0)x, w_0\right)_s \tag{3}$$

for all x in X.

On the other hand, by the use of norm derivatives properties, we have

$$\left(f(x)w_0 - f(w_0)x, w_0\right)_p = f(x)\|w_0\|^2 - \left(x, f(w_0)w_0\right)_q, \qquad x \in X$$

where $p \neq q$, $p, q \in \{i, s\}$.

We conclude, by (3), that

$$\left(x, f(w_0)w_0/\|w_0\|^2\right)_i \leq f(x) \leq \left(x, f(w_0)w_0/\|w_0\|^2\right)_s, \qquad x \in X$$

from where results

$$(x, u)_i \leq f(x) \leq (x, u)_s \qquad \text{for all} \quad x \in X,$$

where $u := f(w_0)w_0/\|w_0\|^2$.

To prove the fact that $\|f\| = \|u\|$, we observe that:

$$-\|x\|\,\|u\| \leq -(x, -u)_s = (x, u)_i \leq f(x) \leq (x, u)_s \leq \|x\|\,\|u\|, \qquad x \in X$$

and

$$\|f\| \geq f(u)/\|u\| \geq (u, u)_i/\|u\| = \|u\|.$$

The theorem is thus proven.

REMARK 1. If u is an "interpolation" element satisfying the relation (1), then u is a maximal element for the functional f.

Indeed, we have $f(u) = \|u\|^2$ and since $\|u\| = \|f\|$ we obtain $f(u) = \|f\| \, \|u\|$.

REMARK 2. The above theorem is a natural generalization of Riesz's representation theorem which works in Hilbert spaces via a result of R.A. TAPIA [7] for smooth spaces which is embodied in the following corollary.

COROLLARY. *Let X be a real Banach space. Then the following statements are equivalent:*

(i) *X is reflexive and smooth;*

(ii) *for every continuous linear functional $f\colon X \to \mathbb{R}$ there exists an element u in X such that:*

$$f(x) = (x, u)_s \qquad \text{for all} \quad x \in X$$

and $\|f\| = \|u\|$.

Further on, we shall point out other approximations of continuous linear functionals on real normed spaces in terms of norm derivatives.

3 – Approximation of continuous linear functionals

Let $f \in X^*$ with $\|f\| = 1$ and let $k \geq 0$. Define [1, p. 1]:

$$K(f, k) := \{x \in X | \; \|x\| \leq k f(x)\};$$

$K(f, k)$ is a closed convex cone. If $k > 1$ then the interior of $K(f, k)$ is nonempty.

THEOREM 4. *Let X be a real normed space, $\varepsilon \in (0, 1)$, $f \in X^*$ with $\|f\| = 1$ and $u \in X$, $\|u\| = 1$ such that the norm derivative $(\cdot, u)_p$ ($p = s$ or $p = i$) is linear on X. If $k > 1 + 2/\varepsilon$ and $(x, u)_p \geq 0$ on $K(f, k)$ then we have the estimation:*

$$|f(x) - (x, u)_p| \leq \varepsilon \|x\| \qquad \text{for all} \quad x \quad \text{in} \quad X.$$

PROOF. The proof follows from Lemma 3, [1, p. 3] for the continuous linear functional $g\colon X \to \mathbb{R}$, $g(x) := (x,u)_p$ and we shall omit the details.

The following approximation theorem for the continuous linear functionals on a general normed linear space also holds.

THEOREM 5. *Let $f\colon X \to \mathbb{R}$ be a continuous linear functional such that for all $\delta \in (0,1)$ there exists a nonzero element $x_{f,\delta}$ in X with the property:*

$$\text{(A)} \qquad (x, x_{f,\delta})_i \le \delta \|x\|\, \|x_{f,\delta}\| \qquad \textit{for all} \quad x \quad \textit{in} \quad \mathrm{Ker}(f).$$

Then for all $\varepsilon > 0$ there exists a nonzero element $u_{f,\varepsilon}$ in X so that the following estimation holds:

$$(4) \qquad -\varepsilon\|x\| + (x, u_{f,\varepsilon})_i \le f(x) \le (x, u_{f,\varepsilon})_s + \varepsilon\|x\|$$

for all x in X.

PROOF. Since f is nonzero, it follows that $\mathrm{Ker}(f)$ is closed in X and $\mathrm{Ker}(f) \ne X$.

Let $\varepsilon > 0$ and put $\delta(\varepsilon) := \varepsilon/(2\|f\|)$. If $\delta(\varepsilon) \ge 1$, then there exists an element $x_{f,\delta(\varepsilon)}$ in $X \setminus \mathrm{Ker}(f)$ such that

$$(5) \qquad \left(y, x_{f,\delta(\varepsilon)}\right)_i \le \delta(\varepsilon)\|y\|\, \|x_{f,\delta(\varepsilon)}\| \qquad \text{for all} \quad x \quad \text{in} \quad \mathrm{Ker}(f).$$

If $0 < \delta(\varepsilon) < 1$ and since the functional f has the (A)-property, then there exists an element $x_{f,\delta(\varepsilon)}$ in $X \setminus \mathrm{Ker}(f)$ (the fact that $x_{f,\delta(\varepsilon)}$ is not in $\mathrm{Ker}(f)$ follows from (A)) such that (5) is valid too.

Put in all cases, $z_{f,\varepsilon} := x_{f,\delta(\varepsilon)}/\|x_{f,\delta(\varepsilon)}\|$. Then for all x in X we have $y := f(x)z_{f,\varepsilon} - f(z_{f,\varepsilon})x$ belongs to $\mathrm{Ker}(f)$ which implies, by (5), that:

$$\left(f(x)z_{f,\varepsilon} - f(z_{f,\varepsilon})x, z_{f,\varepsilon}\right)_i \le \delta(\varepsilon)\|f(x)z_{f,\varepsilon} - f(z_{f,\varepsilon})x\| \le$$

$$\le 2\delta(\varepsilon)\|f\|\, \|x\| \le \varepsilon\|x\|$$

for all x in X.

On the other hand, as above, we have:

$$\left(f(x)z_{f,\varepsilon} - f(z_{f,\varepsilon})x, z_{f,\varepsilon}\right)_i = f(x) - \left(x, f(z_{f,\varepsilon})z_{f,\varepsilon}\right)_s$$

for all x in X and denoting $u_{f,\varepsilon} := f(z_{f,\varepsilon}) \neq 0$, we obtain:

$$f(x) \leq (x, u_{f,\varepsilon})_s + \varepsilon\|x\| \qquad \text{for all} \quad x \quad \text{in} \quad X.$$

Now, if we replace x by $-x$ in the above estimation, we derive

$$f(x) \geq (x, u_{f,\varepsilon})_i - \varepsilon\|x\| \qquad \text{for all} \quad x \quad \text{in} \quad X$$

and the proof is finished.

COROLLARY. *Let X be a smooth normed space over the real number field and denote $[x,y] = (x,y)_i = (x,y)_s$, $x,y \in X$. If $f \in X^*$ is a nonzero functional such that for all $\delta \in (0,1)$ there exists an element $x_{f,\delta} \in X \setminus \{0\}$ with the property*

(A′) $$\left|[x, x_{f,\delta}]\right| \leq \delta\|x\| \, \|x_{f,\delta}\| \qquad \textit{for all} \quad x \in \mathrm{Ker}(f),$$

then for any $\varepsilon > 0$ there is an element $u_{f,\varepsilon} \in X \setminus \{0\}$ so that

(4′) $$\left|f(x) - [x, u_{f,\varepsilon}]\right| \leq \varepsilon\|x\| \qquad \textit{for all} \quad x \in X$$

The proof is obvious from the above theorem and to the fact that $[\ ,\ ]$ is linear in the first variable.

To give the main result of our paper, we need the famous theorem of Bishop-Phelps which says [1, p. 3]:

THEOREM 6. *Let C be a closed bounded convex set in the Banach space X, then the collection of functionals that achieve their maximum on C is dense in X^*.*

Now, we can state and prove our main result.

THEOREM 7. *Let X be a real Banach space. Then for every continuous linear functional $f\colon X \to \mathbb{R}$ and for any $\varepsilon > 0$ there exists an element $u_{f,\varepsilon}$ in X such that the estimation* (4) *holds.*

PROOF. By the use of Bishop-Phelps' theorem for $C = \overline{B}(0,1)$ it follows that the collection of functionals which achieve their norm on unit closed ball is dense in X^*, i.e., for every $f \in X^*$ and $\varepsilon > 0$ there exists a continuous linear functional f_ε on X which achieve their norm on $\overline{B}(0,1)$ and such that

$$(6) \qquad |f(x) - f_\varepsilon(x)| \le \varepsilon\|x\| \qquad \text{for all} \quad x \quad \text{in} \quad X\,.$$

Suppose $f_\varepsilon \neq 0$ and $f_\varepsilon(v_{f,\varepsilon}) = \|f_\varepsilon\|$ with $v_{f,\varepsilon} \in \overline{B}(0,1)$. Then

$$0 < \|v_{f,\varepsilon}\| \le 1 = f_\varepsilon(v_{f,\varepsilon})/\|f_\varepsilon\| = f_\varepsilon(v_{f,\varepsilon} + \lambda y)/\|f_\varepsilon\| \le \|v_{f,\varepsilon} + \lambda y\|$$

for all $\lambda \in \mathbb{R}$ and $y \in \operatorname{Ker}(f_\varepsilon)$, i.e., $v_{f,\varepsilon} \perp_B \operatorname{Ker}(f_\varepsilon)$.

By a similar argument as in Theorem 3 we get:

$$\left(x, f_\varepsilon(v_{f,\varepsilon})v_{f,\varepsilon}/\|v_{f,\varepsilon}\|^2\right)_i \le f_\varepsilon(x) \le \left(x, f_\varepsilon(v_{f,\varepsilon})v_{f,\varepsilon}/\|v_{f,\varepsilon}\|^2\right)_s$$

for all $x \in X$. Denoting $u_{f,\varepsilon} := f_\varepsilon(v_{f,\varepsilon})v_{f,\varepsilon}/\|v_{f,\varepsilon}\|^2$ we obtain

$$(7) \qquad (x, u_{f,\varepsilon})_i \le f_\varepsilon(x) \le (x, u_{f,\varepsilon})_s \qquad \text{for all} \quad x \in X\,.$$

If $f_\varepsilon = 0$ then (4) holds with $u_{f,\varepsilon} = 0$.

Now, we observe that the relations (6) and (7) give the desired evaluation and the proof is finished.

COROLLARY. *Let X be a smooth Banach space. Then for every $f \in X^*$ and for any $\varepsilon > 0$ there exists an element $u_{f,\varepsilon}$ in X such that:*

$$\left|f(x) - [x, u_{f,\varepsilon}]\right| \le \varepsilon\|x\| \qquad \textit{for all} \quad x \quad \textit{in} \quad X\,,$$

where $[\ ,\ ]$ is as above.

REFERENCES

[1] J. DIESTEL: *Geometry of Banach Spaces - Selected Topics*, Springer Verlag, Berlin, Heidelberg, New York, 485, 1975.

[2] S.S. DRAGOMIR: *A characterization of best approximation element in real normed linear spaces*, Studia Univ. Babes-Bolyai, Mathematica, 33 (3) (1988), 74-80.

[3] R.C. JAMES: *Characterizations of reflexivity*, Studia Math., 23 (1964), 205-216.

[4] R.C. JAMES: *Reflexivity and the supremum of linear functionals*, Israel J. Math., 13 (1972), 298-300.

[5] P. PAVEL: *Differential Equations Associated of Some Nonlinear Operators on Banach Spaces*, (Romanian), Ed. Acad., Bucureşti, 1977.

[6] I. SINGER: *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, (Romanian), Bucureşti, Ed. Acad., 1967.

[7] R.A. TAPIA: *A characterization of inner product spaces*, Proc. Amer. Math. Soc., 41 (1973), 569-574.

Lavoro pervenuto alla redazione il 20 febbraio 1991
ed accettato per la pubblicazione il 24 gennaio 1992
su parere favorevole di A. Avantaggiati e di G. Da Prato

INDIRIZZO DELL'AUTORE:

Sever Silvestru Dragomir - Department of Mathematics - University of Timişoara - B-dul V. Pârvan No. 4 - 1900 Timişoara - Romania