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On zeros of Sobolev-type orthogonal polynomials

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RIASSUNTO – *In questo lavoro si analizzano alcune proprietà sugli zeri dei polinomi ortogonali* $Q_n(x)$ *associati al prodotto interno definito da*

$$\langle f,g\rangle = \int_I f(x)g(x)d\mu(x) + \lambda f'(c)g'(c)$$

dove I *è un intervallo reale (non necessariamente limitato),* μ *è una misura positiva,* $c \in \mathbb{R}$ *e* $\lambda \geq 0$. *In particolare si ottengono alcune proprietà di localizzazione e separazione per le radici di* $Q_n(x)$. *Si studia il comportamento degli zeri rispetto a* λ *quando* $c \notin I$.

ABSTRACT – *In this paper we analyze some properties concerning the zeros of orthogonal polynomials* $Q_n(x)$ *associated to the inner product defined by*

$$\langle f,g\rangle = \int_I f(x)g(x)d\mu(x) + \lambda f'(c)g'(c)$$

where I *is a (not necessarily bounded) real interval,* μ *is a positive measure on* I, $c \in \mathbb{R}$ *and* $\lambda \geq 0$. *In particular, some properties of localization and separation for the roots of* $Q_n(x)$ *are obtained. The behavior of the zeros with respect to* λ *is studied when* $c \notin I$.

KEY WORDS – *Orthogonal polynomials - Sobolev spaces - Zeros.*

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1 – Introduction

The study of orthogonal polynomials related to inner products defined on Sobolev spaces appears in a natural fashion when dealing with least square problems of approximation for functions of class $C^{(k)}(I)$, where I is a (not necessarily bounded) real interval. These problems were introduced in [11] and analyzed in detail in later papers (see [7] and [10]).

It is a well known fact that certain properties (for instance, real nature, interlacing) of the zeros of standard orthogonal polynomials , (i.e., those associated to a finite and positive Borel measure defined on a real interval), are connected to the self-adjoint character of the shift operator, or, equivalently, to the fact that these zeros are eigenvalues of certain Jacobi symmetric matrices.

Most of these properties do not appear in the case of inner products defined on Sobolev spaces. However, sometimes it is possible to find properties similar to the above mentioned ones for standard polynomials.

In [2] and [5], orthogonal polynomials Q_n with respect to the inner product in $\mathbb{P}$ given by

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx + \lambda \int_{-1}^{1} f'(x)g'(x)dx$$

have been considered and it has been shown that the zeros of the orthogonal polynomials Q_n are simple and belong to [-1,1]; moreover, they interlace with the zeros of Legendre polynomials P_{n-1} if $\lambda \geq 2/n$. Analogously, Brenner (see [4]) has studied the distribution for the zeros of the orthogonal polynomials according to the inner product in $\mathbb{P}$ given by

$$\langle f,g\rangle = \int_{0}^{\infty} f(x)g(x)e^{-x}dx + \lambda \int_{0}^{\infty} f'(x)g'(x)e^{-x}dx .$$

It is shown that these zeros are real, positive and simple.

In a different framework, in [9] orthogonal polynomials with respect to

$$\langle f,g\rangle = \int_{0}^{\infty} f(x)g(x)e^{-x}dx + \lambda f'(0)g'(0)$$

have been considered. It has been proved that, all the roots of orthogonal polynomials, with the possible exception of one, are positive, and also simple.

For the symmetric measure $d\mu = (1-x^2)^\alpha dx$ in $[-1,1]$, BAVINCK and MEIJER (see [3]) have shown that zeros of the orthogonal polynomials with respect to:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)^\alpha dx + M[f(1)g(1) + f(-1)g(-1)] + \\ + N[f'(1)g'(1) + f'(-1)g'(-1)]$$

are real and simple. If $N \neq 0$, and for n sufficiently large, Q_n has exactly two opposite real zeros lying outside of $(-1,1)$.

In a wider context, in [13] algebraic and differential properties for orthogonal polynomials with respect to

$$\langle f, g\rangle = \int_I f(x)g(x)d\mu(x) + \lambda f^{(r)}(c)g^{(r)}(c)$$

are studied, when $r \in \mathbb{N}$, $\lambda \in \mathbb{R}^+$ and $c \in \mathbb{R}$. We must note that the preceding authors impose a restriction on the location of I and c. More recently, H. MEIJER (see [13]) analyzes the distribution of the zeros for such polynomials in the particular case: $I = \mathbb{R}^+$ and $c = 0$, as a natural generalization of [8].

Finally, in [1] the inner product

$$\langle f, g\rangle = \int_I f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c)$$

is considered, in such a way that, for $c = \sup I$ or $c = \inf I$, the real and simple nature of the roots is proved. Separation properties are obtained, when I is a symmetric interval, $c = 0$ and μ is a symmetric measure.

The aim of this paper is to study the properties of the zeros of the orthogonal polynomials Q_n, for the inner product

$$\langle f, g\rangle = \int_I f(x)g(x)d\mu(x) + \lambda f'(c)g'(c) \tag{1.1}$$

where I is a (not necessarily bounded) interval, μ is a positive Borel measure on I, $\lambda \in \mathbb{R}^+$ and c is a real number.

In section 2, we recall some algebraic properties for these polynomials. These properties have been obtained in [12] for a weight function and extended in a natural way to positive Borel measures. In particular, a representation in terms of the orthogonal polynomials associated to μ is obtained. These formulas will be very useful in the next sections.

In section 3, the zeros of these Sobolev type orthogonal polynomials are studied; we show that, for $c \notin I$, $Q_n(x)$ has n real and simple zeros, and at least $n-1$ of them are on I.

Next, for a bounded interval, we show that for n sufficiently large and $c \geq \sup I$, $Q_n(x)$ has one root at the right side of c. This root converges to c when n tends to $+\infty$. Thus c becomes an attractor for this root of $Q_n(x)$.

In the next section, we prove that the zeros of $Q_n(x)$ separate the zeros of $P_n(x)$, the n-th orthogonal polynomial associated to μ; moreover, they are separated by the zeros of the generalized kernel $K_{n-1}^{(0,1)}(x,c)$.

Finally, we consider the behavior of the zeros of $Q_n(x)$ with respect to λ, showing that, for $c \geq \sup I$, the zeros of $Q_n(x)$ are an increasing and bounded function of λ.

2 – The representation of the polynomials

Let μ be a positive Borel measure on the interval I, let $c \in \mathbb{R}$ and $\lambda \in \mathbb{R}_0^+$. Consider the inner product

$$\langle f,g\rangle = \int_I f(x)g(x)d\mu(x) + \lambda f'(c)g'(c) \tag{2.1}$$

defined on the linear space $\mathbb{P}$ of the real polynomials.

Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be the sequences of monic orthogonal polynomials (MOPS) associated to μ and $\langle\cdot,\cdot\rangle$, respectively.

We will denote by $K_n^{(r,s)}(x,y)$ the generalized kernel

$$K_n^{(r,s)}(x,y) = \frac{\partial^r}{\partial x^r}\frac{\partial^s}{\partial y^s}K_n(x,y)\,.$$

where

$$K_n(x,y)=\sum_{j=0}^{n}\frac{P_j(x)P_j(y)}{\|P_j\|^2}.$$

From the orthogonality conditions we are able to obtain a representation of $Q_n(x)$ in terms of the $P_n(x)$:

PROPOSITION 2.1. (F. MARCELLÁN and A. RONVEAUX, [12])

$$Q_n(x)=P_n(x)-\frac{\lambda P_n'(c)}{1+\lambda K_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c). \tag{2.2}$$

From the Christoffel-Darboux relation

$$(x-y)K_{n-1}(x,y)=\frac{1}{\|P_{n-1}\|^2}[P_n(x)P_{n-1}(y)-P_{n-1}(x)P_n(y)] \tag{2.3}$$

(see [6], page 23, th. 4.5) by derivation with respect to the variable y, and evaluating in $y=c$, we obtain

$$\begin{aligned}&(x-c)^2K_{n-1}^{(0,1)}(x,c)=\\ &=\frac{1}{\|P_{n-1}\|^2}[P_n(x)T_1(P_{n-1},c)(x)-P_{n-1}(x)T_1(P_n,c)(x)]\end{aligned} \tag{2.4}$$

where $T_i(P_j,c)(x)$ denotes the Taylor polynomial of degree i associated to $P_j(x)$ in c.

By substitution in (2.2), we get a formula relating $Q_n(x)$, $P_n(x)$ and $P_{n-1}(x)$

PROPOSITION 2.2. (F. MARCELLÁN and A. RONVEAUX, [12])

$$(x-c)^2Q_n(x)=q_2(x,n)P_n(x)+q_1(x,n)P_{n-1}(x) \tag{2.5}$$

with

$$q_2(x,n)=(x-c)^2-\lambda\frac{Q_n'(c)}{\|P_{n-1}\|^2}T_1(P_{n-1},c)(x)$$

$$q_1(x,n)=\lambda\frac{Q_n'(c)}{\|P_{n-1}\|^2}T_1(P_n,c)(x)$$

Finally, substituting n by $n+1$ in (2.4) and by elimination of $K_{n-1}^{(0,1)}(x,c)$, we obtain a determinantal relation which can be used to compute recursively the polynomials $Q_n(x)$

PROPOSITION 2.3. (F. MARCELLÁN and A. RONVEAUX, [12])

$$\begin{vmatrix} P_{n+1}(x) & P_n(x) \\ P'_{n+1}(c) & P'_n(c) \end{vmatrix} = \lambda_{n-1} \begin{vmatrix} Q_{n+1}(x) & Q_n(x) \\ Q'_{n+1}(c) & Q'_n(c) \end{vmatrix} \tag{2.6}$$

where $\lambda_{n-1} = 1 + \lambda K_{n-1}^{(1,1)}(c,c)$.

3 – The zeros of Q_n

The standard properties for the zeros of orthogonal polynomials are not longer valid for this inner product. However, we can state some particular results.

PROPOSITION 3.1. *For $n \geq 3$, the polynomial $Q_n(x)$ has at least $n-2$ different zeros with odd multiplicity in $\overset{0}{I}$.*

PROOF. Let $y_{n.1}, \ldots, y_{n,k}$ denote the different zeros of $Q_n(x)$ of odd multiplicity which are in $\overset{0}{I}$. Define

$$p(x) = (x - y_{n,1}) \ldots (x - y_{n.k}),$$

then the polynomial $Q_n(x)p(x)(x-c)^2$ does not change sign in the interval I, and hence:

$$\langle Q_n(x), p(x)(x-c)^2 \rangle = \int_I Q_n(x)p(x)(x-c)^2 d\mu(x) \neq 0 .$$

Since Q_n is an orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$, it follows that $\deg(p) = k \geq n-2$. □

PROPOSITION 3.2. *If $c \notin \overset{0}{I}$, the zeros of $Q_n(x)$ are real, simple and at least $n-1$ of them are on $\overset{0}{I}$.*

PROOF. Suppose $c \geq \sup I$. Let $y_{n,1}, \dots, y_{n,k}$ denote the different zeros of $Q_n(x)$ of odd multiplicity which are in $\overset{0}{I}$ and

$$p(x) = (x - y_{n,1}) \dots (x - y_{n,k}).$$

Let α be a constant such that the polynomial $\omega(x) = p(x)(x - \alpha)$ satisfies $\omega'(c) = 0$; i.e., $\alpha = c + p(c)/p'(c)$.

If $c \geq \sup I$, then $p(c)/p'(c) > 0$ and $\alpha > c$. Hence, $\alpha \notin I$ and the polynomial $Q_n(x)\omega(x)$ does not change sign in I, and

$$\langle Q_n(x), \omega(x) \rangle = \int_I Q_n(x)\omega(x) d\mu(x) \neq 0.$$

Therefore $\deg(\omega) \geq n$, and at least $n-1$ of the zeros are real, simple and are on $\overset{0}{I}$; obviously, the remaining root is real and simple too. □

REMARK. Thus, all of the roots are real and simple; $n - 1$ of them are contained in I and, although we cannot state the position of the remaining root, it is possible, however, to give his situation with respect to the interval I.

For simplicity, from now on, we will suppose that $c \geq \sup I$, but we can obtain analogous results for $c \leq \inf I$.

PROPOSITION 3.3. *If $c \geq \sup I$ and $Q_n(x)$ has a root which is not on I, then this root is greater than* $\sup I$.

PROOF. Let $y_{n,1}, \dots, y_{n,n}$ be the zeros of $Q_n(x)$, and denote by $y_{n,1}, \dots, y_{n,n-1}$ those contained in I.

Define $\omega(x) = (x - y_{n,1}) \dots (x - y_{n,n-1})$, then

$$\langle Q_n(x), \omega(x) \rangle = \int_I Q_n(x)\omega(x) d\mu(x) + \lambda Q_n'(c)\omega'(c) = 0$$

and from

$$Q_n'(c) = \frac{P_n'(c)}{1 + \lambda K_{n-1}^{(1,1)}(c, c)} > 0$$

we get

$$\int_I Q_n(x)\omega(x)d\mu(x) = -\lambda Q_n'(c)\omega'(c) < 0 .$$

Since $Q_n(x)\omega(x)$ does not change sign in I, we have $Q_n(x)\omega(x) < 0$ for x in I, that is: $x - y_{n,n} < 0$ for $x \in I$ and then $y_{n,n} > \sup I$. □

4 – The greatest root of Q_n

The study of the position of the greatest zero of $Q_n(x)$ needs the knowledge of the character of the sequence $\left\{\frac{Q_n(c)}{Q_n'(c)}\right\}_n$

PROPOSITION 3.4. *If $c \geq \sup I$, then $\{\frac{Q_n(c)}{Q_n'(c)}\}_n$ is a decreasing sequence.*

PROOF. From the relation (2.7) we get

$$P_n(c)P_{n+1}'(c) - P_{n+1}(c)P_n'(c) = \lambda_{n-1}\Big[Q_n(c)Q_{n+1}'(c) - Q_{n+1}(c)Q_n'(c)\Big] .$$

From the confluent form for the Christoffel-Darboux relation we deduce the positivity of the previous expression and from the positivity of λ_{n-1}, $Q_n'(c)$ and $Q_{n+1}'(c)$ the result follows. □

An interesting consequence of this proposition is that if there exists a N such that $Q_N(c) < 0$, then $Q_n(c) < 0$ for $n \geq N$; that is, if some polynomial has a root at the right side of c, the same will occur for the rest of the polynomials.

We will show that for a finite interval and n sufficiently large, $Q_n(x)$ satisfies $Q_n(c) < 0$; and hence, $Q_n(x)$ has a root greater than c.

From (2.2) we deduce

$$Q_n(c) = P_n(c) - \frac{\lambda P_n'(c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(c,c) =$$

$$= \frac{P_n(c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)}\left\{1 - \lambda\left[\frac{P_n'(c)}{P_n(c)}K_{n-1}^{(0,1)}(c,c) - K_{n-1}^{(1,1)}(c,c)\right]\right\}$$

We denote by

$$A_n = \frac{P_n'(c)}{P_n(c)} K_{n-1}^{(0,1)}(c,c) - K_{n-1}^{(1,1)}(c,c) \tag{4.1}$$

therefore

$$Q_n(c) = \frac{P_n(c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)} [1 - \lambda A_n] \tag{4.2}$$

and the sign of $Q_n(c)$ depends only on the value $1-\lambda A_n$. Now we consider the sequence $\{A_n\}$.

PROPOSITION 4.2. *$\{A_n\}_n$ is an increasing and positive sequence.*

PROOF. Let $P_{n-1}^c(x)$ denote the $(n-1)$-th monic orthogonal polynomial with respect to the modification of the measure $(x-c)^2 d\mu(x)$; it is a well know fact that $P_{n-1}^c(x)$ can be expressed as

$$P_{n-1}^c(x) = \frac{P_n'(c) K_{n-1}(x,c) - P_n(c) K_{n-1}^{(0,1)}(x,c)}{K_{n-1}(c,c)}.$$

Then

$$\begin{aligned} A_n &= \frac{1}{P_n(c)} \left[P_n'(c) K_{n-1}^{(0,1)}(c,c) - P_n(c) K_{n-1}^{(1,1)}(c,c) \right] = \\ &= \frac{1}{P_n(c)} (P_{n-1}^c)'(c) K_{n-1}(c,c) > 0 \end{aligned} \tag{4.3}$$

and the sequence is positive.

To show the increasing character of the sequence, it is enough to recall the confluent form of the Christoffel-Darboux relation; from this we deduce that the sequence

$$\left\{ \frac{P_n'(c)}{P_n(c)} \right\}_n$$

is increasing. Thus

$$\frac{P_n'(c)}{P_n(c)} K_n^{(0,1)}(c,c) < \frac{P_{n+1}'(c)}{P_{n+1}(c)} K_n^{(0,1)}(c,c)$$

$$\frac{P_n'(c)}{P_n(c)} \left[K_{n-1}^{(0,1)}(c,c) + \frac{P_n(c)P_n'(c)}{\|P_n\|^2} \right] < \frac{P_{n+1}'(c)}{P_{n+1}(c)} K_n^{(0,1)}(c,c)$$

therefore,

$$\frac{P_n'(c)}{P_n(c)} K_{n-1}^{(0,1)}(c,c) - K_{n-1}^{(1,1)}(c,c) < \frac{P_{n+1}'(c)}{P_{n+1}(c)} K_n^{(0,1)}(c,c) - K_n^{(1,1)}(c,c)$$

that is, $A_n < A_{n+1}$. □

LEMMA 4.3. *Let* $B_n = \dfrac{P_n'(c)}{P_{n-1}^c(c) K_{n-1}(c,c)}$. *Then* $\{B_n\}_n$ *is a decreasing and positive sequence.*

PROOF. Obviously $B_n > 0$, $\forall n \in \mathbb{N}$.
On the other hand, as in the previous proof, we have

$$\frac{P_{n+1}(c)}{P_{n+1}'(c)} K_n^{(0,1)}(c,c) < \frac{P_n(c)}{P_n'(c)} K_n^{(0,1)}(c,c)$$

$$\frac{P_{n+1}(c)}{P_{n+1}'(c)} K_n^{(0,1)}(c,c) < \frac{P_n(c)}{P_n'(c)} \left[K_{n-1}^{(0,1)}(c,c) + \frac{P_n(c)P_n'(c)}{\|P_n\|^2} \right]$$

$$\frac{P_{n+1}(c)}{P_{n+1}'(c)} K_n^{(0,1)}(c,c) - K_n(c,c) < \frac{P_n(c)}{P_n'(c)} K_{n-1}^{(0,1)}(c,c) - K_{n-1}(c,c)$$

Thus, the sequence $\left\{ K_{n-1}(c,c) - \frac{P_n(c)}{P_n'(c)} K_{n-1}^{(0,1)}(c,c) \right\}$ is increasing. And from this

$$B_n = \frac{P_n'(c)}{P_{n-1}^c(c) K_{n-1}(c,c)} = \frac{1}{K_{n-1}(c,c) - \frac{P_n(c)}{P_n'(c)} K_{n-1}^{(0,1)}(c,c)} >$$

$$> \frac{1}{K_n(c,c) - \frac{P_{n+1}(c)}{P_{n+1}'(c)} K_n^{(0,1)}(c,c)} = B_{n+1} .$$ □

LEMMA 4.4. $\{A_n B_n\}_n$ *is an increasing and positive sequence.*

PROOF. By using (4.3), we have

$$A_n B_n = \frac{P_n'(c)}{P_n(c)} \frac{(P_{n-1}^c)'(c)}{P_{n-1}^c(c)}.$$

And it suffices to remember that the sequences

$$\left\{\frac{P_n'(c)}{P_n(c)}\right\}_n, \quad \left\{\frac{(P_n^c)'(c)}{P_n^c(c)}\right\}_n$$

are both increasing. □

THEOREM 4.5. *If the orthogonality interval of the MOPS $\{P_n(x)\}_n$ is bounded, then:*

$$\exists n_0 \in \mathbb{N} : \mathrm{Q_n(c)} < 0 \quad \forall \mathrm{n} \geq \mathrm{n_0}.$$

That is, there exists a non-negative integer n_0, such that for $n \geq n_0$ the polynomials $Q_n(x)$ have their greatest zero at the right side of c.

PROOF. If we denote by

$$x_{n,1} < x_{n,2} < \ldots < x_{n,n}$$

the roots of $P_n(x)$ in increasing order, we have

$$\frac{P_n'(c)}{P_n(c)} > \sum_{i=1}^{n} \frac{1}{c - x_{n,i}} > \frac{n}{c - \xi_1}$$

where ξ_1 denotes the lower bound of the true interval of orthogonality for the polynomials $P_n(x)$. Thus, the sequence

$$\left\{\frac{P_n'(c)}{P_n(c)}\right\}$$

diverges positively, and the sequence $\{A_n B_n\}_n$ diverges positively, too.

By the previous results, $\{A_n\}_n$ is increasing and $\{B_n\}_n$ is decreasing, therefore $\{A_n\}_n$ diverges. Finally, from the expression (4.2):

$$Q_n(c) = \frac{P_n(c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)}[1 - \lambda A_n]$$

we get the result. □

REMARK. The conclusions of the above theorem also can be obtained from the divergence of the sequence

$$\left\{\frac{P_n'(c)}{P_n(c)}\right\}_n .$$

From the previous proof, we deduce that this occurs when the true interval of orthogonality for the polynomials $P_n(x)$ is bounded or if $x_{n,1}$ is an infinite of order smaller than n.

In these conditions, it is possible to prove that c is an attractor for the greatest root of $Q_n(x)$

PROPOSITION 4.6. *If $Q_n(c) < 0$, the following inequality holds:*

$$c < y_{n,n} < c + \frac{c-\xi_1}{n-1}$$

and thus $\{y_{n,n}\}$ converges to c.

PROOF. From the inequality

$$0 > \frac{Q_n(c)}{Q_n'(c)} = \sum_{j=1}^{n-1} \frac{1}{c-y_{n,j}} - \frac{1}{y_{n,n}-c}$$

we deduce

$$\frac{1}{y_{n,n}-c} > \sum_{j=1}^{n-1} \frac{1}{c-y_{n,j}} > \frac{n-1}{c-y_{n,1}} > \frac{n-1}{c-\xi_1}$$

and thus

$$y_{n,n} < c + \frac{c-\xi_1}{n-1}. \qquad \square$$

Analogous results for $I = [0, +\infty]$ and $c = 0$ have been obtained in [13]. In the case under consideration, c has not to be the end of the interval.

It is still an open question if, for an unbounded interval, it can be found a N such that $Q_N(c) < 0$. For this problem, we have a partial answer: for an arbitrary non-negative integer N, there is a $\lambda_0 \in \mathbb{R}^+$, such that for $\lambda \geq \lambda_0$ the polynomial $Q_N(x)$, orthogonal with respect to the corresponding inner product, satisfies $Q_N(c) < 0$. Thus, we can always find inner products such that the last root of $Q_n(x)$ is greater than c, even when I is not bounded.

PROPOSITION 4.7. *For a fixed $N \in \mathbb{N}$, there exist a λ such that $Q_n(c) < 0$ for $n \geq N$.*

PROOF. From the expression (4.2) we have:

$$Q_n(c) = \frac{P_n(c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)}[1 - \lambda A_n]$$

and the result becomes evident since $A_n > 0$. □

5 – Separation properties for the roots of Q_n

LEMMA 5.1. *The polynomial $K_{n-1}^{(0,1)}(x,c)$ has $n-1$ real and simple zeros which separate those of $P_n(x)$ (and thus they are on I).*

PROOF. From the Christoffel-Darboux relation we get

$$(x-c)^2 K_{n-1}^{(0,1)}(x,c) = \frac{1}{\|P_{n-1}\|^2}\Big[P_n(x)T_1(P_{n-1},c)(x) - P_{n-1}(x)T_1(P_n,c)(x)\Big].$$

It will be sufficient to prove that the right term changes sign between any two consecutive zeros of $P_n(x)$ and since $P_n(x)$ has n real and simple roots, the result will be obvious.

Let $x_{n,1} < \ldots < x_{n,n}$ be the zeros of $P_n(x)$. For $x > x_{n,n}$, $P_n(x)$ is a convex function and therefore:

$$T_1(P_n,c)(x) \leq P_n(x) \qquad \text{for} \quad x \geq x_{n,n}$$

in particular $T_1(P_n,c)(x_{n,n}) < 0$ and the single root of $T_1(P_n,c)(x)$ is at the right side of $x_{n,n}$. Thus

$$T_1(P_n,c)(x_{n,i}) < 0 \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

Consequently

$$(x_{n,i} - c)^2 K_{n-1}^{(0,1)}(x_{n,i}, c) = -\frac{1}{\|P_{n-1}\|^2} P_{n-1}(x_{n,i}) T_1(P_n, c)(x_{n,i})$$

and

$$(x_{n,i+1} - c)^2 K_{n-1}^{(0,1)}(x_{n,i+1}, c) = -\frac{1}{\|P_{n-1}\|^2} P_{n-1}(x_{n,i+1}) T_1(P_n, c)(x_{n,i+1})$$

have opposite sign. In fact $P_{n-1}(x_{n,i})P_{n-1}(x_{n,i+1}) < 0$, because of the separation properties for the zeros of the standard orthogonal polynomials $P_n(x)$ and $P_{n-1}(x)$. □

PROPOSITION 5.2. *The zeros of $Q_n(x)$ separate those of $P_n(x)$ in the following way*

$$x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} < \ldots < x_{n,n} < y_{n,n} .$$

Moreover, the zeros of $Q_n(x)$ are separated by the zeros of $K_{n-1}^{(0,1)}(x, c)$

PROOF. Denote:

$x_{n,1} < x_{n,2} < \ldots < x_{n,n}$	the zeros of	$P_n(x)$
$y_{n,1} < y_{n,2} < \ldots < y_{n,n}$	the zeros of	$Q_n(x)$
$z_{n-1,1} < z_{n-1,2} < \ldots < z_{n-1,n-1}$	the zeros of	$K_{n-1}^{(0,1)}(x, c)$

From (2.2) and Lemma 5.1 we deduce that

$$Q_n(x_{n,i}) = -\lambda Q_n'(c) K_{n-1}^{(0,1)}(x_{n,i}, c)$$

and

$$Q_n(x_{n,i+1}) = -\lambda Q_n'(c) K_{n-1}^{(0,1)}(x_{n,i+1}, c)$$

have opposite sign, because between $x_{n,i}$ and $x_{n,i+1}$ there exists a unique zero of $K_{n-1}^{(0,1)}(x, c)$.

Then, each interval $]x_{n,i}, x_{n,i+1}[$ contains one zero of $Q_n(x)$.

Finally
$$Q_n(x_{n,n}) = -\lambda Q'_n(c) K_{n-1}^{(0,1)}(x_{n,n}, c) < 0$$
since $K_{n-1}^{(0,1)}(x, c) > 0$, $\forall x > z_{n-1,n-1}$.

Hence, we have proved

$$x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} < \dots < x_{n,n} < y_{n,n}\,.$$

This relation implies that

$$P_n(y_{n,i}) P_n(y_{n,i+1}) < 0$$

and therefore,
$$K_{n-1}^{(0,1)}(y_{n,i}, c) K_{n-1}^{(0,1)}(y_{n,i+1}, c) < 0\,.$$
We conclude that between any two consecutive zeros of $Q_n(x)$ there exists a zero of $K_{n-1}^{(0,1)}(x, c)$. □

REMARK. The ordering of the zeros is

$$x_{n,1} < y_{n,1} < z_{n-1,1} < x_{n,2} < y_{n,2} < z_{n-1,2} < \dots < z_{n-1,n-1} < x_{n,n} < y_{n,n}$$

PROPOSITION 5.3. *The zeros $\{x_{n-1,i}\}$ of $P_{n-1}(x)$ separate those of $Q_n(x)$, in the following way*

$$x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < x_{n-1,2} < \dots$$
$$\dots < y_{n,n-1} < x_{n-1,n-1} < x_{n,n} < y_{n,n}\,.$$

PROOF. Consider equation (2.5) relating $Q_n(x)$, $P_n(x)$ and $P_{n-1}(x)$

$$Q_n(x) = P_n(x) - Q'_n(c) \frac{1}{\|P_{n-1}\|^2} \frac{P_n(x) T_1(P_{n-1}, c)(x) - P_{n-1}(x) T_1(P_n, c)(x)}{(x-c)^2}.$$

By evaluation in $x_{n-1,i}$ we get

$$Q_n(x_{n-1,i}) = P_n(x_{n-1,i}) \left[1 - Q'_n(c) \frac{1}{\|P_{n-1}\|^2} \frac{T_1(P_{n-1}, c)(x_{n-1,i})}{(x_{n-1,i} - c)^2} \right]$$

therefore $Q_n(x_{n-1,i})$ and $P_n(x_{n-1,i})$ have the same sign. And from the separation properties of the roots of $P_{n-1}(x)$ with respect to the roots of $P_n(x)$, we conclude that the zeros of $P_{n-1}(x)$ separate those of $Q_n(x)$.

□

6 – Behavior of the zeros of Q_n with respect to λ

PROPOSITION 6.1. *The zeros of $Q_n(x)$ are an increasing function of λ.*

PROOF. Let $0 < \lambda < \mu$ be two positive real numbers. We will denote $Q_n(x,\lambda)$ and $Q_n(x,\mu)$ the corresponding orthogonal polynomials with respect to (2.1). Then

$$Q_n(x,\lambda) = P_n(x) - \frac{\lambda P'_n(c)}{1+\lambda K^{(1,1)}_{n-1}(c,c)} K^{(0,1)}_{n-1}(x,c)$$

$$Q_n(x,\mu) = P_n(x) - \frac{\mu P'_n(c)}{1+\mu K^{(1,1)}_{n-1}(c,c)} K^{(0,1)}_{n-1}(x,c).$$

Since $P'_n(c) \neq 0$, by eliminating $K^{(0,1)}_{n-1}(x,c)$ in both expressions we get

$$Q_n(x,\lambda) - P_n(x) = \frac{\mu^{-1} + K^{(1,1)}_{n-1}(c,c)}{\lambda^{-1} + K^{(1,1)}_{n-1}(c,c)} \left[Q_n(x,\mu) - P_n(x)\right].$$

On the other hand, if $0 < \lambda < \mu$, then $0 < \mu^{-1} < \lambda^{-1}$, and:

$$0 < \frac{\mu^{-1} + K^{(1,1)}_{n-1}(c,c)}{\lambda^{-1} + K^{(1,1)}_{n-1}(c,c)} < 1.$$

Thus, if $Q_n(x,\lambda) - P_n(x) \geq 0$, we have

$$Q_n(x,\lambda) - P_n(x) \leq Q_n(x,\mu) - P_n(x)$$

hence $Q_n(x,\lambda) \leq Q_n(x,\mu)$.

But, if $Q_n(x,\lambda) - P_n(x) \leq 0$, we have

$$Q_n(x,\lambda) - P_n(x) \geq Q_n(x,\mu) - P_n(x)$$

hence $Q_n(x,\lambda) \geq Q_n(x,\mu)$.

From proposition 5.3, if we denote by $y_{n,i}(\lambda)$, $y_{n,i}(\mu)$ $1 \leq i \leq n$ the roots of $Q_n(x,\lambda)$ and $Q_n(x,\mu)$ respectively, we have

$$y_{n,i}(\lambda), y_{n,i}(\mu) \in]x_{n,i}, z_{n-1,i}[, \qquad 1 \leq i \leq n-1$$

and from the preceding inequalities, the roots are ordered in the increasing order of the parameter

$$x_{n,i} < y_{n,i}(\lambda) < y_{n,i}(\mu) < z_{n-1,i}, \qquad 1 \leq i \leq n-1 .$$

Finally, for $x > x_{n,n}$, $Q_n(x,\lambda) > Q_n(x,\mu)$ and

$$x_{n,n} < y_{n,n}(\lambda) < y_{n,n}(\mu) . \qquad \square$$

In spite of the increasing character of the last zero of $Q_n(x)$, this zero does not grow indefinitely, in fact it lies between the last zero of $P_n(x)$ and the last zero of a polynomial which depends on c.

DEFINITION. *We denote by $R_n(x)$ the polynomial of degree n given by*

$$R_n(x) = P_n(x) - \frac{P_n'(c)}{K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(x,c)$$

PROPOSITION 6.2. *The polynomial $R_n(x)$ has n real and simple roots, which separate those of $P_n(x)$ and $K_{n-1}^{(0,1)}(x,c)$ in the following way*

$$x_{n,1} < \xi_{n,1} < z_{n-1,1} < x_{n,2} < \xi_{n,2} < z_{n-1,2} < \cdots$$

$$\cdots < z_{n-1,n-1} < x_{n,n} < c < \xi_{n,n}$$

where $\xi_{n,1} < \xi_{n,2} < \ldots < \xi_{n,n}$ denote the roots of $R_n(x)$.

PROOF. Analogous to proposition 5.2, but, in this case, the last root of $R_n(x)$ is at the right side of c, since $R'_n(c) = 0$. □

PROPOSITION 6.3. *The zeros $\{y_{n,i}\}$ of the polynomial $Q_n(x)$ are distributed each one in the corresponding interval $]x_{n,i}, \xi_{n,i}[$, $i = 1, \dots, n$.*

PROOF. Eliminating $K_{n-1}^{(0,1)}(x,c)$ from the expressions

$$Q_n(x) = P_n(x) - \frac{\lambda P'_n(c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(x,c)$$

and

$$R_n(x) = P_n(x) - \frac{P'_n(c)}{K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(x,c)$$

since $P'_n(c) \neq 0$, we get

$$Q_n(x) - P_n(x) = \frac{\lambda K_{n-1}^{(1,1)}(c,c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)} \left[R_n(x) - P_n(x)\right]$$

and taking in account that

$$0 < \frac{\lambda K_{n-1}^{(1,1)}(c,c)}{1 + \lambda K_{n-1}^{(1,1)}(c,c)} < 1$$

an analogous reasoning to that of proposition 6.1 proves the result. □

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