Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 365-380

Equivalence of the spaces of ultradifferentiable functions and its applications to Whitney extension theorem

S.Y. CHUNG - D. KIM - S.K. KIM

RIASSUNTO – *Si dimostra l'equivalenza tra certi spazi di funzioni* $f \in C^\infty(\mathbb{R}^n)$, *caratterizzati dalla crescenza della successione dei massimi moduli delle successioni delle derivate* $(\partial^\alpha f)_{\alpha \in \mathbb{N}^n}$ *nei compatti di* $\mathbb{R}^n$, *e alcuni spazi del tipo di Beurling [1] e Björck [2]. Si dimostra inoltre che in tali spazi vale il teorema di estensione di Whitney.*

ABSTRACT – *We show the equivalence of spaces of ultradifferentiable functions and Whitney extension theorem for* $\mathcal{E}_{\{M_p\}}$ *which refines the result of Bruna. Also, as an application of this we show that Whitney extension theorem can be extended to* $\mathcal{E}_\omega(K)$ *for compact set* K *whenever* ω *satisfies* $(\alpha) \sim (\zeta)$.

KEY WORDS – *Whitney extension theorem - Ultradifferentiable function - Weight function.*

A.M.S. CLASSIFICATION: 46F05 - 26A24 - 46E10

– Introduction

Various classes of non-quasianalytic functions on $\mathbb{R}^n$ are usually defined by imposing conditions on the derivatives of the functions. For example, if $(M_p)_{p \in \mathbb{N}}$ is an appropriate sequence of positive numbers we can define

$$\mathcal{E}_{(M_p)}(\mathbb{R}^n) = \Big\{ f \in C^\infty(\mathbb{R}^n) | \text{ for each compact set } K \text{ in } \mathbb{R}^n \text{ and each } h > 0 \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}^n}} \frac{|\partial^\alpha f(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty \Big\}.$$

Also, $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n)$ is defined by replacing all the quantifier over h by existence quantifiers. Since the classical work of E. BOREL [4] many authors (see BRUNA [6], KANTOR [7], MITYAGIN [15], PETZSCHE [16]) have investigated conditions on sequences $(M_p)_{p\in\mathbb{N}}$ and $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ which imply the existence of a function $f \in \mathcal{E}_{(M_p)}(\mathbb{R}^n)$ (resp. $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n)$) with $\partial^\alpha f(0) = a_\alpha$ for all $\alpha \in \mathbb{N}^n$.

Especially KANTOR [7] and BRUNA [6] gave conditions on jets $(f^\alpha)_{\alpha\in\mathbb{N}^n}$ (of continuous functions on the compact set K) which imply the existence of function $f \in \mathcal{E}_{(M_p)}(\mathbb{R}^n)$ (resp. $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n)$) such that

$$\partial^\alpha f(x) = f^\alpha(x) \text{ on } K,$$

for all $\alpha \in \mathbb{N}^n$.

On the other hand, MEISE and TAYLOR [12] studied the question for the functions f in the classes $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ (resp. $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$) which are modifications of classes introduced by BEURLING [1] and BJÖRCK [2]. But they solved this question only for the case of a singleton or a compact convex set with nonempty interior, and the case for arbitrary compact sets remained open.

In this paper we prove that if ω is a weight function satisfying the conditions $(\alpha) \sim (\zeta)$ then there exists a sequence $(M_p)_{p\in\mathbb{N}^n}$ of positive numbers satisfying the conditions (M.1)$\sim$(M.3) such that

$$\mathcal{E}_{(M_p)}(\Omega) = \mathcal{E}_{(\omega)}(\Omega)$$

and

$$\mathcal{E}_{\{M_p\}}(\Omega) = \mathcal{E}_{\{\omega\}}(\Omega)$$

Also, we give a direct proof of the converse which was stated in MEISE and TAYLOR [12]. We think that the equivalence result in this paper clarifies the results in MEISE and TAYLOR [12] and BONET, MEISE and TAYLOR [3].

Also we show that Whitney extension theorem holds true for $\mathcal{E}_{\{M_p\}}$ whenever (M_p) satisfies (M.1)~(M.3) which refines the result of BRUNA [6].

As an application of these results we show that Whitney extension theorem can be extended to $\mathcal{E}_{(\omega)}(K)$ and $\mathcal{E}_{\{\omega\}}(K)$ for arbitrary compact set K whenever ω satisfies $(\alpha) \sim (\zeta)$ with a counterexample showing that the condition (ζ) is essential.

1 – Ultradifferentiable functions

In introducing the weight functions ω, ultradifferentiable functions and some necessary results on the Young conjugate φ^* we closely follow MEISE and TAYLOR [12].

DEFINITION 1.1. *Let $\omega : \mathbb{R} \longrightarrow [0,\infty)$ be a continuous even function which is increasing on $[0,\infty)$ and satisfies $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$. We consider the following conditions on ω:*
(α) $0 = \omega(0) \le \omega(s+t) \le \omega(s) + \omega(t)$ for all $s, t \in \mathbb{R}$;
(β) $\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} dt < +\infty$;
(γ) $\lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0$;
(δ) $\varphi : t \longrightarrow \omega(e^t)$ is convex on $\mathbb{R}$;
(ϵ) there exists $C > 0$ with $\int_1^{\infty} \frac{\omega(yt)}{t^2} dt \le C\omega(y) + C$ for all $y \ge 0$;
(ζ) there exists $H \ge 1$ with $2\omega(t) \le \omega(Ht) + H$ for all $t \ge 0$.

REMARK 1.2. (a) The conditions $(\alpha), (\beta)$ and (γ) are basically those which are used in BJÖRCK [2] and the conditions (δ) and (ϵ) are introduced by MEISE and TAYLOR [12].
(b) The above definitions involving the weight function ω do not change if ω is replaced by σ, where $\sigma|_{[0,1]} \equiv 0$ and $\sigma(t) = \omega(t)$ for all sufficiently large $t > 0$. Thus we may assume that $\omega|_{[0,1]} \equiv 0$.
(c) The condition (ϵ) implies that there exists an increasing concave function $\kappa : [0,\infty) \longrightarrow [0,\infty)$ with $\kappa(0) = 0$ and

(i) $\omega(y) \le \kappa(y) \le C\omega(y) + C$,
(ii) $\int_1^{\infty} \frac{\kappa(yt)}{t^2} dt \le C\kappa(y) + C$.

Thus we may assume that ω is concave, if necessary.
(d) Note that the constant H in (ζ) is closely related to that of $(M.2)$ in Definition 1.4.

For a weight function ω with the properties $(\alpha) \sim (\epsilon)$ let φ denote the function defined by 1.1(δ). We define its Young conjugate $\varphi^* : [0,\infty) \longrightarrow [0,\infty)$ by

$$\varphi^*(x) = \sup\{xt - \varphi(t) | t \geq 0\}.$$

Let Ω be an open set in $\mathbb{R}^n$ and K a compact subset of Ω. For $f \in C^\infty(\Omega)$ and $\lambda \in \mathbb{N}$ we define

$$\|f\|_{\omega,K,\lambda} = \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}^n}} \frac{|\partial^\alpha f(x)|}{\exp[\lambda\varphi^*(\frac{|\alpha|}{\lambda})]}.$$

A jet on K is a multisequence $F = (f^\alpha)_{\alpha\in\mathbb{N}}$ of continuous functions on K. For the jet $F = (f^\alpha)$ and $\lambda \in \mathbb{N}$ we define

$$\|F\|_{\omega,K,\lambda} = \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}^n}} \frac{|f^\alpha(x)|}{\exp[\lambda\varphi^*(\frac{|\alpha|}{\lambda})]} + \sup_{\substack{x,y\in K\\ x\neq y\\ |k|\leq m\\ m\in\mathbb{N}}} \frac{(m-|k|+1)!|R_y^m F)^k(x)|}{|x-y|^{m-|k|+1}\exp[\lambda\varphi^*(\frac{m+1}{\lambda})]}.$$

Here we note that the second term is not necessary if K is a convex set, since it can be estimated by the first one.

DEFINITION 1.3. *(a) We define the space $\mathcal{E}_{(\omega)}(\Omega)$ ($\mathcal{E}_{\{\omega\}}(\Omega)$ respectively) of ultradifferentiable functions of Beurling type (Roumieu type) is the set of C^∞-functions f in Ω with the property that for each compact set $K \subset \Omega$ and each $\lambda \in \mathbb{N}$ (some $\lambda \in \mathbb{N}$), $\|f\|_{\omega,K,\lambda}$ ($\|f\|_{\omega,K,\frac{1}{\lambda}}$) is finite. $\mathcal{E}_{(\omega)}(\Omega)$ ($\mathcal{E}_{\{\omega\}}(\Omega)$) is given the topology of the projective limit over K and λ (projective limit over K of the inductive limit over λ).*

(b) The space $\mathcal{E}_{(\Omega)}(K)$ of Whitney jets of Beurling type is defined to be the set of jets F with the property that for each $\lambda \in \mathbb{N}$, $\|F\|_{\omega,K,\lambda} < +\infty$. The space $\mathcal{E}_{\{\omega\}}(K)$ of Whitney jet of Roumieu type is similarly defined to be the set of jets F with $\|F\|_{\omega,K,\frac{1}{\lambda}} < +\infty$ for some $\lambda \in \mathbb{N}$.

DEFINITION 1.4. *Let $(M_p)_{p\in\mathbb{N}}$ be a sequence of positive numbers which has the following properties:*

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$ for all $p \in \mathbb{N}$;

(M.2) there exist $A, H > 1$ with $M_{p+q} \leq AH^{p+q}M_pM_q$ for all $p, q \in \mathbb{N}$;

(M.3) there exists $A > 0$ with $\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq Ap\frac{M_p}{M_{p+1}}$ *for all* $p \in \mathbb{N}$.

Then the space $\mathcal{E}_{(M_p)}(\Omega)$ of ultradifferentiable functions of class (M_p) is defined to be

$$\{f \in C^\infty(\Omega) | \text{for each } h > 0 \text{ and each compact set} K \subset \Omega,$$
$$\|f\|_{M_p,K,h} = \sup_{\substack{x \in K \\ \alpha \in \mathbb{N}^n}} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < +\infty\}.$$

Similarly the space $\mathcal{E}_{\{M_p\}}(\Omega)$ is defined to be

$$\{f \in C^\infty(\Omega) | \text{for each compact set } K \subset \Omega,$$
$$\text{there exists } h > 0 \text{ with } \|f\|_{M_p,K,h} < +\infty\}.$$

Also, the space $\mathcal{E}_{(M_p)}(K)$ of Whitney jets of class (M_p) is the set of jets $F = (f^\alpha)_{\alpha \in \mathbb{N}}$ with the property that for each $h > 0$,

$$\|F\|_{M_p,K,h} = \sup_{\substack{x \in K \\ \alpha \in \mathbb{N}^n}} \frac{|f^\alpha(x)|}{h^{|\alpha|} M_{|\alpha|}} + \sup_{\substack{x,y \in K \\ x \neq y \\ |k| \leq m \\ m \in \mathbb{N}}} \frac{(m - |k| + 1)! |(R_y^m F)^k(x)|}{|x - y|^{m-|k|+1} h^{m+1} M_{m+1}}.$$

is finite. The space $\mathcal{E}_{\{M_p\}}(K)$ is similarly defined to be the set of jets F with $\|F\|_{M_p,K,h} < +\infty$ for some $h > 0$.

2 – Some basic results on the Young conjugates

In this section we state and prove some technical results which are essential for our further considerations.

The following lemma is well known and easy to check.

LEMMA 2.1. (BRAUN, MEISE and TAYLOR [5]) *Let* $\varphi : [0, \infty) \longrightarrow [0, \infty)$ *be continuous, increasing, convex function with* $\varphi(0) = 0$ *and* $\lim_{t \to \infty} \frac{t}{\varphi(t)} = 0$. *Then we can define the Young conjugate* φ^* *of* φ *by*

$$\varphi^* : [0, \infty) \longrightarrow [0, \infty), \quad \varphi^*(x) = \sup_{t \geq 0}\{xt - \varphi(t)\},$$

and φ^* is convex, increasing, $\varphi^*(0) = 0$ and $\lim_{x\to\infty} \frac{x}{\varphi^*(x)} = 0$. Moreover, we have $(\varphi^*)^* = \varphi$, which means that

$$\varphi(t) = \sup_{x\geq 0}\{tx - \varphi^*(x)\}. \tag{2.1}$$

LEMMA 2.2. *(i) Let $\varphi(t)$ be the given one in 1.1(δ) and $\lambda > 0$. Then for each $p \in \mathbb{N}$ it follows that*

$$\exp[\lambda\varphi^*(\frac{p}{\lambda})] = \sup_{t\geq 0}\left[\frac{t^p}{\exp\lambda\omega(t)}\right]. \tag{2.2}$$

(ii) Furthermore, we have

$$\exp[\lambda\omega(t)] = \sup_{x\geq 0}\left\{\frac{t^x}{\exp[\lambda\varphi^*(\frac{x}{\lambda})]}\right\}. \tag{2.3}$$

PROOF. (i) From Remark 1.2 (b), we may assume that $\varphi(t) \equiv 0$ on $(-\infty, 0]$. Thus it follows that

$$\begin{aligned}\exp\left[\lambda\varphi^*(\frac{p}{\lambda})\right] &= \exp\left[\lambda\sup_{t\geq 0}\left\{\frac{p}{\lambda}t - \varphi(t)\right\}\right]\\ &= \exp\left[\sup_{t\in\mathbf{R}}\{pt - \lambda\omega(e^t)\}\right]\\ &= \exp\left[\sup_{t\geq 0} p\log t - \lambda\omega(t)\right]\\ &= \sup_{t\geq 0}\left[\frac{t^p}{\exp\lambda\omega(t)}\right].\end{aligned}$$

(ii) It is clear by (2.1).

LEMMA 2.3. *Let $\varphi : [0,\infty) \longrightarrow [0,\infty)$ be continuous, increasing, convex function with $\varphi(0) = 0$ and $\lim_{t\to\infty}\frac{t}{\varphi(t)} = 0$. Then there exists $C > 0$ such that*

$$\sup_{p\in\mathbf{N}}\{xp - \varphi(p)\} \leq \varphi^*(x) \leq C\sup_{p\in\mathbf{N}}\{xp - \varphi(p)\} + C. \tag{2.4}$$

PROOF. The first inequality is trivial from the definition of the Young conjugate φ^*. Now to prove the second we put $h(t) = tx - \varphi(t)$. Then h is concave and has a maximum value at some point $t_0 \geq 0$, on $[0, \infty)$. If x is sufficiently large then so is t_0. Thus we may choose an integer p_0 such that $1 < p_0 - 1 < p_0 < t_0$, $0 < t_0 - p_0 < 1$, $h(p_0) > 0$ since $\lim_{x \to \infty} \varphi^*(x) = \infty$. Then since h is concave,

$$h(t_0) \leq 2h(p_0) - h(p_0 - 1) \leq 3 \sup_{p \in \mathbb{N}} \{xp - \varphi(p)\}. \tag{2.5}$$

Since this inequality holds for large x and φ^* is continuous, it follows that

$$\varphi^*(x) \leq C \sup_{p \in \mathbb{N}} \{xp - \varphi(p)\} + C \tag{2.6}$$

for some $C > 0$, which is required.

LEMMA 2.4. *Let $H \geq 1$, $\lambda > 0$ and $0 < \epsilon < 1$. Then it follows that for each $p \in \mathbb{N}$,*

$$H^p \exp[\lambda \varphi^*(\frac{p}{\lambda})] \leq \exp\left[\frac{\lambda}{H} \varphi^*(\frac{H}{\lambda} p)\right] \tag{2.7}$$

and

$$\epsilon^p \exp[\lambda \varphi^*(\frac{p}{\lambda})] \geq \exp\left[\frac{\lambda}{\epsilon} \varphi^*(\frac{\epsilon}{\lambda} p)\right]. \tag{2.8}$$

PROOF. Note that we may assume that $\omega(t)$ is concave (see Remark 1.2 (c)). Then it follows from the result of Lemma 2.2 (i) that

$$\begin{aligned} H^p \exp[\lambda \varphi^*(\frac{p}{\lambda})] &= H^p \sup_{t \geq 0} \left[\frac{t^p}{\exp \lambda \omega(t)}\right] = \sup_{t \geq 0} \frac{t^p}{\exp[\lambda \omega(\frac{t}{H})]} \\ &\leq \sup_{t \geq 0} \frac{t^p}{\exp[\frac{\lambda}{H} \omega(t)]} = \exp\left[\frac{\lambda}{H} \varphi^*(\frac{H}{\lambda} p)\right]. \end{aligned}$$

The proof of (2.8) is similar to (2.7).

LEMMA 2.5. *Let $\lambda > 0$ and $p, q \in \mathbb{N}$. Then*

$$\exp[\lambda\varphi^*(\frac{p}{\lambda})]\exp[\lambda\varphi^*(\frac{q}{\lambda})] \leq \exp[\lambda\varphi^*(\frac{p+q}{\lambda})] \tag{2.9}$$

and

$$\exp[\lambda\varphi^*(\frac{p+q}{\lambda})] \leq e^{\lambda H}H^{p+q}\exp[\lambda\varphi^*(\frac{p}{\lambda})]\exp[\lambda\varphi^*(\frac{q}{\lambda})] \tag{2.10}$$

where H is the constant in 1.1(ζ).

PROOF. The first inequality is easily obtained from the convexity of φ^*. To prove the second one, we consider the condition (ζ): $2\omega(t) \leq \omega(Ht) + H$ for all $t \geq 0$. Then it follows from Lemma 2.2 that

$$\begin{aligned} H^{-(p+q)}e^{-\lambda H}\exp[\lambda\varphi^*(\frac{p+q}{\lambda})] &= H^{-(p+q)}e^{-\lambda H}\sup_{t\geq 0}\frac{t^{p+q}}{\exp\lambda\omega(t)} \\ &= \sup_{t\geq 0}\frac{t^{p+q}}{\exp[\lambda\omega(Ht)+\lambda H]} \\ &\leq \sup_{t\geq 0}\frac{t^p}{\exp\lambda\omega(t)}\sup_{t\geq 0}\frac{t^q}{\exp\lambda\omega(t)} \\ &= \exp[\lambda\varphi^*(\frac{p}{\lambda})]\exp[\lambda\varphi^*(\frac{q}{\lambda})] \end{aligned}$$

which is required.

The inequality (2.9) is related to the logarithmic convexity of the sequence $(\exp\varphi^*(p))_{p\in\mathbb{N}}$ and the inequality (2.10) is related to (M.2) of Definition 1.4.

3 – Equivalence of the spaces of ultradifferentiable functions

In this section we will show that there is a one to one correspondence between the family of ultradifferentiable functions defined by the sequences M_p and the family of ultradifferentiable functions defined by the weight functions ω. First we give a direct proof of the following theorem which is stated in MEISE and TAYLOR [12].

THEOREM 3.1. *Let $(M_p)_{p\in\mathbb{N}}$ be a sequence satisfying (M.1)~(M.3) in Definition 1.4. Then there exists a weight function $\chi(t)$ satisfying all the conditions $(\alpha)\sim(\zeta)$ in Definition 1.1 with*

$$\mathcal{E}_{(M_p)}(\Omega) = \mathcal{E}_{(\chi)}(\Omega) \tag{3.1}$$

and

$$\mathcal{E}_{\{M_p\}}(\Omega) = \mathcal{E}_{\{\chi\}}(\Omega) \tag{3.2}$$

topologically where Ω is an open subset in $\mathbb{R}^n$.

PROOF. Let $M(t) : \mathbb{R} \longrightarrow [0,\infty)$ be defined by

$$M(t) = \begin{cases} \sup\limits_{p\in\mathbb{N}} \frac{|t|^p M_0}{M_p} & \text{for } |t| > 0 \\ 0 & \text{for } t = 0 \end{cases} \tag{3.3}$$

Then $M(t)$ is a continuous even function with $M(0) = 0$ and $\lim\limits_{t\to\infty} M(t) = \infty$, which satisfies 1.1($\gamma$) and 1.1($\delta$). By Proposition 4.4 in KOMATSU [9] (M.3) implies that $M(t)$ satisfies 1.1(ϵ). Moreover, by Proposition 3.6 in KOMATSU [9] (M.2) implies that $M(t)$ also satisfies 1.1(ζ). The Remark 1.2(c) implies the existence of a weight function $\chi(t)$ which satisfies $(\alpha)\sim(\epsilon)$, which is concave on $[0,\infty)$ and which satisfies

$$M(t) \le \chi(t) \le CM(t) + C \tag{3.4}$$

for some $C > 0$ and for all $t > 0$.

On the other hand, since $M(t)$ satisfies the condition (ζ), (3.4) implies that

$$\begin{aligned} 2\chi(t) &\le 2CM(t) + 2C \le 2^k M(t) + 2^k \text{ for some } k \in \mathbb{N} \\ &\le 2^{k-1}[M(Ht) + H] + 2^k \\ &\le M(H't) + H' \text{ for some } H' > 0 \\ &\le \chi(H't) + H', \end{aligned} \tag{3.5}$$

which means that $\chi(t)$ also satisfies (ζ).

Now to prove (3.1) and (3.2), let φ^* be the Young conjugate of $\varphi(t) = \chi(e^t)$. Then it follows from Lemma 2.2 and (3.4) that for $\lambda \in \mathbb{N}$,

$$\exp[\lambda\varphi^*(\frac{p}{\lambda})] = \sup_{t\geq 0}\left[\frac{t^p}{\exp\lambda\chi(t)}\right] \leq \sup_{t\geq 0}\left[\frac{t^p}{\exp\chi(\lambda t)}\right] =$$

$$= \left(\frac{1}{\lambda}\right)^p \sup_{t\geq 0}\frac{t^p}{\exp\chi(t)} \leq \left(\frac{1}{\lambda}\right)^p \sup_{t\geq 0}\frac{t^p}{\exp M(t)} = \left(\frac{1}{\lambda}\right)^p M_p. \tag{3.6}$$

The last inequality is given by (M.1). This inequality implies that

$$\|\phi\|_{\chi,K,\lambda} \geq \|\phi\|_{M_p,K,\frac{1}{\lambda}}, \qquad \phi \in C^\infty(\Omega). \tag{3.7}$$

Let C be the constant in (3.4) and $k \in \mathbb{N}$ with $\lambda C \leq 2^k$. Then since $M(t)$ satisfies the condition (ζ) it follows that, as in (3.5),

$$\begin{aligned}\lambda CM(t) &\leq 2^k M(t) \leq 2^{k-1}[M(Ht) + H] \\ &\leq M(H^k t) + 2^k H.\end{aligned} \tag{3.8}$$

Hence it follows from (3.4) that

$$\begin{aligned}\exp\left[\lambda\varphi^*(\frac{p}{\lambda})\right] &= \sup_{t\geq 0}\left[\frac{t^p}{\exp\lambda\chi(t)}\right] \geq \sup_{t\geq 0}\frac{t^p}{\exp\{\lambda CM(t) + \lambda C\}} = \\ &= e^{-\lambda C}\sup_{t\geq 0}\frac{t^p}{\exp\{\lambda CM(t)\}} \geq \\ &\geq e^{-\lambda C}\sup_{t\geq 0}\frac{t^p}{\exp\{M(H^k t) + 2^k H\}} = \\ &= C'\left(\frac{1}{H^k}\right)^p \sup_{t\geq 0}\frac{t^p}{\exp M(t)} = C'\left(\frac{1}{H^k}\right)^p M_p\end{aligned} \tag{3.9}$$

for some C' depending on λ. This implies that

$$C'\|\phi\|_{\chi,K,\lambda} \leq \|\phi\|_{M_p,K,\frac{1}{H^k}}, \qquad \phi \in C^\infty(\Omega), \tag{3.10}$$

Therefore, the inequality (3.7) and (3.10) give the topological equivalences of (3.1) and (3.2).

The following theorem is an important result in the following section, which can be considered as the converse of Theorem 3.1.

THEOREM 3.2. *Let $\omega(t)$ be a weight function satisfying the condition $(\alpha) \sim (\zeta)$. Then there exists a sequence $(M_p)_{p\in\mathbb{N}}$ satisfying (M.1)$\sim$(M.3) with*

$$\mathcal{E}_{(\omega)}(\Omega) = \mathcal{E}_{(M_p)}(\Omega), \tag{3.11}$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}_{\{M_p\}}(\Omega), \tag{3.12}$$

topologically where Ω is an open subset of $\mathbb{R}^n$ and K is a compact subset. Furthermore, (M_p) satisfies the strong logarithmic convexity condition:

$$\left(\frac{M_p}{p!}\right)^2 \leq \frac{M_{p-1}}{(p-1)!} \cdot \frac{M_{p+1}}{(p+1)!}, \qquad p = 1, 2, \cdots.$$

PROOF. Let φ^* be the Young conjugate of the convex function $\varphi(t) = \omega(e^t)$. For the candidate for the defining sequence, put $M_p = \exp \varphi^*(p)$ for $p \in \mathbb{N}$. Then taking $\lambda = 1$ in Lemma 2.5 we obtain that M_p satisfies the first two conditions (M.1) and (M.2).

Now to prove (M.3), let $\omega_M(t)$ be an associated function defined as in (3.3) for the sequence (M_p) and $\varphi_M(t) = \omega_M(e^t)$. Then as we know, $\omega_M(t)$ is a continuous even function with $\omega(0) = 0$ and $\lim_{t\to\infty} \omega_M(t) = \infty$, which satisfies 1.1($\gamma$), ($\delta$) and ($\zeta$). Because of BRUNA [6], 2.1, it remains to show that $\omega_M(t)$ satisfies the condition (ϵ).

We note that

$$\varphi_M(t) = \sup_{p\in\mathbb{N}}\{pt - \log M_p\} = \sup_{p\in\mathbb{N}}\{pt - \varphi^*(p)\} \tag{3.13}$$

and

$$\varphi(t) = \sup_{x\geq 0}\{xt - \varphi^*(x)\}. \tag{3.14}$$

Then it follows from Lemma 2.3 that

$$\varphi_M(t) \le \varphi(t) \le C\varphi_M(t) + C \tag{3.15}$$

for $t > 0$ and for some constant $C > 0$. This implies that

$$\omega_M(t) \le \omega(t) \le C\omega_M(t) + C \tag{3.16}$$

for all $t > 0$. Here we may assume that $\omega(t)|_{[0,1]} \equiv 0$. Thus the equivalence of ω_M and ω in the sense of (3.16) means that ω_M also satisfies the condition (ϵ). Thus (M_p) satisfies the condition (M.3). Furthermore, from PETZSCHE [16], 1.1, we may assume that (M_p) satisfies the strong logarithmic convexity.

Then it follows from Theorem 3.1 that there exists a weight function $\chi(t)$ which satisfies $(\alpha) \sim (\zeta)$,

$$\omega_M(t) \le \chi(t) \le C\omega_M(t) + C \tag{3.17}$$

and (3.1), (3.2) are true. Then from the equivalence of (3.16) and (3.17) we obtain the required equality (3.11) and (3.12).

For the Whitney fields $\mathcal{E}_{(\omega)}(K)$ or $\mathcal{E}_{(M_p)}(K)$, we can get similar results related to Theorem 3.1 and 3.2. We will state them here whose proofs are slightly different.

COROLLARY 3.3. *If $(M_p)_{p\in\mathbb{N}}$ is a sequence satisfying (M.1)~(M.3) then there exists a weight function $\chi(t)$ satisfying $(\alpha) \sim (\zeta)$ with*

$$\mathcal{E}_{(M_p)}(K) = \mathcal{E}_{(\chi)}(K) \text{ and } \mathcal{E}_{\{M_p\}}(K) = \mathcal{E}_{\{\chi\}}(K). \tag{3.18}$$

Conversely, if $\omega(t)$ is a weight function satisfying $(\alpha) \sim (\zeta)$ then there exist a sequence $(M_p)_{p\in\mathbb{N}}$ which satisfies $(M.1) \sim (M.3)$, strong logarithmic convexity, and

$$\mathcal{E}_{(\omega)}(K) = \mathcal{E}_{(M_p)} \text{ and } \mathcal{E}_{\{\omega\}}(K) = \mathcal{E}_{\{M_p\}}(K). \tag{3.19}$$

Here we note that the weight function $\chi(t)$ in (3.18) and the sequence (M_p) in (3.19) are the same one as in Theorem 3.1 and 3.2.

4 – Applications to Whitney extension theorems

Now we are ready to extend the results of BRUNA [6].

LEMMA 4.1. *Let $(M_p)_{p\in\mathbb{N}}$ be a sequence satisfying the conditions (M.1)~(M.3). Then the restriction map*

$$\rho_K : \mathcal{E}_{\{M_p\}}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{\{M_p\}}(K),$$

defined by $\rho_K(f) = (\partial^\alpha f)_{\alpha\in\mathbb{N}^n}$, is surjective.

PROOF. In BRUNA [6], he proved the theorem under the conditions (M.1)~ (M.3) and the following conditions

(i) $M_{p+1}^p \le A^p M_p^{p+1}$, $p \in \mathbb{N}$ for some $A > 0$

(ii) (M_p) is strongly logarithmic convex.

But it is easy to show that (i) and (ii) are redundant, since (i) is equivalent to (M.2) (see MATSUMOTO [11], 2.5) and (M.3) guarantees the condition (ii) without disturbing (M.2) (see PETZSCHE [16]).

LEMMA 4.2. (KANTOR [7]). *Let $(M_p)_{p\in\mathbb{N}}$ be a sequence satisfying the condition (M.1)~(M.3). Then the restriction map*

$$\rho_K : \mathcal{E}_{(M_p)}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{(M_p)}(K)$$

is surjective.

In fact, the proof of Lemma 4.2. can also be obtained from the slight variation of that of Lemma 4.1.

Combining Lemma 4.1 and 4.2 and the equivalence results in §3, we obtain the following Whitney extension theorem for arbitrary compact sets K under the condition $(\alpha) \sim (\zeta)$.

THEOREM 4.3. *Let $\omega(t)$ be a weight function satisfying the conditions $(\alpha) \sim (\zeta)$. Then the restriction maps*

(i) $$\rho_K : \mathcal{E}_{\{\omega\}}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{\{\omega\}}(K) \text{ is surjective,}$$

and

(ii) $$\rho_K : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{(\omega)}(K) \text{ is surjective.}$$

PROOF. It follows from the result of Theorem 3.2 and Corollary 3.3 that there exists a sequence $(M_p)_{p\in\mathbb{N}}$ satisfying the conditions (M.1) $\sim$ (M.3) and

$$\mathcal{E}_{\{\omega\}}(\mathbb{R}^n) = \mathcal{E}_{\{M_p\}}(\mathbb{R}^n), \qquad \mathcal{E}_{\{\omega\}}(K) = \mathcal{E}_{\{M_p\}}(K).$$

Then Lemma 4.1. and its remark give the surjectivity of $\rho_K : \mathcal{E}_{\{\omega\}}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{\{\omega\}}(K)$. A similar argument gives also the surjectivity of

$$\rho_K : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{(\omega)}(K).$$

We think that it is an interesting question whether the restriction map ρ_K admits a continuous right inverse, i.e. whether one can do the extension with a continuous linear operator. In general, this question is negatively answered. But MEISE and TAYLOR [12] proved that the condition (ϵ) is sufficient for the case that K is a singleton or $K = \prod_{j=1}^{m} \overline{G}_j$ in $\mathbb{R}^n$ where G_j's are open in $\mathbb{R}^{n_j}$ with real analytic boundary. But for an arbitrary compact set, it is unsolved yet.

Finally we give an example showing that the condition (ζ) is essential for Whitney extension theorem for arbitrary compact set K.

EXAMPLE 4.4. Let $\omega(t) = (\log|t|)^2$ for sufficiently large $t > 0$. then it is easy to check that ω satisfies $(\alpha) \sim (\epsilon)$ but not (ζ) after a suitable change on $[-A, A]$ for some $A > 0$ if necessary and that its Young conjugate $\varphi^* = \frac{x^2}{4}$. Consider a sequence (x_k) of real numbers such that $x_1 > x_2 > x_3 > \cdots \to 0$ and

$$x_{k-1} - x_k \leq \exp(-k^3).$$

Let $K = \{x_1, x_2, \cdots\} \cup \{0\}$ and define

$$f_k = \begin{cases} 1 & , \ x = x_1, x_2, \cdots, x_k \\ 0 & , \ x = x_{k+1}, x_{k+2}, \cdots, \text{ or } 0 \end{cases}$$

then f_k is continuous on K and for each $\lambda > 0$

$$\sup_{x\in K} |f_k(x)| \leq \exp[\lambda\varphi^*(\frac{k}{\lambda})].$$

Now we claim that this jet in the sense of MEISE and TAYLOR [12] cannot be extended to $\mathcal{E}_{(\omega)}(\mathbb{R})$. We suppose that there exists a function $f \in \mathcal{E}_{(\omega)}(\mathbb{R})$ with $\partial^k f = f_k$ on K. Then it follows that for some constant $C > 0$

$$(4.1) \qquad \sup_{0\le x\le x_1} |\partial^k f(x)| \le C \exp[\lambda\varphi^*(\frac{k}{\lambda})] = C\exp[\frac{k^2}{4\lambda}]$$

for each $k \in \mathbb{N}$. Applying the mean value theorem there is t_k in (x_k, x_{k-1}) such that $\partial^k f(t_k) = (x_{k-1} - x_k)^{-1}$. we obtain from this and (4.1) that for each $k \in \mathbb{N}$

$$\exp(k^3) \le \partial^k f(t_k) \le \sup_{0\le x\le x_1} |\partial^k f(x)| \le C\exp[\frac{k^2}{4\lambda}],$$

which leads to a contradiction. Thus the jet (f_k) cannot be extended to $\mathcal{E}_{(\omega)}(\mathbb{R})$.

REFERENCES

[1] A. BEURLING: *Quasi-analyticity and general distributions*, Lectures 4 and 5, AMS Summer Institute, Stanford, 1961.

[2] G. BJÖRCK: *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1985), 351-407.

[3] J. BONET - R. MEISE - B.A. TAYLOR: *Whitney's extension theorem for ultradifferentiable functions of Roumieu type*, Proc. R. Ir. Acad. **89A** (1989), 53-66.

[4] E. BOREL: *Sur quelques points de la théorie des fonctions*, Ann. École Norm. Sup. (3) **12** (1895).

[5] R.W. BRAUN - R. MEISE - B.A. TAYLOR: *Ultradifferentiable functions and Fourier analysis*, manuscript.

[6] J. BRUNA: *An extension theorem of Whitney type for non-quasianalytic class of functions*, J. London Math. Soc. (2), **22** (1980), 495-505.

[7] J.M. KANTOR: *Classes non-quasianalytic et décomposition des support des ultradistributions*, An. Acad. Brasil. Ceincas **44** (1972), 171-180.

[8] D. KIM - S.Y. CHUNG: *Characterizations of ultradistributions with compact support and decomposition by support*, Proc. Amer. Math. Soc. **112** (1991), 65-74.

[9] H. KOMATSU: *Ultradistributions I*, J. Fac. Sci. Univ. Tokyo Sect. IA **20** (1973), 25-105.

[10] B. MALGRANGE: *Ideals of differentiable functions*, Oxford University Press, 1966.

[11] W. MATSUMOTO: *Theory of pseudo-differential operators of ultradifferentiable class*, J. Math. Kyoto Univ. **27** (1987), 453-500.

[12] R. MEISE - B.A. TAYLOR: *Whitney's extension theorem for ultradifferentiable functions of Beurling type*, Ark. Mat. **26** (1988), 256-287.

[13] R. MEISE - B.A. TAYLOR: *Linear extension operator for ultradifferentiable functions of Beurling type on compact sets*, Amer. J. Math. **111** (1989), 309-337.

[14] R. MEISE - B.A. TAYLOR: *Opérateurs linéares continus d'extexsion pour les fonctions ultradifférentiables sur des intervalles compact*, C.R. Acad. Sc. Paris **302** (1986), 219-222.

[15] B.S. MITYAGIN: *An infinitely differentiable functions with the values of its derivatives given at a point*, Soviet Math. Dokl. **2** (1961), 594-597, translated from Dokl. Acad. Nauk SSSR **138** (1961), 289-292.

[16] H.J. PETZSCHE: *On E. Borel's theorem*, Math. Ann. **282** (1988), 299-313.

Lavoro pervenuto alla redazione il 22 febbraio 1991
ed accettato per la pubblicazione il 16 settembre 1991
su parere favorevole di A. Avantaggiati e di L. Rodino

INDIRIZZO DEGLI AUTORI:

Soon-Yeong Chung[1] - Department of Mathematics - Duksung Women's University - Seoul 132-714, Korea

Dohan Kim[1] - Sung Ki Kim[2] - Department of Mathematics - Seoul National University - Seoul 151-742, Korea

This work has been partially supported by the Ministry of Education[1] and by the GARC-KOSEF[2].