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Surfaces with conformal second fundamental form

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RIASSUNTO – *In questo lavoro si studiano le sottovarietà M di una varietà Riemanniana N con seconda forma quadratica conforme; il problema ha interesse solo in codimensione maggiore di uno ed è collegato alla armonicità dell'applicazione sferica di Gauß. Il risultato principale è una classificazione completa delle superficie compatte di una varietà a curvatura costante semplicemente connessa con curvatura media parallela e con seconda forma quadratica conforme; in particolare viene ottenuta una nuova caratterizzazione del toro di Clifford e della superficie di Veronese.*

ABSTRACT – *The subject of the present paper is the study of the submanifolds M of a Riemannian manifold N with conformal second fundamental form. The question is interesting only in codimension greater than one and is related to the harmonicity of the spherical Gauß map. The main result is a complete classification of compact surfaces of a space form with parallel mean curvature and with conformal second fundamental form; in particular a new characterization for the Clifford torus and the Veronese surface is given.*

KEY WORDS – *Spherical Gauß map - Harmonic maps - Submanifolds with conformal second fundamental form - Minimal submanifolds - Submanifolds with parallel mean curvature vector field.*

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1 – Introduction

A Riemannian immersion $f : M \to N$ induces a map of the unit

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normal bundle of M in the unit tangent bundle of N, defined by

$$\nu_f : TM_1^{\perp} \to TN_1$$

$$(x,v) \mapsto (f(x),v); \qquad x \in M, \qquad v \in T_x M_1^{\perp}. \tag{1.1}$$

In other words ν_f sends a unit normal vector to M to itself, considered as a unit vector in N; ν_f is called the spherical Gauß map.

JENSEN and RIGOLI [6] endowed the bundles $TM_1^{\perp}$ and TN_1 with a Sasaki like metric and studied the conditions under which ν_f is harmonic. One of these conditions is that the second fundamental form h of the immersion must be conformal. It can be expressed as follows: let (e_i) $i = 1,..,m = \dim M$ be an orthonormal moving frame on M, then h is conformal if, for any field v and w on $TM^{\perp}$

$$\sum_{i,j=1}^{m} (h(e_i,e_j),v)(h(e_i,e_j),w) = \lambda^2(v,w), \tag{1.2}$$

where (,) denotes the inner product in N and λ is a suitable scalar function on M.

If M is a hypersurface of N, it is trivially true that h is conformal.

An important result on the harmonicity of the spherical Gauß map is the following ([6], compare also [9]): if N is a space form, codim$(M) > 1$ and either

(a) M is minimal in N or

(b) M has parallel mean curvature in N

then the spherical Gauß map is harmonic or vertically harmonic respectively if and only if f has conformal second fundamental form (vertically harmonic means that the component of the tension field of ν_f tangent to the fibre of the bundle TN_1 vanishes).

The aim of the present paper is to characterize all surfaces M isometrically immersed in a n-dimensional space form $\mathbb{R}^n(c)$ of constant curvature c with conformal second fundamental form for which either (a) or (b) holds and codim$(M) > 1$.

An easy codimension argument (see (2.14) and (2.15) in section 2) shows that the only cases to consider are the following: (1) n=4 and M minimal, (2) n=4 and M non minimal, (3) n=5 and M non minimal.

The main result is

MAIN THEOREM. *Let $f: M \to \mathbb{R}^n(c)$ be an isometric immersion of a compact connected surface in a n-dimensional space form of constant curvature c (with $n = 4, 5$). If*

(a) the second fundamental form of M is conformal (and not zero) and

(b) the mean curvature vector field H is parallel or, if $H = 0$, the length of h is constant,

then:

(1) *if $n = 4$ and M is minimal, then M coincides with the Veronese surface in S^4,*

(2) *if $n = 4$ and M is not minimal, then $c = 0$ and M coincides with the Clifford torus immersed in $\mathbb{R}^4$,*

(3) *if $n = 5$ then f is an immersion of $\mathbb{RP}^2$ in S^5, which factors throught the Veronese surface in a suitable S^4 in S^5.*

In any case M is a pseudoumbilical submanifold and it is either minimal in $\mathbb{R}^n(c)$ or minimally immersed in a small hypersphere of $\mathbb{R}^n(c)$.

The proof of the Main Theorem will be given in section 3. In particular (1) is a consequence of theorem 3.2, (2) follows from theorem 3.4 and (3) from theorem 3.6.

In section 2 some algebraic problems concerning the condition that h is conformal are examined and it will be shown that this condition is equivalent to some other notions which were introduced by SIMONS [10] and B.Y. CHEN [1].

Section 3 is devoted to the study of surfaces with codimension greater than one; in particular this yields the proof of the main theorem, which gives a new characterization of the Clifford torus and the Veronese surface.

In the case of m-dimensional submanifolds ($m > 2$) it seems unlikely that a complete classification could be done. In a forthcoming paper some problems concerning the submanifolds with dimension greater than two with conformal second fundamental form will be analized.

2 – Preliminaries and conditions equivalent to h conformal

Let M be a m-dimensional manifold isometrically immersed in a n-dimensional manifold N.

In this section I will use the following sets of indices with the following ranges:

$$A, B, .. = 1, .., n$$
$$i, j, .. = 1, .., m$$
$$\alpha, \beta, .. = m+1, .., n\,.$$

Furthermore repeated indices are summed over the respective ranges. Let $(e_A) = (e_i, e_\alpha)$ be an orthonormal moving frame of N adapted to M (which is called a Darboux frame). This is equivalent to say that, restriced to M, e_i are tangent vector fields to M (in fact they are a local frame on M) and the e_α are normal vector fields to M.

If (ω^A) denotes the dual frame of (e_A) and ω^A_B are the Levi-Civita connection forms on N, then the structure equations of N are given by (compare [3])

$$d\omega^A = -\omega^A_B \wedge \omega^B \qquad (\omega^B_A + \omega^A_B = 0) \tag{2.1}$$

$$d\omega^A_B = -\omega^A_C \wedge \omega^C_B + \frac{1}{2} R^N_{ABCD}\omega^C \wedge \omega^D \tag{2.2}$$

where R^N is the curvature tensor of N. If N is a space of constant curvature c then

$$R^N_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,.$$

If the forms ω^A are restricted to M, then (ω^i) is the orthonormal coframe of M, the Levi Civita connection of M is defined by (ω^i_j), and

$$\omega^\alpha_i = h^\alpha_{ij}\omega^j, \qquad h^\alpha_{ij} = h^\alpha_{ji} = (\nabla^N_{e_i} e_j, e_\alpha) = (h(e_i, e_j), e_\alpha)\,. \tag{2.3}$$

Hence the h^α_{ij} are the components of the second fundamental form, which is a $TM^\perp$ valued bilinear symmetric form on M. In addition, one has the following formulas

$$R^M_{ijhk} = R^N_{ijhk} + h^\alpha_{ih}h^\alpha_{jk} - h^\alpha_{ik}h^\alpha_{jh} \qquad \text{(Gauss equations)}, \tag{2.4}$$

$$R^\perp_{\alpha\beta ij} = R^N_{\alpha\beta ij} + h^\alpha_{ki}h^\beta_{jk} - h^\alpha_{kj}h^\beta_{ki} \qquad \text{(Ricci equations)}, \tag{2.5}$$

where $R^\perp$ is the curvature tensor of the Riemannian connection $\nabla^\perp$ in the normal bundle $TM^\perp$ determined by the forms (ω^α_β). The covariant derivative of the second fundamental form is $\bar\nabla h$ and it has components h^α_{ijk}, where $h^\alpha_{ijk} = ((\bar\nabla_{e_k} h)(e_i, e_j), e_\alpha)$ and

$$(\bar\nabla_{e_k} h)(e_i, e_j) = \nabla^\perp_{e_k}(h(e_i, e_j)) - h(\nabla^M_{e_k} e_i, e_j)h(e_i, \nabla^M_{e_k} e_j).$$

In terms of the forms ω^A_B

$$h^\alpha_{ijk}\omega^k = dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta \qquad (h^\alpha_{ijk} = h^\alpha_{jik}). \tag{2.6}$$

Exterior differentiation of (2.3) yields

$$h^\alpha_{ijk} = h^\alpha_{ikj} + R^N_{\alpha ikj} \qquad \text{(Codazzi equations)}. \tag{2.7}$$

The mean curvature vector field H of the immersion of M in N is defined by

$$H = \frac{1}{m} h^\alpha_{ii} e_\alpha . \tag{2.8}$$

M is minimal in N if $H = 0$; M has parallel mean curvature if H is parallel with respect to the normal connection i.e. $\nabla^\perp H = 0$ or

$$h^\alpha_{iik} = 0 . \tag{2.9}$$

The $(n - m)$ symmetric matrices of order m which are determined by h, are

$$H_\alpha = (h^\alpha_{ij}) . \tag{2.10}$$

The norm (or length) of h is given by

$$\|h\|^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2 = \sum_\alpha \|H_\alpha\|^2 , \tag{2.11}$$

where the scalar product and the norm of matrices are defined in the usual way: if $A = (a_{ij})$ and $B = (b_{ij})$ then

$$(A, B) = \operatorname{trace}({}^t AB) = \sum a_{ij} b_{ij} .$$

DEFINITION 2.1. *The second fundamental form of the immersion of M in N is conformal if there exists a function λ on M such that*

$$(H_\alpha, H_\beta) = \lambda^2 \delta_{\alpha\beta}. \tag{2.12}$$

In other words the matrices (H_α) are orthogonal and they have the same length.

(2.12) implies, in particular, that, if h is conformal, then

$$\|h\|^2 = (n-m)\lambda^2. \tag{2.13}$$

I will suppose $n - m \geq 2$, as it is trivially true that a hypersurface has conformal second fundamental form. It also will be assumed that λ is not identically zero, or , in other terms, that M is not totally geodesic in N. As the dimension of the vector space of the symmetric matrices of order m is $\frac{m(m+1)}{2}$, one can notice that the second fundamental form is conformal (and not zero) only if the H_α are linear independent, hence

$$n - m \leq \frac{m(m+1)}{2}. \tag{2.14}$$

Furthermore, if M is minimal in N, the matrices (H_α) belong to the hyperplane of the traceless symmetric matrices. Hence h is conformal only if

$$n - m \leq \frac{m(m+1)}{2} - 1. \tag{2.15}$$

In the last part of this chapter it will be shown that 'h conformal' is equivalent to some other conditions.

Let $S(M)$ be the fibre bundle of symmetric endomorphisms on TM and let $A \in \mathrm{hom}(TM^\perp, S(M))$ be the operator associated to h, i.e.

$$(h(X,Y), w) = (A_w(X), Y).$$

For the Darboux frame (e_i, e_α)

$$A_{e_\alpha} = H_\alpha.$$

J. SIMONS, [10], introduced the operator $\tilde{A} \in \hom(TM^{\perp}, TM^{\perp})$, defined by

$$\tilde{A} = {}^{t}AA,$$

where ${}^{t}A$ is the adjoint operator of A (i.e. $({}^{t}A(s), w) = (A(w), s)$).

It can be easily verified that

$$(\tilde{A}(e_{\alpha}), e_{\beta}) = (A_{e_{\alpha}}, A_{e_{\beta}}) = h^{\alpha}_{ij} h^{\beta}_{ij}\,.$$

Hence h is conformal if and only if

$$\tilde{A} = \lambda^{2}\mathrm{I}\,, \tag{2.16}$$

where I is the identity.

On the other hand, B.Y. CHEN, [1], introduced, for any normal vector field ξ, the allied vector field $a(\xi)$ defined as follows: if (e_{α}) is an orthonormal moving frame such that $e_{m+1} = \frac{\xi}{\|\xi\|}$ then one defines, if $\xi \neq 0$

$$a(\xi) = \frac{\|\xi\|}{m} \sum_{\beta=m+2}^{n} \operatorname{trace}(H_{m+1} H_{\beta}) e_{\beta} = \frac{\|\xi\|}{m} \sum_{\beta=m+2}^{n} (H_{m+1}, H_{\beta}) e_{\beta}\,, \tag{2.17}$$

otherwise, if $\xi = 0$, $a(\xi) = 0$.

From (2.17) follows that h is conformal if and only if, for any $\xi \in TM^{\perp}$, $a(\xi) = 0$. Hence

THEOREM 2.1. *If M is a submanifold of a Riemannian manifold N then the following are equivalent:*

(1) *the second fundamental form is conformal,*
(2) *the Simons operator $\tilde{A} \in \hom(TM^{\perp}, TM^{\perp})$ is proportional to the identity,*
(3) *for any normal vector field, the allied vector field vanishes.*

REMARK. Any submanifold for which $a(H) = 0$ is called a A-submanifold or Chen-submanifold ([1], [4]). The class of Chen submanifolds includes the hypersurfaces, the minimal submanifolds and, more generally, the pseudoumbilical submanifolds -that is the submanifolds such that for any $X, Y \in TM$

$$(h(X,Y), H) = (X,Y)\|H\|^2,$$

or, in other words, such that A_H is proportional to the identity (i.e. the section H is umbilical). In fact, if M is pseudoumbilical, H_{n+1} is a multiple of the identity matrix and all H_β for $\beta \geq m+2$ are traceless, hence $a(H)$ vanishes.

3 – Surfaces with conformal second fundamental form

If $\operatorname{codim}(M) > 1$, from all what was remarked in section 2 it follows that the study of the condition that the second fundamental form of a surface is conformal can be done considering separately the following situations:

(1) M is a minimal surface in a 4-dimensional manifold
(2) M is a non-minimal surface in a 4-dimensional manifold
(3) M is a non-minimal surface in a 5-dimensional manifold.

In this section, examples of any of these three cases will be examined. It will be given particular attention to the case in which N is a space of constant curvature and M has parallel mean curvature.

3.1 – Minimal surfaces in a 4-dimensional manifold

Whatever is the Darboux frame (e_A), the matrices H_3 and H_4 (defined by (2.10) have zero trace. With a suitable choice of the frame (e_1, e_2) of M, it can be assumed that one of these matrices, say H_3, is diagonal. The condition that h is conformal implies that, if the orientation of e_4 is appropriately chosen, the matrices H_3 and H_4 are

$$(3.1) \qquad H_3 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix},$$

where μ is a function on M, not identically zero, unless M is totally geodesic.

For any $X = X_1e_1 + X_2e_2$ tangent to M

$$h(X,X) = (\mu X_1^2 - \mu X_2^2)e_3 + 2\mu X_1X_2e_4$$

hence

$$\|h(X,X)\| = |\mu|\|X\|^2 \tag{3.2}$$

which means that M is a isotropic minimal submanifold of N as introduced by O' Neill [8]. It can be verified easily that the converse is also true, hence

THEOREM 3.1. *A surface minimally immersed in a 4-dimensional manifold has conformal second fundamental form if and only if it is isotropic.*

If N is a space of constant curvature c, then, from (2.4), (2.5) and (3.1) follows that the Gaussian curvature K and the normal curvature $K^\perp$ of N are given by

$$K = R^M_{1212} = c - 2\mu^2, \qquad K^\perp = R^\perp_{3412} = 2\mu^2, \tag{3.3}$$

and

$$d\omega^1_2 = K\omega^1 \wedge \omega^2, \qquad d\omega^3_4 = K^\perp\omega^1 \wedge \omega^2. \tag{3.4}$$

A significative example of minimal surface in a space of constant curvature with conformal second fundamental form is the Veronese surface in S^4. If $S^2(R)$ is the sphere in $\mathbb{R}^3$ with centre in the origin and radius R, then the mapping

(3.5)

$$f\colon \mathbb{R}^3 \to \mathbb{R}^5 : (x,y,z) \mapsto \frac{1}{R}(xy, xz, yz, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2))$$

induces an isometric immersion of $S^2(R)$ in $S^4(\frac{R}{\sqrt{3}})$. As antipodal points are mapped by f to the same point of $S^4(\frac{R}{\sqrt{3}})$, one obtains an isometric immersion of the real projective plane $\mathbb{RP}^2$ in $S^4(\frac{R}{\sqrt{3}})$, which is called the Veronese surface.

An easy computation in local coordinates shows that one can choose a Darboux frame (e_1, e_2, e_3, e_4) so that

$$(3.6) \qquad H_3 = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 0 & \frac{1}{R} \\ \frac{1}{R} & 0 \end{pmatrix},$$

and setting $\mu = \frac{1}{R}$ one can see that (3.6) coincides with (3.1). Conversely, if one supposes that in (3.1) μ is constant (i.e that the second fundamental form has constant length) and if one takes exterior differentiation of the formulas (which are just (3.1) restated)

$$(3.1') \qquad \omega_1^3 = \mu\omega^1, \qquad \omega_2^3 = -\mu\omega^2, \qquad \omega_1^4 = \mu\omega^2, \qquad \omega_2^4 = \mu\omega^1,$$

then it follows that

$$(3.7) \qquad \omega_4^3 = 2\omega_2^1,$$

whence, using (3.3) and (3.4), one obtains

$$K^{\perp} = 2\mu^2 = 2K = 2c - 4\mu^2,$$

hence

$$(3.8) \qquad c = 3\mu^2, \qquad K = \mu^2.$$

In particular (3.8) implies that M is a space of constant curvature. Hence the same argument as in the proof of theorem 3, page 72 of [3] shows that M coincides locally (globally if M is compact) with the Veronese surface. (Formulas (4.12) and (4.13) of [3] are the same as (3.7) and (3.8)).

This proves the following

THEOREM 3.2. *Let N be a space form of dimension 4 and let M be a connected minimal surface of N with second fundamental form conformal and of constant length; then*

(1) *the curvature of N is positive*
(2) *M coincides locally with the Veronese surface in S^4; if M is compact it coincides with the Veronese surface.*

One can achieve the same result starting from different assumptions. As a matter of fact it will be proved the following

THEOREM 3.3. *Let N be a space form of dimension 4 and let M be a compact and connected minimal surface of M with conformal second fundamental form. If K is the Gaussian curvature of M, $K^{\perp}$ is the normal curvature of M and if*

$$2K \geq K^{\perp} \quad \text{or} \quad 2K \leq K^{\perp} \tag{3.9}$$

everywhere on M, then $2K = K^{\perp}$, N has positive sectional curvature and M coincides with the Veronese surface.

PROOF. It will be proved that (compare also [5])

$$\Delta \log \mu = 2K - K^{\perp}, \tag{3.10}$$

where Δ is the Laplace-Beltrami operator on M, that acts on scalars according to the following formula

$$\Delta f = \nabla^2_{e_i e_i} f = e_i e_i f - \nabla_{e_i} e_i f. \tag{3.11}$$

Taking exterior differentiation of (3.1) and using the equations of Codazzi it follows

$$e_1 \mu = -\mu(\omega_4^3 - 2\omega_2^1)(e_2), \qquad e_2 \mu = \mu(\omega_4^3 - 2\omega_2^1)(e_1).$$

An easy computation shows that

$$\Delta \log \mu = 2d\omega_2^1(e_1, e_2) - d\omega_4^3(e_1, e_2) = 2K - K^{\perp}.$$

As M is compact

$$0 = \int_M \Delta log\mu dM = \int_M (2K - K^{\perp}) dM$$

applying (3.9), $\Delta log\mu = 0$: hence μ is constant.

REMARK. Theorems 3.2 and 3.3 show a remarkable feature of the Veronese surface. On the other hand, it is possible to find examples of surfaces that are minimally immersed in a space form of dimension 4 with second fundamental form conformal but not of constant length.

For instance, if one sets $z = x + iy$, the surface immersed in $\mathbb{R}^4$, which is the image of $\mathbb{C} - \{0\}$ by the map

$$(\mathrm{Re}(z^3), \mathrm{Im}(z^3), \mathrm{Re}(z^2), \mathrm{Im}(z^2))$$

$$\text{or} \qquad (x^3 - 3xy^2, 3x^2y - y^3, x^2 - y^2, 2xy),$$

has conformal second fundamental form.

To see this, set

$$P_x = (3x^2 - 3y^2, 6xy, 2x, 2y), \qquad P_y = (-6xy, 3x^2 - 3y^2, -2y, 2x),$$

so that $e_1 = \frac{P_x}{\|P_x\|}$ and $e_2 = \frac{P_y}{\|P_y\|}$ is an orthonormal frame. If

$$e_3 = \frac{1}{\|P_x\|}(2x, 2y, -3(x^2 + y^2), 0),$$

$$e_4 = \frac{1}{\|P_x\|}(-2y, 2x, 0, -3(x^2 + y^2)),$$

one can readily compute the second fundamental form, finding

$$h^3_{11} = -h^3_{22} = h^4_{12} = \frac{1}{\|P_x\|^5}(x^2 + y^2)^2(24 + 54(x^2 + y^2)),$$

$$h^3_{12} = h^4_{11} = h^4_{22} = 0\,.$$

This example can be found in [7], page 41.

3.2 – Non minimal surfaces in a 4-dimensional manifold

It can be easily verified that one can choose the Darboux frame so that e_4 is parallel to the mean curvature vector H of M in N and the matrix H_4 is diagonal: that is

$$H_4 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \alpha + \beta \neq 0\,.$$

The trace of H_3 is zero, so H_3 is of the kind

$$H_3 = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}.$$

The orthogonality of H_3 and H_4 implies

$$(\alpha - \beta)\gamma = 0, \qquad \alpha^2 + \beta^2 = 2\gamma^2 + 2\delta^2 .$$

It follows that two case are possible

(1) if $\alpha \neq \beta$, i.e. M is not pseudoumbilical, then $\gamma = 0$, so

$$(3.12) \qquad H_3 = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}, \qquad H_4 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

(2) if M is pseudoumbilical, $\alpha = \beta$ and one can choose (e_1, e_2) so that H_3 and H_4 are

$$(3.13) \qquad \begin{array}{l} H_3 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{or} \quad H_3 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \\ H_4 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}. \end{array}$$

It can be remarked that the two different expressions of of H_3 in (3.13) are reducible one to the other by means of a reflection with respect to bisector of the angle (e_1, e_2). Hence no geometric distinction between this last two cases holds. In both cases (1) and (2) H_3 and H_4 can be expressed as follows:

(3.14)

$$H_3 = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}, \qquad H_4 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \alpha + \beta \neq 0, \qquad \alpha^2 + \beta^2 = 2\delta^2.$$

cases (1) or (2) occurring according to $\alpha \neq \beta$ or $\alpha = \beta$ respectively.

An example of a non pseudoumbilical surface satisfing (3.12) is the following surface of $\mathbb{R}^4$:

$$(3.15) \qquad (f(u)\cos v, f(u)\sin v, \cos(\sqrt{2}v), \sin(\sqrt{2}v)),$$

where f is any function with $f' \neq 0$.

Such a surface is a particular case of a type of Chen-submanifolds, which can be found in [4].

An important example of pseudoumbilical surface with conformal second fundamental form is the Clifford torus immersed in $\mathbb{R}^4$:

$$(3.16)\qquad \begin{aligned} &P: S^1(R) \times S^1(R) \to \mathbb{R}^4 \\ &\qquad (u,v) \mapsto P(u,v) = (R\cos u, R\sin u, R\cos v, R\sin v). \end{aligned}$$

which is, as well known, a flat minimal surface in $S^3(\sqrt{2}R)$.

Set

$$e_1 = P_u/\|P_u\| = (-\sin u, \cos u, 0, 0),$$

$$e_2 = P_v/\|P_v\| = (0, 0, -\sin v, \cos v).$$

Then $H = -\frac{1}{R}(\cos u, \sin u, \cos v, \sin v)$.

Hence H is orthogonal to $S^3(\sqrt{2}R)$, which means that the torus is minimal in the sphere. If $e_4 = \frac{H}{\|H\|}$, $e_3 = \frac{1}{\sqrt{2}}(-\cos u, -\sin u, \cos v, \sin v)$ it can be easily seen that

$$H_3 = \frac{1}{\sqrt{2}R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_4 = \frac{1}{\sqrt{2}R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The importance of the immersion of the Clifford torus in $\mathbb{R}^4$ is clarified by the following

THEOREM 3.4. *Let N be a space form of dimension 4 with curvature c and let M be a compact and connected surface with conformal second fundamental form and with parallel and non vanishing mean curvature vector field. Then:*

(1) *N is flat (i.e. $N = \mathbb{R}^4$)*

(2) *M coincides with the Clifford torus.*

PROOF. By (3.14), (2.4) and (2.5) yield

$$(3.17)\qquad K = R^M_{1212} = c + \alpha\beta - \delta^2,$$

$$(3.18)\qquad K^\perp = R^\perp_{3412} = (\beta - \alpha)\delta,$$

from which follows that the normal curvature of M vanishes if and only if M is pseudoumbilical.

(2.6) implies, using the parallelism of the mean curvature vector and (3.16),

$$(\alpha+\beta)\omega_4^3 = 0, \qquad d(\alpha+\beta) = 0.$$

Hence $(\alpha+\beta)$ is constant, $\omega_4^3 = 0$ and therefore $K^\perp = 0$, i.e. $\alpha = \beta$.

Thus, M is a pseudoumbilical submanifold of N and (by (3.17) and the conformity of h) its Gaussian curvature K is given by:

$$K = c.$$

On the other hand the section e_3 is parallel, isoperimetric and umbilical hence ([1] prop. 5.1 page 124)

$$K = 0,$$

whence, as N is simply connected, $N = \mathbb{R}^4$. Let $\tilde{X}$ be the position vector field of $\mathbb{R}^4$, and let Y be any tangent vector to M; as $H = \alpha e_4$ and α is constant

$$\nabla_Y^{\mathbb{R}^4}(\tilde{X} + \frac{1}{\alpha}e_4) = Y - \frac{1}{\alpha}A_{e_4}(Y) = Y - Y = 0$$

hence $(\tilde{X} + \frac{1}{\alpha}e_4)$ is a constant vector a: therefore M is cointained in the 3-sphere centered at a and with radius $\frac{1}{\alpha} = \frac{1}{\|H\|}$, H is orthogonal to the sphere, which implies that M is minimal in the sphere. Thus, if M is compact, it follows by the same argument as in the proof of Theorem 2, page 70 of [3] (in the case of a surface) that M coincides with the Clifford torus in $\mathbb{R}^4$.

REMARK. One can notice that by (3.18) (if N has constant curvature and M is a non minimal surface with conformal second fundamental form) the vanishing of the normal curvature of M is equivalent to the pseudoumbilicity of M. On the other hand ([1] prop. 2.4 page 179), for a pseudoumbilical submanifolds of codimension 2 in a space form $\|H\|$ is constant if and only H is parallel. The following example shows that there exist pseudoumbilical surfaces of codimension 2 with conformal second fundamental form and H non parallel. An easy computation shows that the surface in $\mathbb{R}^4$

(3.19) $$(u,v) \mapsto e^u(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v)$$

has mean curvature vector parallel to

$$e_4 = -\frac{1}{\sqrt{2}}(\cos v(\sin u + \cos u), \sin v(\sin u + \cos u),$$
$$\cos v(\sin u - \cos u), \sin v(\sin u - \cos u)),$$

and $\|H\| = e^{-u}/\sqrt{2}$.

Hence, if $e_3 = (\sin u \sin v, -\sin u \cos v, -\cos u \sin v, \cos u \cos v)$,

$$H_3 = \frac{e^{-u}}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad H_4 = \frac{e^{-u}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.3 – Surfaces of a 5-dimensional manifold

We choose the Darboux frame $(e_1, e_2, e_3, e_4, e_5)$, so that e_5 is parallel to H (which cannot be zero, as noticed previously), (e_1, e_2) is an orthonormal frame diagonalizing H_5. Thus

$$H_3 = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}, \qquad H_4 = \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix}, \qquad H_5 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

The orthogonality of this matrices yields

$$(\alpha - \beta)\gamma = 0, \qquad (\alpha - \beta)\lambda = 0.$$

$\gamma = \lambda = 0$ being impossible (unless H_3, H_4 and H_5 are not independent), $\alpha = \beta$, i.e. the section H is umbilical. Hence

THEOREM 3.5. *A surface of a 5-dimensional manifold with conformal second fundamental form is pseudoumbilical.*

As this property does not depend on the choice of (e_1, e_2), one can suppose the orhonormal frame (e_1, e_2) such that H_3 is diagonal. Because of the orthogonality of H_3 and H_4 and the equality of the norms of H_3, H_4 and H_5, one can assume that the second fundamental form is represented by

$$(3.20) \quad H_3 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}, \qquad H_5 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

Let now N be a space of constant curvature. (3.20) and (2.5) yield

$$R^{\perp}_{3412} = 2\alpha^2.$$

Hence the normal curvature is not zero. If the mean curvature vector field is parallel, then M is minimal in a small sphere $S^4(r)$ of N (see Proposition 4.2 of [2]), and the section e_5, parallel to H, is normal to $S^4(r)$ on M, whereas e_3 and e_4 are tangent to the 4-sphere. As M has conformal second fundamental form in $S^4(r)$, Theorem 3.2 implies that M is locally immersed in $S^4(r)$ as Veronese surface. By (3.20) and (2.4) the Gaussian curvature K of M as submanifold of N is

$$K = c - \alpha^2 .$$

On the other hand, as H_3 and H_4 represent the second fundamental form of the immersion of M in $S^4(r)$, if M is considered as submanifold of $S^4(r)$, by (3.8):

$$K = \frac{1}{r^2} - 2\alpha^2, \qquad K = \alpha^2, \qquad \frac{1}{r^2} = 3\alpha^2.$$

Hence, set $\alpha = 1/R$, it follows that $r = R/\sqrt{3}$, $c = 2/R^2$. This proves the following

THEOREM 3.6. *Let N be a space form of dimension 5 and let M be a compact surface with conformal second fundamental form and with parallel mean curvature vector field. Then:*

(1) *N is 5-sphere*
(2) *M is a sphere $S^2(R)$ immersed in $S^4(\frac{R}{\sqrt{3}})$ as Veronese surface and $S^4(\frac{R}{\sqrt{3}})$ is immersed in $S^5(\frac{R}{\sqrt{2}})$ as a totally umbilical submanifold, i.e. $S^4(\frac{R}{\sqrt{3}})$ is a section of $S^5(\frac{R}{\sqrt{2}})$ by a hyperplane at a distance $\frac{R}{\sqrt{6}}$ from the centre of $S^5(\frac{R}{\sqrt{2}})$.*

REMARK 1. It is trivial that the parallelism of H implies that $\alpha = \|H\|$ is constant and therefore that the length of h is constant. Conversely, if N has constant sectional curvature and if M has conformal second fundamental form, then $\|H\| =$ constant implies that H is parallel. This result can be proven by means of a straightforward computation, differentiating the forms ω_i^α $(i = 1, 2; \alpha = 3, 4, 5)$ whose coefficients are expressed by (3.20) with $\alpha = \|H\| =$ constant (hence the result is a consequence of the Codazzi conditions). Therefore the Veronese surface immersed in a 5-sphere as in Theorem 5 is the only example of surface immersed in 5-dimensional space of constant curvature with second fundamental form conformal and of constant length.

REMARK 2. A more general version of Theorem 5 can be stated: Let $q: M \longrightarrow \mathbb{R}^5(c)$ be an isometric immersion of a surface with conformal second fundamental form. If N is any Riemannian 4-dimensional manifold and q factors throught a minimal immersion $f: M \longrightarrow N$ and an immersion $g: N \longrightarrow \mathbb{R}^5(c)$ then the length of the second fundamental form of q, h_q, is constant and q is locally the immersion of the 2-sphere in the 5-sphere of Theorem 5.

To prove this, one notices that the mean curvature vector H of q is orthogonal to N. Hence, using the well known identity $h_q = h_f + f^*h_g$, it can be recognized that one can choose a Darboux frame $(e_1, e_2, e_3, e_4, e_5)$ adapted to the two immersions f and g such that the matrices H_3 and H_4 (which represent h_f) and H_5 (which is the matrix of f^*h_g) are in the form (3.20). In particular, for any X and Y tangent to M, $h_g(X,Y) = (X,Y)H$, and the equations of Codazzi for the immersion g yield $\|H\| = \alpha =$ constant.

REMARK 3. If one considers the immersion of $S^2(R)$ in $\mathbb{R}^5$ defined by (3.5), it can be easily verified that, chosen e_5 parallel to H, which is orthogonal to $S^4(\frac{R}{\sqrt{3}})$ one obtains that H_5 is given by

$$H_5 = \frac{\sqrt{3}}{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and that H_3 and H_4 are given by (3.6). Hence the H_i are orthogonal but with different norms. By modifing the canonical metric of $\mathbb{R}^5$ properly, one can obtain an immersion with conformal second fundamental form. If ρ denotes the radial coordinate and (θ^i) $(i = 1, .., 4)$ is an orthonormal coframe of $S^4(1)$, then the canonical metric of $\mathbb{R}^5$ is given by

$$ds^2 = d\rho^2 + \rho^2(\sum(\theta^i)^2).$$

If one considers the metric of $\mathbb{R}^5$

$$ds_1^2 = 3d\rho^2 + \rho^2(\sum(\theta^i)^2),$$

then (3.5) define an immersion in $(\mathbb{R}^5, ds_1^2)$ with conformal second fundamental form. It must be remarked, however, that $(\mathbb{R}^5, ds_1^2)$ is not a space of constant curvature.

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