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Homogeneous representations of the hyperbolic spaces related to homogeneous structures of class $\mathcal{T}_1 \oplus \mathcal{T}_3$

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RIASSUNTO – *Si descrivono le rappresentazioni omogenee degli spazi iperbolici* $\mathbb{H}^n$, $n \geq 4$, *determinate da una struttura Riemanniana omogenea propria appartenente alla classe* $\mathcal{T}_1 \oplus \mathcal{T}_3$ *della classificazione dovuta a F. Tricerri e L. Vanhecke.*

ABSTRACT – *We describe the homogeneous representations of the hyperbolic spaces* $\mathbb{H}^n$, $n \geq 4$, *determined by a proper homogeneous Riemannian structure belonging to the class* $\mathcal{T}_1 \oplus \mathcal{T}_3$ *in the classification given by F. Tricerri and L. Vanhecke.*

KEY WORDS – *Homogeneous Riemannian structures - Hyperbolic spaces.*

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– Introduction

TRICERRI and VANHECKE proved that a simply connected Riemannian manifold admitting a homogeneous Riemannian structure $T \neq 0$, $T \in \mathcal{T}_1$ is isometric to the hyperbolic space $\mathbb{H}^n$, $n \geq 2$, [6].

The related representation is the Lie group with the following product:

$$(x^1, \dots, x^n)(y^1, \dots, y^n) = (x^1y^1, x^1y^2 + x^2, \dots, x^1y^n + x^n)$$

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i.e. a solvable Lie group which is a semidirect product of the multiplicative group $\mathbb{R}^+$ and the additive group $\mathbb{R}^n$, [6].

In loc.cit., the same authors proved that $n = 4$ is the minimum value of the dimension of (M, g) to have the existence of a proper $T \in \mathcal{T}_1 \oplus \mathcal{T}_3$. Here, proper means that $T \notin \mathcal{T}_1$, $T \notin \mathcal{T}_3$. They also gave $\mathbb{H}^4$ as example of simply connected manifold admitting such a structure.

The related representation is the semidirect product

$$\mathbb{H}^4 = \mathbb{R}^3 \rtimes C\mathbb{O}(3)/S\mathbb{O}(3)$$

where $\mathbb{R}^3$ is an abelian group and $C\mathbb{O}(3)$ is the conformal group.

Finally, in [3], we proved that a simply connected Riemannian manifold admitting a proper homogeneous structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_3$ is isometric to the hyperbolic space $\mathbb{H}^n$, $n \geq 4$.

In this paper we are interested in the related representations of $\mathbb{H}^n$, $n \geq 4$, and we will see that they involve the conformal groups.

1 – Preliminaries

We refer to [1], [4], [6] for the following definitions and results.

Let us recall that a homogeneous Riemannian structure is a tensor field T of type (1,2) satisfying the following equations of AMBROSE and SINGER, [1], [6], where $T_X Y$ stands for $T(X, Y)$.

(AS)
$$\text{i)}\, g(T_X Y, Z) + g(T_X Z, Y) = 0$$
$$\text{ii)}\, (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}$$
$$\text{iii)}\, (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$$

for any X, Y, Z belonging to the Lie algebra $\mathcal{H}(M)$ of the tangent vector fields on M. Here, ∇ denotes the Riemannian connection and R its curvature tensor field defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Now, $T \in \mathcal{T}_1 \oplus \mathcal{T}_3$ if and only if the following condition holds, [6],

$$T(X, Y) + T(Y, X) = 2g(X, Y)\xi - g(Y, \xi)X - g(X, \xi)Y \tag{1}$$

where the vector field ξ is defined by $\xi = \frac{1}{n-1} \sum_{i=1}^{n} T(E_i, E_i)$, $(E_1, \ldots, E_n)$ being a local orthonormal basis of $T(M)$, [6].

Condition (1) allows us to write:

$$T(X,Y) = g(X,Y)\xi - g(Y,\xi)X + \pi(X,Y)\,,$$

where π is an alternanting tensor field of type (1,2) on M, [4]. Since we suppose that T is proper, we have $\xi \neq 0$ and $\pi \neq 0$.

Furthermore, the 1-form ω, which is dual of ξ with respect to the metric g, is closed, [4].

We refer to [4] for the following results:

$$\pi(X,\xi) = 0 = \pi(\xi,X) \tag{2}$$

$$g(\pi(X,Y),\xi) = 0 \tag{3}$$

$$g(\pi(X,Y),Z) + g(\pi(X,Z),Y) = 0 \tag{4}$$

for any $X,Y,Z \in \mathcal{H}(M)$.

We know that the maximal integral manifold N of the distribution $\mathcal{D}$, orthogonal to ξ, are isometric to $\mathbb{R}^{n-1}$ and they inherit a homogeneous Riemannian structure $\overline{T} \in \mathcal{T}_3$, $\overline{T} \neq 0$ defined by

$$\overline{T}(X,Y) = \pi(X,Y) \quad \text{for any} \quad X,Y \in \mathcal{H}(N)\,.$$

Now, the focal point is that π verifies the condition

$$\mathop{\mathfrak{S}}_{X,Y,Z} \pi(\pi(X,Y),Z) = 0 \quad \text{for} \quad X,Y,Z \in \mathcal{H}(N) \tag{5}$$

and then N, or equivalently $\mathbb{R}^{n-1}$, becomes a naturally reductive space with Ambrose-Singer connection having an algebraic curvature tensor field, [4], [5]. On the other hand, in [4], we proved that a proper structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_3$ on $\mathbb{H}^n$, $n \geq 4$, can be reconstructed starting from a non vanishing structure $\overline{T} \in \mathcal{T}_3$ on $\mathbb{R}^{n-1}$ whose Ambrose-Singer connection has an algebraic curvature.

This fact allows us to describe the representations of $\mathbb{H}^n$, $n \geq 4$.

2 – Links between the transvection algebras

We need to fix some notation. Consider on $\mathbb{H}^n$, $n \geq 4$, a structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_3$, $T \notin \mathcal{T}_1$, $T \notin \mathcal{T}_3$. Denote by ∇ the Riemannian connection on $\mathbb{H}^n$ and by $\tilde{\nabla} = \nabla - T$ the related Ambrose-Singer connection.

Let $\tilde{\Sigma}, \tilde{R}$ be the torsion and the curvature tensor fields of $\tilde{\nabla}$, respectively.

PROPOSITION 1. *For any X, Y belonging to the distribution $\mathcal{D}$ orthogonal to ξ, we have:*

1) $\tilde{R}(X,\xi) = -\|\xi\|^2 \pi_X$
2) $\tilde{R}(X,Y) = \pi_{\pi(X,Y)}$
3) $[\tilde{R}(X,\xi), \tilde{R}(Y,\xi)] = \|\xi\|^4 \tilde{R}(X,Y)$.

Namely, 1) has been proved in [4]. The second Bianchi identity, $\underset{X,Y,Z}{\mathfrak{S}} \tilde{R}(\tilde{\Sigma}(X,Y),Z) = 0$, applied to the vector fields $X, Y \in \mathcal{D}$ and to ξ, gives

$$\tilde{R}(\tilde{\Sigma}(X,Y),\xi) + \tilde{R}(\tilde{\Sigma}(Y,\xi),X) + \tilde{R}(\tilde{\Sigma}(\xi,X),Y) = 0\,.$$

Since $\tilde{\Sigma}(X,Y) = -2\pi(X,Y)$ and $\tilde{\Sigma}(Y,\xi) = \|\xi\|^2 Y$, we have:

$$\tilde{R}(\pi(X,Y),\xi) + \|\xi\|^2 \tilde{R}(X,Y) = 0\,.$$

Now, (3) implies that $\pi(X,Y) \in \mathcal{D}$, and using 1) we obtain 2).

Finally, 3) follows from 1), 2) and the condition (5) which is equivalent to $[\pi_X, \pi_Y] = \pi_{\pi(X,Y)}$.

Now, fix a point $O \in \mathbb{H}^n$ and the maximal integral manifold N through O. We know that N is isometric to $\mathbb{R}^{n-1}$ and we denote by $\overline{T} \in \mathcal{T}_3$ the induced homogeneous Riemannian structure given by $\overline{T}(X,Y) = \pi(X,Y)$ for any $X, Y \in \mathcal{H}(N)$. Let $\tilde{D} = D - \overline{T}$ be the related Ambrose-Singer connection, where D is the Levi-Civita connection on N (or $\mathbb{R}^{n-1}$). $\tilde{D}$ has torsion tensor $\Sigma_{\tilde{D}} = -2\overline{T}$ and curvature tensor $R_{\tilde{D}}$ which can be expressed as, [3],

$$R_{\tilde{D}}(X,Y)Z = R_D(X,Y)Z + \underset{X,Y,Z}{\mathfrak{S}} \overline{T}(\overline{T}(X,Y),Z) + \overline{T}(\overline{T}(X,Y),Z)\,.$$

Since $R_D = 0$ and the cyclic sum vanishes, it follows that

$$R_{\tilde{D}}(X,Y) = \overline{T}_{\overline{T}(X,Y)}\,. \tag{6}$$

Now, we put $V = T_O(\mathbb{H}^n)$, $V' = T_O(N)$ so that we have the orthogonal decomposition $V = V' \oplus \operatorname{span}\{\xi\}$.

Denote by G/H and G'/H' the homogeneous representations of $\mathbb{H}^n$ and $\mathbb{R}^{n-1}$ which come from the existence of the homogeneous structures T and $\overline{T}$, respectively. Then, for the related Lie algebras, we have the linear direct decompositions, [6]:

I. $\mathfrak{g} = \mathfrak{k} + m$ where $m = V$, and $\mathfrak{k} = \operatorname{span}\{\tilde{R}(X,Y) | X, Y \in V\}$ is a subalgebra of $\mathfrak{so}(n)$. The Lie bracket on $\mathfrak{g}$ is given by:

$$
\begin{aligned}
&[X,Y] = -\tilde{\Sigma}_X Y - \tilde{R}(X,Y) && X,Y \in V\\
&[A,X] = -[X,A] = A(X) && X \in V, A \in \mathfrak{k}\\
&[A,B] = AB - BA && A,B \in \mathfrak{k}.
\end{aligned}
$$

II. $\mathfrak{g}' = \mathfrak{k}' + m'$ where $m' = V'$, $\mathfrak{k}' = \operatorname{span}\{R_{\tilde{D}}(X,Y) | X, Y \in V'\}$ is a subalgebra of $\mathfrak{so}(n-1)$, and the Lie Bracket is given by:

$$
\begin{aligned}
&[X,Y]' = -\Sigma_{\tilde{D}}(X,Y) - R_{\tilde{D}}(X,Y) = \\
&\qquad = 2\overline{T}(X,Y) - \overline{T}_{\overline{T}(X,Y)} && X,Y \in V'\\
&[A,X]' = -[X,A]' = A(X) && X \in V', A \in \mathfrak{k}'\\
&[A,B]' = AB - BA && A,B \in \mathfrak{k}'.
\end{aligned}
$$

Now, put $L = (V, \pi, g)$ and $L' = (V', \overline{T}, g')$ where g' is the induced scalar product on V', and $\overline{T}, g', \pi, g$ are valued at the point O.

PROPOSITION 2. *L and L' are compact real Lie algebras, and L' is a subalgebra of L. Furthermore $Z(L) = Z(L') \oplus \operatorname{span}\{\xi\}$.*

We already know that L' is a compact Lie algebra, [5]. Now, consider L. Since π is an alternanting tensor of type (1,2) on V, satisfying the conditions:

$$\pi_\xi = 0 \quad \text{and} \quad \underset{X,Y,Z \in V'}{\mathfrak{S}} \pi(\pi(X,Y), Z) = 0,$$

we have $\mathfrak{S}_{X,Y,Z\in V}\, \pi(\pi(X,Y),Z) = 0$ and so L is a Lie algebra. Now, condition (4) implies the compactness.

Furthermore, for any $X, Y \in V'$, we have $\overline{T}(X,Y) = \pi(X,Y)$ and L' is a subalgebra of L. Finally, it is easy to shows that we have $Z(L) = Z(L') \oplus \mathrm{span}\{\xi\}$, for the corresponding centers.

PROPOSITION 3. *$\mathfrak{k}'$ is a non trivial ideal of $\mathfrak{k}$.*

Observe that, for any $X, Y, Z \in V'$, $\tilde{R}(X,Y)$ is an endomorphism of V such that $\tilde{R}(X,Y)\xi = 0$, [4], and $\tilde{R}(X,Y)Z = \pi(\pi(X,Y),Z) \in V'$, since (3) holds. It follows that $(\tilde{R}(X,Y)_{|V'})_{\#} = R_{\tilde{D}}(X,Y)\colon V' \longrightarrow V'$. We define $j\colon \mathfrak{k}' \to \mathfrak{k}$ putting $j(R_{\tilde{D}}(X,Y)) = \tilde{R}(X,Y)$ for any $X, Y \in V'$ and extending by linearity.

It is easy to see that j is a monomorphism of Lie algebras, so that we can identify $\mathfrak{k}'$ with $j(\mathfrak{k}')$ in $\mathfrak{k}$.

Now, to show that $\mathfrak{k}'$ is an ideal in $\mathfrak{k}$, it suffices to verify that $[A,B] \in \mathfrak{k}'$ for any $A = \tilde{R}(X,Y)$ and $B = \tilde{R}(Z,\xi)$ with $X, Y, Z \in V'$.

Namely, by proposition 1, we have

$$[\tilde{R}(X,Y), \tilde{R}(Z,\xi)] = -\|\xi\|^2[\pi_{\pi(X,Y)}, \pi_Z] = -\|\xi\|^2\tilde{R}(\pi(X,Y),Z) \in \mathfrak{k}'\,.$$

Obviously, $\mathfrak{k}'$ is not trivial. In fact, from [5] we know that $R_{\tilde{D}} = 0$ would imply $\overline{T} = 0$ and then, since $\pi_\xi = 0$, we would have $\pi = 0$ and $T \in \mathcal{T}_1$, a contradiction.

PROPOSITION 4. *The Lie algebras $\mathfrak{k}$ and $\mathfrak{k}'$ coincide.*

Condition 3) of proposition 1 implies that $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}'$.

On the other hand, L and L' are compact Lie algebras, then their adjoint algebras are semisimple. Since proposition 1 implies that $\mathfrak{k} = \mathrm{ad}(L)$, we obtain $\mathfrak{k} = [\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}'$ and $\mathfrak{k} = \mathfrak{k}'$.

3 – Representations of $\mathbb{H}^n$

We refer to the decomposition I in section 2.

Since L and L' are compact Lie algebras, we have the decomposition:

$$L = L_1 \oplus L_2 \oplus \ldots L_s \oplus Z(L') \oplus \mathrm{span}\{\xi\}$$

where L_i, $i = 1, \ldots, s$, are simple ideals and the center $Z(L')$ may be trivial. Note that the above decomposition is orthogonal with respect to the scalar product g.

The corresponding decomposition of $\mathrm{ad}(L) = \mathfrak{k}$ in simple ideals is given by

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \ldots \oplus \mathfrak{k}_s$$

where each $\mathfrak{k}_i$ can be considered as the adjoint algebra of L_i, so that it acts trivially on each L_j, $j \neq i$, and on $Z(L') \oplus \mathrm{span}\{\xi\}$.

We put

$$n_i = \dim \mathfrak{k}_i = \dim L_i\,, \quad i = 1, \ldots, s \quad \text{and}$$

$$r = \sum_{i=1}^{s} n_i = \dim \mathfrak{k} = \dim \mathfrak{k}' \leq n - 1\,.$$

We choose an orthonormal basis $(E_1, \ldots, E_r, E_{r+1}, \ldots, E_{n-1}, E_n)$ of L such that

$$(E_1, \ldots, E_r) \qquad \text{is a basis of} \quad L_1 \oplus \ldots \oplus L_s$$

$$(E_{r+1}, \ldots, E_{n-1}) \quad \text{is a basis of} \quad Z(L')\,, \text{and} \quad E_n = \xi / \|\xi\|\,.$$

It follows easily that $(\pi_{E_1}, \ldots, \pi_{E_r})$ is a basis of the Lie algebra $\mathfrak{k}$ and then $(E_1, \ldots, E_n, \pi_{E_1}, \ldots, \pi_{E_r})$ is a basis of the Lie algebra $\mathfrak{g}$.
Now, we construct a new basis $(X_1, \ldots, X_n, \pi_{E_1}, \ldots, \pi_{E_r})$ of $\mathfrak{g}$, putting:

$$X_i = E_i - \pi_{E_i} \quad \text{for any} \quad i = 1, \ldots, r$$

$$X_i = E_i \qquad \text{for any} \quad i = r + 1, \ldots, n\,.$$

Consequently, the Lie bracket in $\mathfrak{g}$ is determined by:

$$[X_i, X_j] = 0 \qquad i, j = 1, \ldots, r$$

$$[X_i, X_j] = \pi(E_i, E_j) - \pi_{\pi(E_i, E_j)} = 0 \quad i = 1, \ldots, r,\ j = r + 1, \ldots, n - 1$$
$$\text{since } X_j = E_j \in Z(L)$$

$$[X_i, X_j] = 0 \qquad i,j = r+1,\dots,n-1$$

$$[X_n, X_i] = \|\xi\| X_i \qquad i = 1,\dots,n-1$$

$$[\pi_{E_i}, X_j] = \pi(E_i, E_J) - \pi_{\pi(E_i,E_j)} \qquad i,j = 1,\dots,r$$

$$[\pi_{E_i}, X_j] = \pi(E_i, E_j) = 0 \qquad i = 1,\dots,r,\ j = r+1,\dots,n-1,$$
since $E_J \in Z(L)$

$$[\pi_{E_i}, X_n] = 0 \qquad i = 1,\dots,r$$

$$[\pi_{E_i}, \pi_{E_j}] = \widetilde{R}(E_i, E_j) \qquad i,j = 1,\dots,r.$$

Now, observe that for $i,j = 1,\dots,r$, we have:

$$\begin{cases} \pi(E_i, E_j) = 0 & \text{if} \quad E_i \in L_h \quad \text{and} \quad E_j \in L_k \quad \text{with} \quad h \neq k \\ \pi(E_i, E_j) \in L_h & \text{if} \quad E_i, E_j \in L_h \end{cases}$$

In any case, we can write $\pi(E_i, E_j) = \sum_{t=1}^{r} a_{ij}^t E_t$ and we obtain

$$[\pi_{E_i}, X_j] = \sum_{t=1}^{r} a_{ij}^t (E_t - \pi_{E_t}) = \sum_{t+1}^{r} a_{ij}^t X_t$$

so that

$$[\pi_{E_i}, X_j] \in \operatorname{span}\{X_1, \dots, X_{n-1}\}.$$

Therefore, we have that $\mathrm{I} = \operatorname{span}\{X_1, \dots, X_{n-1}\}$ is an abelian ideal of $\mathfrak{g}$.

Now, we know that the conformal group $C\mathbb{O}(n-1) = \mathbb{R}^+ \times \mathbb{O}(n-1)$, where $\mathbb{R}^+$ denotes the multiplicative group of the real positive numbers, has Lie algebra $\mathfrak{co}(n-1) = \mathbb{R} \oplus \mathfrak{so}(n-1)$ where $\mathbb{R}$ is the additive group of the real numbers. Since $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{so}(n-1)$, we denote by $\mathfrak{c}(\mathfrak{t})$ the Lie subalgebra $\mathbb{R} \oplus \mathfrak{t}$ of $\mathfrak{co}(n-1)$, by H the connected Lie group having $\mathfrak{t}$ as Lie algebra and by $C(H)$ the connected Lie subgroup of $C\mathbb{O}(n-1)$ having $\mathfrak{c}(\mathfrak{t})$ as its Lie algebra.

Therefore, the above table of the Lie bracket of $\mathfrak{g}$ says that $\mathfrak{g}$ is a semidirect sum of the abelian ideal I and $\mathfrak{c}(\mathfrak{t})$ i.e.

$$\mathfrak{g} = \mathbb{R}^{n-1} \oplus_\phi \mathfrak{c}(\mathfrak{t})$$

where $\mathbb{R}^{n-1}$ is abelian.

The corresponding representation for $\mathbb{H}^n$ is given by the semidirect product of groups $\mathbb{H}^n = \mathbb{R}^{n-1} \rtimes C(H)/H$.

Finally, note that, for $n = 4$, the Lie algebra $\mathfrak{k}$ coincides with $\mathfrak{so}(3)$ since it has to be compact and semisimple in $\mathfrak{so}(3)$, and we reobtain the representation of $\mathbb{H}^4$ given by Tricerri and Vanhecke.

In dimension $n \geq 7$, we can have more than one representation of $\mathbb{H}^n$, depending on the representations of $\mathbb{R}^{n-1}$ as naturally reductive space with Ambrose-Singer connection having an algebraic curvature tensor field, [5].

REFERENCES

[1] W. Ambrose - I.M. Singer: *On homogeneous Riemannian manifolds*, Duke Math. J. 25, (1958), 647-669.

[2] M. Goto - F.D. Grosshans: *Semisimpe Lie algebras*, Lect. Notes in Pure and Appl. Math. 38, (1978).

[3] O. Kowalski - L. Vanhecke: *Four-dimensional naturally reductive homogeneous spaces*, Rend. Sem. Mat. Univ. Torino, fascicolo speciale (Sept. 1983).

[4] A.M. Pastore: *On the homogeneous Riemannian structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$*, Geom. Ded. 30, (1989), 235-246.

[5] A.M. Pastore: *Canonical connections with an algebraic curvature tensor field on naturally reductive spaces*, preprint.

[6] F. Tricerri - L. Vanhecke: *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lect. Note Series 83, (1983).

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