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Iteration methods for the solution of operator equations and their application to ordinary and partial differential equations

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RIASSUNTO – *Lo scopo del presente lavoro è quello di dimostrare una nuova generalizzazione del teorema di punto fisso di Banach-Caccioppoli per operatori in spazi funzionali con K-norma. La differenza essenziale di tale generalizzazione consiste nel fatto che al posto della costante di Lipschitz nella condizione di contrazione si mettono certi operatori lineari o nonlineari che agiscono in uno spazio vettoriale appropriato parzialmente ordinato. Oltre ai casi classici, il nuovo principio è applicabile anche nel caso di spazi localmente convessi oppure di scale di spazi di Banach. I risultati astratti vengono illustrati tramite applicazioni ai problemi di Cauchy o di Goursat per equazioni differenziali con operatori nonlimitati. In tale maniera, si ottengono generalizzazioni di alcuni risultati classici del tipo Cauchy-Kovalevskaja dovuti a T. Nishida, L.V. Ovsjannikov, F. Treves, ed altri.*

ABSTRACT – *The purpose of this paper is to prove a new generalization of the Banach-Caccioppoli fixed point theorem to operators in function spaces with K-norm. The essential difference of this generalization consists in the fact that the Lipschitz constant in the contraction condition is replaced by certain linear or nonlinear operators which act in a suitable partially ordered vector space. Apart from classical cases, the new principle also covers the case of locally convex spaces, or scales of Banach spaces. The abstract results are illustrated by means of applications to the Cauchy and Goursat problems for differential equations with unbounded operators. In this way, one obtains generalizations of some classical results of Cauchy-Kovalevskaja type due to T. Nishida, L.V. Ovsjannikov, F. Treves, and others.*

KEY WORDS – *K-normed spaces - Banach-Caccioppoli principle - Cauchy problem - Goursat problem - Cauchy-Kovalevskaja theorem - Ovsjannikov-Nishida theorem.*

A.M.S. CLASSIFICATION: 47H10 - 47H17 - 46B40 - 46E30 - 34G20 - 35A10

The present paper is concerned with a new general approach to the study of fixed points of deteriorating operators, i.e. operators which decrease certain qualitative properties (usually, the smoothness) of their arguments. There is no general fixed point theory for such operators so far, although a large number of special existence results for fixed points of deteriorating operators are known. In this connection, one should mention a large number of results on the solvability of the Cauchy problem for differential equations with deteriorating right-hand side and, in particular, for partial differential equations (leading to the classical Cauchy-Kovalevskaja theorem, see e.g. [2-6, 8, 9, 15-23]), as well as several papers on the Goursat problem for partial differential equations (see e.g. [10, 12]), or some work on integral and integro-differential equations.

One basic tool in the study of such problems is the method of successive approximations; we point out, however, that the corresponding results cannot be obtained by applying the classical Banach-Caccioppoli contraction mapping theorem. Thus, one could hope to apply other fixed point principles which are different from the Banach-Caccioppoli theorem, but cover deteriorating operators. This idea was carried out in fact in the papers [26-28]. More precisely, in [26] and [27] a new generalization of the Banach-Caccioppoli theorem to K-spaces is formulated which covers, in particular, the well-known Ovsyannikov-Nishida theorem (see [2, 5, 18, 23]). In [28] a fixed point principle is given which includes solvability theorems for the Cauchy problem in Roumieu spaces of test functions and generalized functions (see e.g. [19]). We also mention the papers [1, 24, 25] which basically contain a new approach to studying the Cauchy problem for differential equations involving deteriorating operators (see also [13, 14]).

The purpose of the present paper is two-fold. First, we discuss a new general approach to the fixed point theory for deteriorating operators; the notions and results of [1, 24, 25] may be viewed as special aspects of this general approach. Second, we describe several classes of problems (for both ordinary and partial differential equations) where our approach allows us to obtain basic new results.

1 – Fixed point principles for deteriorating operators

Let $\mathcal{X}$ be a locally convex Hausdorff space, B some Banach space which is ordered by some cone K, and $\mathcal{X}(\omega) \subseteq \mathcal{X}$ $(\omega \in \Omega)$ a family of spaces, equipped with a K-valued K-norm $]\cdot[_{\mathcal{X}(\Omega)}$ and the usual norm

$$\|x\|_{\mathcal{X}(\omega)} = \|]x[_{\mathcal{X}(\omega)}\|_B \, . \tag{1}$$

A typical example of such a family of spaces in the theory of differential equations is the following: let $X(\omega)(\omega \in \Omega)$ be a family of Banach spaces which is continuously imbedded in a locally convex Hausdorff space X, and let $\mathcal{X}(\omega) = C(M, X(\omega))(\omega \in \Omega)$ be the space of all continuous functions on some set M with values in $X(\omega)$. In this example, it is natural to take B as the space $C(M, \mathbb{R})$ of all continuous real functions on M, and to define the K-norm on $\mathcal{X}(\omega)$ by

$$]x(t)[_{\mathcal{X}(\omega)} = \|x(t)\|_{X(\omega)} \, . \tag{2}$$

The notions developed in this paper carry over as well to the family $\mathcal{X}(\omega) = L_p(M, X(\omega))(\omega \in \Omega, 1 \leq p \leq \infty)$ of spaces of Bochner-integrable functions on M with values in $X(\omega)$, or even to the family $\mathcal{X}(\omega) = \mathcal{L}(M, X(\omega))$, with $\mathcal{L}$ being some space of measurable functions which may even depend on ω.

We remark that in many applications (see e.g. [1, 5, 15, 16, 18, 23, 25]) one takes $\Omega = [0, 1]$ and $X(s') \subseteq X(s'')$ for $0 \leq s'' < s' \leq 1$. On the other hand, in the papers [19, 26-28] it is assumed that $\Omega = \{1, 2, 3, \ldots, \infty\}$ and either $X_1 \supseteq X_2 \supseteq \ldots \supseteq X_\infty$ or $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \ldots \subseteq X_\infty$.

Suppose we are given an operator A which is defined on the union $\mathcal{X}_0$ of some of the spaces $\mathcal{X}(\omega)(\omega \in \Omega_0)$ and satisfies a Lipschitz condition

$$\begin{gathered}]Ax_1 - Ax_2[_{\mathcal{X}(\omega'')} \leq Q(\omega', \omega'')]x_1 - x_2[_{\mathcal{X}(\omega')} \\ \Big(x_1, x_2 \in \mathcal{X}(\omega');\, (\omega', \omega'') \in W\Big); \end{gathered} \tag{3}$$

here W is some set of pairs $(\omega', \omega'') \in \Omega_0 \times \Omega_0$, and $Q(\omega', \omega'')((\omega', \omega'') \in W)$ is a family of positive monotone (and usually also positively homogeneous and semi-additive, or even linear) operators in the space B.

By $W_n(\omega', \omega'')$ $((\omega', \omega'') \in W; n = 1, 2, \ldots)$ we denote the family of all chains $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ such that $\omega_0 = \omega'$, $\omega_n = \omega''$, and

$(\omega_{j-1}, \omega_j) \in W$; moreover, we put

$$W^*(\omega', \omega'') = \bigcup_{n=1}^{\infty} W_n(\omega', \omega''). \tag{4}$$

At this point, we have to introduce a new notion. Let M be some subset of the cone K. By s-inf M (called the "semi-infimum" of M) we denote the set of all elements $z \in K$ with the property that the relations $0 \leq x \leq \xi$ for $x \in K$ and some $\xi \in M$ imply that $x \leq z$. Obviously, $s\text{-}\inf M \neq \emptyset$ if $M \neq \emptyset$. If B is a K-space in Kantorovich's sense, or if the set M admits an infimum in the usual sense, then $s\text{-}\inf M = \inf M + K$. In general, s-inf M is the set of all least upper bounds of the set of all greatest lower bounds of M.

Let

$$\nu(M) = \inf\{\|z\| : z \in s\text{-}\inf M\}, \tag{5}$$

$$E(\omega', \omega'') = \left\{z \in B : \sum_{n=1}^{\infty} \nu(H_n(\omega', \omega''; z)) < \infty\right\}, \tag{6}$$

$$\mathcal{U}(\omega', \omega'') = \left\{z \in B : \lim_{n \to \infty} \nu(H_n(\omega', \omega''; z)) = 0\right\}, \tag{7}$$

and

$$T(\omega', \omega'') = \left\{z \in B : \overline{\lim_{n \to \infty}} \sqrt[n]{\nu(H_n(\omega', \omega''; z))} < 1\right\}, \tag{8}$$

where

$$H_n(\omega', \omega''; z) = \left\{\prod_{j=1}^{n} Q(\omega_{j-1}, \omega_j) z : \omega \in W_n(\omega', \omega'')\right\}. \tag{9}$$

Since the superposition of operators is non-commutative, we have to define the order of superposition: we do this by putting $\prod_{j=1}^{n} Q_j = Q_n \dots Q_1$.

Obviously,

$$(10)\qquad T(\omega',\omega'') \subseteq E(\omega',\omega'') \subseteq \mathcal{U}(\omega',\omega'').$$

Finally, let

$$(11)\qquad \psi_n(\omega',\omega'';z) = \sum_{k=n}^{\infty} \nu(H_k(\omega',\omega'';z)) \qquad (n=1,2,\ldots).$$

THEOREM 1. *Let* $(\omega',\omega'') \in W$. *Suppose that* A, *considered as an operator in* $\mathcal{X}(\omega'')$, *is closed (this holds, for instance, if* A *is continuous from* $\mathcal{X}(\omega'')$ *into* $\mathcal{X}$*). Assume, moreover, that*

$$(12)\qquad]x_0 - Ax_0[_{\mathcal{X}(\omega')} \in E(\omega',\omega'').$$

Then A *has at least one fixed point* $x^* \in \mathcal{X}(\omega'')$; *this fixed point may be obtained as limit of the successive approximations* $x_{n+1} = Ax_n$ *in* $\mathcal{X}(\omega'')(n=0,1,\ldots)$. *Moreover, the estimate*

$$(13)\qquad]x_n - x^*[_{\mathcal{X}(\omega'')} \le \psi_n(\omega',\omega'';]x_0 - Ax_0[_{\mathcal{X}(\omega')})$$

holds. Finally, the fixed point is unique in the set

$$(14)\qquad \mathcal{E}(\omega',\omega'';x_0) = \Big\{ \lim_{n\to\infty} A^n x \colon]x - x_0[_{\mathcal{X}(\omega')} \in \mathcal{U}(\omega',\omega'') \Big\};$$

in particular, A *cannot have two fixed points* x^* *and* x^{**} *for which* $]x^* - x^{**}[_{\mathcal{X}(\omega')} \in \mathcal{U}(\omega',\omega'')$.

PROOF. The proof of this theorem is very simple. In fact, putting $x_n = A^n x_0 (n=0,1,\ldots)$ we get from (3) for $\omega = (\omega_0,\ldots,\omega_n) \in W_n(\omega',\omega'')$ that

$$\begin{aligned}]x_n - x_{n+1}[_{\mathcal{X}(\omega'')} &=]A^n x_0 - A^n(Ax_0)[_{\mathcal{X}(\omega_n)} \le \\ &\le Q(\omega_{n-1},\omega_n)]A^{n-1}x_0 - A^{n-1}(Ax_0)[_{\mathcal{X}(\omega_{n-1})} \le \cdots \\ &\cdots \le Q(\omega_{n-1},\omega_n)\ldots Q(\omega_0,\omega_1)]x_0 - Ax_0[_{\mathcal{X}(\omega_0)}. \end{aligned}$$

This implies that

$$]x_n - x_{n+1}[_{\mathcal{X}(\omega'')} \in H_n(\omega',\omega'';]x_0 - Ax_0[_{\mathcal{X}(\omega')}).$$

Consequently,

$$\|x_n - x_{n+1}\|_{\mathcal{X}(\omega'')} \le \nu\Big(H_n(\omega',\omega'';]x_0 - Ax_0[_{\mathcal{X}(\omega')})\Big),$$

which shows that, by (12), the series $\sum_{n=0}^{\infty}(x_n - x_{n+1})$ is absolutely convergent in the norm of $\mathcal{X}(\Omega'')$. Thus, the sequence x_n converges in the norm of $\mathcal{X}(\omega'')$ to some limit x^*; obviously, the inequality (13) holds. Since A is closed in $\mathcal{X}(\omega'')$, x^* is a fixed point for A in $\mathcal{X}(\omega'')$.

It is clear that $x^* \in \mathcal{E}(\omega',\omega'';x_0)$. If $x^{**} \in \mathcal{E}(\omega',\omega'';x_0)$ is another fixed point of A, we have $x^{**} = \lim_{n\to\infty} A^n x_{00}$ for some $x_{00} \in \mathcal{U}(\omega',\omega'')$. Again, by (3), we get for $\omega = (\omega_0,\ldots,\omega_n) \in W_n(\omega',\omega'')$ that

$$\begin{aligned}]A^n x_0 - A^n x_{00}[_{\mathcal{X}(\omega'')} =]A^n x_0 - A^n x_{00}[_{\mathcal{X}(\omega_n)} \le \\ \le Q(\omega_{n-1},\omega_n)]A^{n-1}x_0 - A^{n-1}x_{00}[_{\mathcal{X}(\omega_{n-1})} \le \ldots \\ \ldots \le Q(\omega_{n-1},\omega_n)\ldots Q(\omega_0,\omega_1)\Big]x_0 - x_{00}[_{\mathcal{X}(\omega_0)}, \end{aligned}$$

hence

$$]A^n x_0 - A^n x_{00}[_{\mathcal{X}(\omega'')} \in H_n\Big(\omega',\omega'';]x_0 - x_{00}[_{\mathcal{X}(\omega')}\Big)$$

and

$$\|A^n x_0 - A^n x_{00}\|_{\mathcal{X}(\omega'')} \le \nu\Big(H_n(\omega',\omega'';]x_0 - x_{00}[_{\mathcal{X}(\omega')})\Big).$$

This shows that

$$\|x^* - x^{**}\|_{\mathcal{X}(\omega'')} = \lim_{n\to\infty}\|A^n x_0 - A^n x_{00}\|_{\mathcal{X}(\omega'')} = 0,$$

and thus $x^* = x^{**}$. ∎

We remark that, if the family $\mathcal{X}(\omega)(\omega \in \Omega)$ is not "large" enough, Theorem 1 may "degenerate" and turn into a well-known classical result. For example, in case $\Omega = \{1, 2, \dots, \infty\}$ one may arrive at results which are similar to those obtained in [28]. However, the statement of Theorem 1 applies in its full strength already in the case when $\mathcal{X}(\omega)$ $(\omega \in \Omega)$ is a "fan" of spaces; this means that $\Omega = \{(m,n)\colon m = 0, \dots, n;\, n = 1, 2, \dots\}$, $\mathcal{X}_{(0,n)} = \mathcal{U}$, and $\mathcal{X}_{(n,n)} = V$, where $(\mathcal{U}, V)$ is a fixed pair of K-normed spaces. A similar situation was discussed in [14].

In spite of the rather cumbersome formulation of Theorem 1, the verification of its hypotheses is in general not difficult. We restrict ourselves to just some general remarks which cover the most important cases.

Suppose, first of all, that

$$Q(\omega', \omega'') = c(\omega', \omega'')J\,, \tag{15}$$

where J is some positive linear operator in the space B, and $c(\omega', \omega'')$ is a function on W with values in $[0, \infty]$. If we define

$$c_n(\omega', \omega'') = \inf\Big\{\prod_{j=1}^{n} c(\omega_{j-1}, \omega_j)\colon \omega \in W_n(\omega', \omega'')\Big\}\,, \tag{16}$$

it is easy to see that

$$\nu(H_n(\omega', \omega''; z)) = c_n(\omega', \omega'')\|J^n z\| \qquad (n = 1, 2, \dots)\,. \tag{17}$$

Consequently, studying the sets (6) - (8) reduces to studying the sequence of functions (17). In particular, the following holds.

THEOREM 2. *Let $Q(\omega', \omega'')$ $((\omega', \omega'') \in W)$ be the family of operators defined by (15). Then*

$$\Big\{z \in B\colon \overline{\lim_{n\to\infty}} \sqrt[n]{c_n(\omega', \omega'')\|J^n z\|} < 1\Big\} \subseteq T(\omega', \omega'')\,. \tag{18}$$

We shall say that a family of operators $Q(\omega', \omega'')$ $((\omega', \omega'') \in W)$ has the property (S) if, given any $(\omega', \omega'') \in W$, for $n = 1, 2, \dots$ one may find a chain $\omega = (\omega_0, \omega_1, \dots, \omega_n) \in W_n(\omega', \omega'')$ such that

$$Q(\omega_0, \omega_1) = Q(\omega_1, \omega_2) = \dots = Q(\omega_{n-1}, \omega_n).$$

Denote by $S_n(\omega', \omega'')$ $(n = 1, 2, \dots)$ the set of these chains; in most examples, $S_n(\omega', \omega'')$ consists of just one chain. If a family of operators $Q(\omega', \omega'')$ $((\omega', \omega'') \in W)$ has the property (S) and $\omega = (\omega_0, \omega_1, \dots, \omega_n) \in S_n(\omega', \omega'')$, we put

$$Q_n(\omega', \omega''; \omega) = Q(\omega_{j-1}, \omega_j) \qquad (j = 1, \dots, n). \tag{19}$$

Obviously,

$$Q_n(\omega', \omega''; \omega)^n z \in H_n(\omega', \omega''; z),$$

hence

$$\nu(H_n(\omega', \omega''; z)) \le q_n(z) = \inf\left\{\|Q_n(\omega', \omega''; \omega)^n z\| : \omega \in S_n(\omega', \omega'')\right\}.$$

Thus, the following holds.

THEOREM 3. *Suppose that $Q(\omega', \omega'')$ $((\omega', \omega'') \in W)$ is a family of operators with the property (S). Then*

$$\left\{z \in B : \varlimsup_{n \to \infty} \sqrt[n]{q_n(z)} < 1\right\} \subseteq T(\omega', \omega''). \tag{20}$$

The most important special case of the previous discussion is when

$$Q(\omega', \omega'') = \sum_{k=1}^{m} c_k(\omega', \omega'') J_k \tag{21}$$

with $J_1, \dots J_m$ being positive linear (and usually commuting) operators in the space B, and $c_k(\omega', \omega'')$ $(k = 1, \dots, m)$ are functions on W with values in $[0, \infty]$. The property (S) holds in this case, for instance, if these functions have the form

$$c_k(\omega', \omega'') = h_k(c(\omega', \omega'')) \qquad (k = 1, \dots, m),$$

where $c(\omega',\omega'')$ is a fixed function on W with values in $[0,\infty]$, and $h_k(\xi)$ $(k=1,\ldots,m)$ are monotone functions mapping $[0,\infty)$ into itself.

If the family (21) has the property (S), we may define the functions $(\omega=(\omega_0,\omega_1,\ldots,\omega_n)\in S_n(\omega',\omega''))$

$$c_{k,n}(\omega',\omega'';\omega)=c_k(\omega_{j-1},\omega_j) \tag{22}$$
$$(j=1,\ldots,n;\, k=1,\ldots,m;\, n=1,2,\ldots).$$

The operators (19) are then given by

$$Q_n(\omega',\omega'';\omega)=\sum_{k=1}^{m}c_{k,n}(\omega',\omega'';\omega)J_k. \tag{23}$$

THEOREM 4. *Suppose that the family of operators $Q(\omega',\omega'')((\omega',\omega'')\in W)$ is defined by* (21) *and has the property (S). Then*

$$\left\{z\in B:\ \overline{\lim_{n\to\infty}}\sqrt[n]{\gamma_n(z)}<1\right\}\subseteq T(\omega',\omega'') \tag{24}$$
$$\left(\gamma_n(z)=\inf\left\{\left\|\Big(\sum_{k=1}^{m}c_{k,n}(\omega',\omega'';\omega)J_k\Big)^n z\right\|:\ \omega\in S_n(\omega',\omega'')\right\}\right).$$

Let us make some remarks on Theorems 1 - 4. First of all, in case $\mathcal{X}(\omega')\subset\mathcal{X}(\omega'')$ (strict inclusion) there is a "gap" between the existence and uniqueness of a fixed point: existence holds in the large space $\mathcal{X}(\omega'')$, and uniqueness in the narrow space $\mathcal{X}(\omega')$. This flaw, however, may be removed rather easily, since one may establish both existence and uniqueness in one and the same space $\mathcal{X}(\omega)$, just by considering the pair $(\omega',\omega)\in W$ in the existence proof, and the pair $(\omega,\omega'')\in W$ in the uniqueness proof.

Further, Theorem 1 establishes the convergence of any successive approximation to the fixed point of A in the norm of the space $\mathcal{X}(\omega'')$. Similarly, completely analogous results to Theorem 1 (and Theorems 2–4) may be formulated, where the convergence of successive approximations is understood in the sense of weak convergence or o-convergence. Passing to these types of convergence would allow us to weaken the hypotheses on the family of operators $Q(\omega', \omega'')$ $((\omega', \omega'') \in W)$; observe, however, that one has to suppose then completeness of the spaces $\mathcal{X}(\Omega)$ $(\omega \in \Omega)$. We point out that Theorem 1 is a particular case of a rather simple generalization of the Banach-Caccioppoli principle for contraction mappings to K-metric spaces; the scalar case has been studied in [28]. Similar results may be found in [7, 11, 26].

Let $\mathcal{U}$ and V be two K-metric spaces whose K-metrics take values in a Banach space B ordered by some cone K. We suppose that V, equipped with the metric $\rho(v_1, v_2) = \|d_V(v_1, v_2)\|_B$, is a complete metric space. Let A be some operator between $\mathcal{U}$ and V, which is closed in the metric of V, and satisfies

$$(25) \qquad \begin{gathered} d_V(A^n x_1, A^n x_2) \leq Q d_{\mathcal{U}}(x_1, x_2) \\ \Big(Q \in Q_n(r); d_{\mathcal{U}}(x_1, x_0), d_{\mathcal{U}}(x_2, x_0) \leq r; r \in B\Big), \end{gathered}$$

where $Q_n(r)(n = 1, 2, \ldots)$ is a family of positive monotone (and usually also positively homogeneous and semi-additive, or even linear) operators in the space B. Let

$$H_n(r; z) = \{Qz \colon Q \in Q_n(r)\} \qquad (n = 1, 2, \ldots),$$

and

$$\sum_{n=0}^{\infty} M_n = \Big\{ \sum_{n=0}^{\infty} m_n \colon m_n \in M_n, \sum_{n=0}^{\infty} \|m_n\| < \infty \Big\}.$$

We shall write $M \sqsubseteq r$ if there exists an element $m \in M$ such that $m \leq r$.

THEOREM 5. *Suppose that the condition*

$$(26) \qquad \sum_{n=0}^{\infty} s - \inf H_n(r; d_{\mathcal{U}}(x_0, Ax_0)) \sqsubseteq r$$

holds. Then A has at least one fixed point x^* in the space V; this fixed point may be obtained as limit of the successive approximations $x_{n+1} = Ax_n (n = 1, 2, \ldots)$. Moreover, the estimate

$$(27) \qquad \rho(x_n, x^*) \leq \sum_{k=n}^{\infty} \nu\Big(H_n(r; d_{\mathcal{U}}(x_0, Ax_0))\Big)$$

holds. Finally, the fixed point is unique in the set

$$(28)\qquad \mathcal{E}(x_0,r)=\Big\{\lim_{n\to\infty}A^n x:\ \lim_{n\to\infty}\nu\Big(H_n(r;d_{\mathcal{U}}(x,x_0))\Big)=0\Big\}\,;$$

in particular, A cannot have two fixed points x^* and x^{**} for which

$$(29)\qquad \lim_{n\to\infty}\nu\Big(H_n(r;d_{\mathcal{U}}(x^*,x^{**}))\Big)=0\,.$$

The proof of Theorem 5 is literally the same as that of Theorem 1, and therefore we shall not present it. Theorem 1, in turn, is a special case of Theorem 5, as may be seen by taking $\mathcal{U}=\mathcal{X}(\omega')$ and $V=\mathcal{X}(\omega'')$; here the family of operators $Q_n(r)(n=1,2,\dots)$ is even independent of r.

2 – Applications to differential equations

In what follows, $X(\omega)$ $(\omega\in\Omega)$ is a family of Banach spaces which are continuously imbedded in some locally convex Hausdorff space X.

The simplest and most important applications of Theorem 1 in the theory of differential equations refer to the Cauchy problem

$$(30)\qquad \frac{dx}{dt}=f(t,x)\,,\qquad x(0)=x_0$$

with some deteriorating (singular) right-hand side. The latter means that f satisfies a Lipschitz condition

$$(31)\quad \|f(t,u)-f(t,v)\|_{X(\omega'')}\le c(\omega',\omega'')\|u-v\|_{X(\omega')}\quad (u,v\in X(\omega'))\,,$$

where $c(\omega',\omega'')$ is some functions with values in $[0,\infty]$. It is well-known that the Cauchy problem (30) is equivalent to a fixed point problem for the operator

$$(32)\qquad Ax(t)=x_0+\int_0^t f(\tau,x(\tau))\,d\tau$$

which satisfies a Lipschitz condition (3) in the family of spaces $\mathcal{X}(\omega) = C([0,T], X(\omega))(\omega \in \Omega)$. Here the family of operators $Q(\omega',\omega'')((\omega',\omega'') \in W)$ is given by (15), where

$$Jz(t) = \int_0^t z(\tau)d\tau \tag{33}$$

is simply the operator of indefinite integration. Let the functions $c_n(\omega',\omega'')$ $(n = 1, 2, \ldots)$ be defined by (16). As a consequence of Theorems 1 and 2, we get the following.

THEOREM 6. *Suppose that the inequality*

$$T \varlimsup_{n\to\infty} \sqrt[n]{(n!)^{-1}c_n(\omega',\omega'')} < 1 \tag{34}$$

holds. Then the Cauchy problem (30) *has, for any* $x_0 \in X(\omega')$, *at least one solution* $x^*(t)$ *in* $X(\omega'')(0 \le t \le T)$, *and cannot have a second solution* $x^{**}(t)$ *such that the difference* $x^*(t) - x^{**}(t)$ *belongs to* $X(\omega')$ *and is bounded.*

Theorem 6 contains the classical Ovsyannikov-Nishida theorem [15-18], as may be seen by choosing $\Omega = [0,1]$ and $c(\omega',\omega'') = c(\omega' - \omega'')^{-1} + \infty H(\omega'' - \omega')$, with H being the Heaviside jump function; here the estimate (34) is equivalent to the inequality $ecT < \omega' - \omega''$. We point out that in the case $c(\omega',\omega'') = c(\omega' - \omega'')^{-\delta} + \infty H(\omega'' - \omega')(0 < \delta < 1)$, the estimate (34) holds for any T.

Similarly, one may consider the Cauchy problem

$$\frac{d^m x}{dt^m} = f\Big(t, x, \frac{dx}{dt}, \ldots, \frac{d^{m-1}x}{dt^{m-1}}\Big), x(0) = x_0, \ldots, x^{(m-1)}(0) = x_{m-1} \tag{35}$$

where the right-hand side satisfies an estimate

(36)

$$\|f(t,u_0,\ldots,u_{m-1}) - f(t,v_0,\ldots,v_{m-1})\|_{X(\omega'')} \le$$

$$\le \sum_{j=0}^{m-1} c_j(\omega',\omega'')\|u_j - v_j\|_{X(\omega')} \qquad (u_j, v_j \in X(\omega'); j = 0,\ldots,m-1).$$

The solutions of this problem are the fixed points of the operator

$$(37)\quad Av(t) = f\Big(t, \sum_{j=0}^{m-1} \frac{x_j t^j}{j!} + \frac{1}{m!}\int_0^t (t-\tau)^m v(\tau)d\tau, \dots, x_{m-1} + \int_0^t v(\tau)d\tau\Big)$$

($v = x^{(m)}$), which satisfies a Lipschitz condition (3) in the family of spaces $\mathcal{X}(\omega) = C([0,T], X(\omega))(\omega \in \Omega)$ with coefficients (21) and $J_k = J^{m-k}$, J given by (33). Suppose that the operators (21) have the property (S), and $c_{k,n}(\omega', \omega'')$ ($k = 1, \dots, m; n = 1, 2, \dots$) are given by (22). As a consequence of Theorems 1 and 4, we get the following.

THEOREM 7. *Suppose that the inequality*

$$(38)\quad \overline{\lim_{n\to\infty}} \sqrt[n]{\sum_{\alpha_1+\dots+\alpha_m=n} \frac{n! T^{\alpha_1+\dots+m\alpha_m}}{\alpha_1!\dots\alpha_m!(\alpha_1+\dots+m\alpha_m)!} \prod_{k=1}^m c_{k,n}^{\alpha_k}(\omega', \omega'')} < 1$$

holds. Then the Cauchy problem (35) *has, for any* $x_0, \dots, x_{m-1} \in X(\omega')$, *at least one solution* $x^*(t)$ *in* $X(\omega'')$ $(0 \le t \le T)$, *and cannot have a second solution* $x^{**}(t)$ *such that the difference* $x^*(t) - x^{**}(t)$ *belongs to* $X(\omega')$ *and is bounded.*

The most important special case of Theorem 7 is again $\Omega = [0,1]$ and $c_k(\omega', \omega'') = c_k(\omega' - \omega'')^{k-m} + \infty H(\omega'' - \omega')$ $(k = 1, \dots, m)$; here the estimate (38) is satisfied if

$$e\sum_{j=1}^m c_j((\omega' - \omega'')^{-1}T)T^j < 1.$$

In case $m = 1$ we get again the classical Ovsyannikov-Nishida theorem.

Finally, let us consider the Goursat problem

$$(39)\qquad \frac{\partial^2 x}{\partial t \partial s} = f\Big(t, s, x, \frac{\partial x}{\partial t}, \frac{\partial x}{\partial s}\Big), x(0,s) = x_1(s), x(t,0) = x_2(t),$$

where $x_1(s)$ and $x_2(t)$ are given functions with $x_1(0) = x_2(0) = x_0$. We suppose that the function f satisfies the condition

$$(40)\qquad \begin{aligned} &\|f(t,s,u_0,u_1,u_2) - f(t,s,v_0,v_1,x_2)\|_{X(\omega'')} \le \\ &\le \sum_{j=0}^2 c_j(\omega', \omega'')\|u_j - v_j\|_{X(\omega')} \qquad (u_j, v_j \in X(\omega'); j = 0,1,2). \end{aligned}$$

It is not hard to see that the Goursat problem (39) is equivalent to a fixed point problem for the operator

$$\begin{aligned}Av(t,s) =&f\Big(t,s,v(t,s),x_1(s)+x_2(t)-x_0+\int_0^t\int_0^s v(\tau,\sigma)d\tau d\sigma,\\ &x_2(t)+\int_0^s v(t,\sigma)d\sigma, x_1(s)+\int_0^t v(\tau,s)d\tau\Big)\end{aligned} \tag{41}$$

($v=\partial^2 x/\partial t\partial s$), which satisfies a Lipschitz condition (3) in the family of spaces $\mathcal{X}(\omega)=C([0,T]\times[0,S],X(\omega))$ ($\omega\in\Omega$) with coefficients (21) and operators

$$J_1 z(t,s)=\int_0^s z(t,\sigma)d\sigma, J_2 z(t,s)=\int_0^t z(\tau,s)d\tau, J_0=J_1J_2\,. \tag{42}$$

As a consequence of Theorems 1 and 4, we get the following.

THEOREM 8. *Suppose that the inequality*

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\sum_{\alpha_0+\alpha_1+\alpha_2=n}\frac{n!T^{\alpha_0+\alpha_2}S^{\alpha_0+\alpha_1}}{\alpha_0!\alpha_1!\alpha_2!(\alpha_0+\alpha_1)!(\alpha_0+\alpha_2)!}\prod_{k=0}^{2}c_{k,n}^{\alpha_k}(\omega',\omega'')}<1 \tag{43}$$

holds, and assume that the functions $x_1(s)$ and $x_2(t)$ in (39) are differentiable functions in $X(\omega')$ with bounded derivatives. Then the Goursat problem (39) has at least one solution $x^*(t,s)$ in $X(\omega'')$ ($0\le t\le T, 0\le s\le S$), and cannot have a second solution $x^{**}(t,s)$ such that the difference $x^*(t,s)-x^{**}(t,s)$ belongs to $X(\omega')$ and is bounded.

In the most important special case $\Omega=[0,1]$, $c_0(\omega',\omega'')=c_0(\omega'-\omega'')^{-2}+\infty H(\omega''-\omega')$ and $c_j(\omega',\omega'')=c_j(\omega'-\omega'')^{-1}+\infty H(\omega''-\omega')$ ($j=1,2$), the estimate (43) is satisfied if

$$e\Big(\frac{4c_0TS}{(\omega'-\omega'')^2}+\frac{2c_1T}{\omega'-\omega''}\frac{2c_2S}{\omega'-\omega''}\Big)<1\,.$$

The most important applications of the abstract Theorems 6 - 8 refer to partial differential equations and integro-differential equations. By means of a standard reasoning (see e.g. [15]), one may obtain the classical Cauchy-Kovalevskaya theorem and its generalizations in this way; to this end, as spaces $X(\omega)$ ($\omega \in \Omega$) one has to choose scales of certain spaces of analytic functions. Essentially more interesting results, however, may be obtained by choosing as $X(\omega)$ ($\omega \in \Omega$) certain spaces of test functions and generalized functions of finite or infinite order (in particular, Roumieu and Gevrey classes). The results obtained in this way are essential generalizations and extensions of theorems proved in [3, 4, 19, 24, 25].

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