Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 505-520

Periodic solutions of some forced nonlinear nonautonomous second order ordinary differential equations at resonance

C.P. GUPTA - M.N. NKASHAMA(*)

RIASSUNTO – *Si ottengono dei risultati sull'esistenza di soluzioni 2π-periodiche per l'equazione differenziale non lineare del secondo ordine* $x''(t)+m^2x(t)+g(t,x(t))=e(t)$ *con condizioni ai bordi periodiche* $x(0)-x(2\pi)=x'(0)-x'(2\pi)=0$ *purché il rapporto* $x^{-1}g(t,x)$ *incontri asintoticamente (in senso opportuno) infiniti autovalori del problema lineare* $x''(t)+m^2x(t)+\lambda x(t)=0,\ x(0)-x(2\pi)=x'(0)-x'(2\pi)=0.$

ABSTRACT - *Existence results for the forced nonlinear second order ordinary differential equation* $x''(t)+m^2x(t)+g(t,x(t))=e(t)$ *with periodic boundary conditions* $x(0)-x(2\pi)=x'(0)-x'(2\pi)=0$ *are obtained when the ratio* $x^{-1}g(t,x)$ *asymptotically crosses in some sense infinitely many eigenvalues of the linear eigenvalue problem* $x''(t)+m^2x(t)+\lambda x(t)=0,\ x(0)-x(2\pi)=x'(0)-x'(2\pi)=0.$

KEY WORDS - *Resonance and non-resonance conditions - Crossing of infinitely many eigenvalues - Nonautonomous, asymptotic behavior.*

A.M.S. CLASSIFICATION: 34B15 - 34C25

1 – Introduction

In recent years much work has been devoted to existence results for

(*)This work was supported in part by US National Science Foundation under grant DMS-9006134.

forced nonlinear second order ordinary differential equation

$$(1.1)\qquad \begin{gathered} x''(t) + m^2 x(t) + g(t, x(t)) = e(t), \\ x(0) = x(2\pi),\, x'(0) = x'(2\pi) \end{gathered}$$

where m is a nonnegative integer, $e \in L^1(0, 2\pi)$, $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function; that is, $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbb{R}$, $g(t, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $t \in [0, 2\pi]$, and for each constant $r > 0$, there exists a real valued function $\gamma_r \in L^1(0, 2\pi)$ such that

$$(1.2)\qquad |g(t, x)| \le \gamma_r(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ with $|x| \le r$.

Initiated by Lazer and Leach [6] many authors have studied the existence of solutions to Eq.(1.1) when the ratio $x^{-1}g(t, x)$ stays asymptotically between two consecutive eigenvalues 0 and $2m + 1$ of the linear periodic eigenvalue value problem

$$(1.3)\qquad \begin{gathered} x''(t) + m^2 x(t) + \lambda x(t) = 0, \quad \lambda \in \mathbb{R}, \\ x(0) = x(2\pi),\, x'(0) = x'(2\pi)\,. \end{gathered}$$

We refer to [5, 6, 7, 8, 9, 10] and references therein for more details. (We also refer to [2] for the case when the ratio $x^{-1}g(t, x)$ stays asymptotically between two consecutive eigenvalue-branches related to the Fučik spectrum.)

It is the purpose of this paper to show that by exploiting the nonautonomous character of the function g one can obtain existence results for Eq.(1.1) when the ratio $x^{-1}g(t, x)$ asymptotically crosses in some sense infinitely many eigenvalues of the linear periodic problem (1.3). Of course, the aforementioned results, concerning the asymptotic behavior of the ratio $x^{-1}g(t, x)$ between two consecutive eigenvalues, can be derived as special cases of our main result herein. Our results were motivated by those in [3, 4, 8] where crossing of eigenvalues is considered near the first two consecutive eigenvalues. Note that herein we are concerned with the case dealing with crossing of eigenvalues near any two consecutives eigenvalues. Resonance and nonresonance conditions are considered.

The paper is organized as follows. In Section 2 we give our notations and prove some preliminary results that we shall need. In Section 3, we state and prove our main result on crossing of infinitely many eigenvalues, subsequently followed by some corollaries related to the resonance and nonresonance cases. To illustrate our results, we conclude the paper with a very elementary example.

2 – Preliminary Results

Besides the classical function spaces $C([0,2\pi])$, $C^p([0,2\pi])$ of respectively continuous, p-times continuously differentiable functions and Lebesgue spaces $L^p(0,2\pi)$, we shall make use of the Sobolev spaces $H^1(0,2\pi)$, $W^{2,1}(0,2\pi)$ and its subspace of 2π-periodic functions defined by

$$W^{2,1}_{2\pi}(0,2\pi) \overset{\text{def}}{=} \{x \in W^{2,1}(0,2\pi) : x(0) = x(2\pi),\ x'(0) = x'(2\pi)\}. \tag{2.1}$$

We refer to Brézis [1] for definitions and properties of these function spaces.

Let $x \in W^{2,1}_{2\pi}(0,2\pi)$. If the Fourier expansion of x is

$$x = a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt),$$

we shall write

$$\bar{x} = a_0 + \sum_{k=1}^{m-1}(a_k \cos kt + b_k \sin kt),\ \bar{x} = 0 \text{ if } m = 0,$$

$$x^0 = a_m \cos mt + b_m \sin mt,\ x^0 = a_0 \text{ if } m = 1,$$

$$\tilde{x} = \sum_{k=m+1}^{\infty}(a_k \cos kt + b_k \sin kt), \text{ and}$$

$$x^{\perp} = x - x^0.$$

In order to prove our main result, we shall need some *a priori* estimates obtained from some bilinear forms on appropriate function spaces.

The following result was proved in [5].

LEMMA 1. *Let $\Gamma_0 \in L^\infty(0, 2\pi)$ be such that for a.e. $t \in [0, 2\pi]$*

$$0 \leq \Gamma_0(t) \leq 2m + 1 \tag{2.2}$$

with $\Gamma_0(t) < (2m+1)$ on a subset of $[0, 2\pi]$ of positive measure.

Then, there exists a constant $\delta = \delta(\Gamma_0) > 0$ such that for all $x \in W^{2,1}_{2\pi}(0, 2\pi)$, one has

$$B_{\Gamma_0}(x) \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} [x''(t) + m^2 x(t) + \Gamma_0(t) x(t)][\overline{x}(t) + x^0(t) - \tilde{x}(t)]dt \geq \delta |x^\perp|^2_{H^1}.$$

The following result will allow crossing of infinitely many eigenvalues of the linear periodic boundary value problem.

LEMMA 2. *Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty(0, 2\pi)$, $\Gamma_1 \in L^1(0, 2\pi)$ with $\Gamma_\infty(t) + \Gamma_1(t) \geq 0$ for a.e. $t \in [0, 2\pi]$, and $\Gamma_0 \in L^\infty(0, 2\pi)$ is such that $0 \leq \Gamma_0(t) \leq (2m+1)$ for a.e. $t \in [0, 2\pi]$ with $\Gamma_0(t) < (2m+1)$ on a subset of $[0, 2\pi]$ of positive measure. Let $\delta = \delta(\Gamma_0) > 0$ be given by Lemma 1.*

Then, for all $x \in W^{2,1}_{2\pi}(0, 2\pi)$, one has

$$B_\Gamma(x) \geq [\delta - \alpha |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty}] |x^\perp|^2_{H^1}, \tag{2.3}$$

where $\alpha = \dfrac{\pi^2}{3}$.

PROOF. By using the definition of Γ and Lemma 1, we get

$$
\begin{aligned}
B_\Gamma(x) &= \frac{1}{2\pi}\int_0^{2\pi}[x'' + (m^2+\Gamma_0(t))x(t)][\overline{x}(t)+x^0(t)-\tilde{x}(t)]dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi}[(\Gamma_1(t)+\Gamma_\infty(t))x(t)][\overline{x}(t)+x^0(t)-\tilde{x}(t)]dt \\
&\geq \delta|x^\perp|^2_{H^1} + \frac{1}{2\pi}\int_0^{2\pi}(\Gamma_1(t)+\Gamma_\infty(t))(\overline{x}(t)+x^0(t))^2dt \\
&- \frac{1}{2\pi}\int_0^{2\pi}(\Gamma_1(t)+\Gamma_\infty(t))\tilde{x}^2(t)dt \\
&\geq \delta|x^\perp|^2_{H^1} - |\Gamma_1|_{L^1}|\tilde{x}|^2_{L^\infty} - |\Gamma_\infty|_{L^\infty}|\tilde{x}|^2_{L^2} \\
&\geq \delta|x^\perp|^2_{H^1} - |\Gamma_1|_{L^1}\alpha|\tilde{x}'|^2_{L^2} - |\Gamma_\infty|_{L^\infty}|\tilde{x}'|^2_{L^2} \\
&\geq \delta|x^\perp|^2_{H^1} - |\Gamma_1|_{L^1}\alpha|(x^\perp)'|^2_{L^2} - |\Gamma_\infty|_{L^\infty}|(x^\perp)'|^2_{L^2} \\
&\geq (\delta - \alpha|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty})|x^\perp|^2_{H^1}
\end{aligned}
$$

where, we also have used the inequalities (see e.g [11, p. 208])

$$|\tilde{x}|_{L^2} \leq |\tilde{x}'|_{L^2} \quad \text{and} \quad |\tilde{x}|_{L^\infty} \leq \sqrt{\alpha}|\tilde{x}'|_{L^2} \quad \text{where} \quad \alpha = \frac{\pi^2}{3}.$$

The proof is complete.

LEMMA 3. *Let $\Gamma = \Gamma_0+\Gamma_1+\Gamma_\infty$ be as in Lemma 2 and $\delta = \delta(\Gamma_0) > 0$ be as given by Lemma 1. Let $\varepsilon > 0$ be given.*

Then, for all $p \in L^1(0,2\pi)$ satisfying

$$0 \leq p(t) \leq \Gamma(t) + \varepsilon \tag{2.4}$$

a.e. on $[0,2\pi]$ and all $x \in W^{2,1}_{2\pi}(0,2\pi)$, one has

$$
\begin{aligned}
B_p(x) &= \frac{1}{2\pi}\int_0^{2\pi}[x''(t) + (m^2+p(t))x(t)][\overline{x}(t)+x^0(t)-\tilde{x}(t)]dt \\
&\geq [\delta - \alpha|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \varepsilon]|x^\perp|^2_{H^1}
\end{aligned}
$$

PROOF. For $x \in W^{2,1}_{2\pi}(0,2\pi)$, by using integration by parts, inequalities (2.4) and Lemma 2, we get

$$\begin{aligned}
B_p(x) &= \frac{1}{2\pi}\int_0^{2\pi}[(\tilde{x}'(t))^2 - (m^2+p(t))(\tilde{x}(t))^2]dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi}[m^2(\overline{x}(t))^2 - (\overline{x}'(t))^2]dt + \frac{1}{2\pi}\int_0^{2\pi} p(t)(\overline{x}(t)+x^0(t))^2dt \\
&\geq \frac{1}{2\pi}\int_0^{2\pi}[(\tilde{x}'(t))^2 - (m^2+\Gamma(t)+\varepsilon)(\tilde{x}(t))^2]dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi}[m^2(\overline{x}(t))^2 - (\overline{x}'(t))^2]dt. \\
&\geq \frac{1}{2\pi}\int_0^{2\pi}[(\tilde{x}'(t))^2 - (m^2+\Gamma_0(t))(\tilde{x}(t))^2]dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi}[m^2(\overline{x}(t))^2 - (\overline{x}'(t))^2]dt \\
&- \frac{1}{2\pi}\int_0^{2\pi}[\Gamma_1(t)+\Gamma_\infty(t)+\varepsilon](\tilde{x}(t))^2dt \\
&\geq [\delta - \alpha|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} - \varepsilon]|x^\perp|^2_{H^1}.
\end{aligned}$$

The proof is complete.

3 – Main Result and Consequences

Throughout this section we shall assume that g is a Carathéodory function (see Section 1 for definition). We have the following main result.

THEOREM 1. *Assume that for all $\varepsilon > 0$ there exist a constant $B = B_\varepsilon > 0$ and a function $b_\varepsilon \in L^\infty(0,2\pi)$ such that*

$$|g(t,x)| \leq (\Gamma(t)+\varepsilon)|x| + b_\varepsilon(t) \tag{3.1}$$

for a.e. $t \in [0, 2\pi]$ *and all* $x \in \mathbb{R}$ *with* $|x| \geq B$*, where* Γ *is as in Lemma 2, and*

$$\alpha|\Gamma_1|_{L^1} + |\Gamma_\infty|_{L^\infty} < \delta(\Gamma_0). \tag{3.2}$$

Furthermore, suppose there exist functions a*,* $A \in L^1(0, 2\pi)$ *and constants* r*,* $R \in \mathbb{R}$ *with* $r < 0 < R$ *such that*

$$g(t, x) \geq A(t) \tag{3.3}$$

for a.e. $t \in [0, 2\pi]$ *and all* $x \geq R$*,*

$$g(t, x) \leq a(t) \tag{3.4}$$

for a.e. $t \in [0, 2\pi]$ *and all* $x \leq r$*.*

Then the periodic boundary value problem

$$\begin{gathered} x''(t) + m^2 x(t) + g(t, x(t)) = e(t), \\ x(0) = x(2\pi),\, x'(0) = x'(2\pi) \end{gathered} \tag{3.5}$$

has at least one solution for each $e \in L^1(0, 2\pi)$ *that is such that*

$$\int_0^{2\pi} e(t)v(t) < \int_{v>0} g_+(t)v(t)dt + \int_{v<0} g_-(t)v(t)dt \tag{3.6}$$

for all $v \in \operatorname{Span}\{\cos mt, \sin mt\} \setminus \{0\}$*, where*

$$g_+(t) = \liminf_{x \to \infty} g(t, x) \quad \textit{and} \quad g_-(t) = \limsup_{x \to -\infty} g(t, x). \tag{3.7}$$

PROOF. Let $\eta = \delta - \alpha|\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} > 0$, where $\delta = \delta(\Gamma_0) > 0$ and $\alpha > 0$ are given in Lemma 2. Then, by assumption (3.1), there exist a constant $B = B_\eta > 0$ and a function $b = b_\eta \in L^\infty(0, 2\pi)$ such that

$$|g(t, x)| \leq \left(\Gamma(t) + \frac{1}{4}\eta\right)|x| + b_\eta(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ with $|x| \geq B$.

By using Lemma 3 in [5], we can write

$$g(t,x) = q_1(t,x) + g_1(t,x)$$

with

$$q_1(t,x)x \geq 0$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Furthermore, there exists a function $\sigma_1 \in L^1(0, 2\pi)$ such that

$$|g_1(t,x)| \leq \sigma_1(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Moreover, by Lemma 4 in [5], we have

$$|q_1(t,x)| \leq \left(\Gamma(t) + \frac{\eta}{4}\right)|x| + b_\eta(t) + 1$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ with $|x| \geq \max(1, B)$.

Choose $\overline{B} > \max(1, B)$ such that

$$(b_\eta(t) + 1)/|x| < \frac{1}{4}\eta$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ with $|x| \geq \overline{B}$. It follows that

$$0 \leq q_1(t,x)/x \leq \Gamma(t) + \frac{1}{2}\eta$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ with $|x| \geq \overline{B}$.

Now, define $\tilde{\gamma} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\gamma}(t,x) = \begin{cases} \dfrac{q_1(t,x)}{x} & \text{for } |x| \geq \overline{B}, \\ \dfrac{1}{\overline{B}} q_1(t, \overline{B}) \dfrac{x}{\overline{B}} + \left(1 - \dfrac{x}{\overline{B}}\right)\Gamma(t) & \text{for } 0 \leq x < \overline{B}, \\ \dfrac{1}{\overline{B}} q_1(t, -\overline{B}) \dfrac{x}{\overline{B}} + \left(1 + \dfrac{x}{\overline{B}}\right)\Gamma(t) & \text{for } -\overline{B} < x \leq 0. \end{cases}$$

$\tilde{\gamma}$ is a Carathéodory function since q_1 is. Moreover,

$$0 \leq \tilde{\gamma}(t,x) \leq \Gamma(t) + \frac{\eta}{2} \tag{3.8}$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$.

Define $h : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ by

$$h(t,x) = g_1(t,x) + q_1(t,x) - \tilde{\gamma}(t,x)x.$$

Then, it follows that there exists a function $\sigma \in L^1(0, 2\pi)$ such that

$$|h(t,x)| \leq \sigma(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$, where σ depends only on Γ.

The periodic boundary value problem (3.5) is equivalent to

$$\begin{aligned} x''(t) + m^2 x(t) + \tilde{\gamma}(t, x(t))x(t) + h(t, x(t)) &= e(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0. \end{aligned} \tag{3.9}$$

In order to apply Mawhin's coincidence degree (more precisely Theorem I.2 in [7]), we have to prove the existence of an *a priori* bound for the possible solutions in $W^{2,1}_{2\pi}(0, 2\pi)$ to the family of equations

$$\begin{aligned} x''(t) + m^2 x(t) + [(1-\lambda)(\eta/2) + \lambda\tilde{\gamma}(t, x(t))]x(t) \\ + \lambda h(t, x(t)) - \lambda e(t) = 0, \quad \lambda \in [0,1). \end{aligned} \tag{3.10}$$

It is clear that for $\lambda = 0$, Eq. (3.10) has only the trivial solution in $W^{2,1}_{2\pi}(0, 2\pi)$. Now, if $x \in W^{2,1}_{2\pi}(0, 2\pi)$ is a solution of Eq.(3.10) for some $\lambda \in (0, 1)$, then using inequalities (3.8) and Lemma 3, we obtain

$$\begin{aligned} 0 &= (2\pi)^{-1} \int_0^{2\pi} (\bar{x}(t) + x^0(t) - \tilde{x}(t)) \\ &\quad \times \{x''(t) + m^2 x(t) + [(1-\lambda)(\eta/2) + \lambda\tilde{\gamma}(t, x(t))]x(t)\}dt \\ &\quad + (2\pi)^{-1} \int_0^{2\pi} (\bar{x}(t) + x^0(t) - \tilde{x}(t))(\lambda h(t, x(t)) - \lambda e(t))dt \\ &\geq (\eta/2)|x^{\perp}|^2_{H^1} - (2\pi)^{-1}(|\bar{x}|_C + |x^0|_C + |\tilde{x}|_C)(|h|_{L^1} + |e|_{L^1}). \end{aligned}$$

Therefore, by the continuous imbedding of $H^1(0,2\pi)$ into $C([0,2\pi])$, one gets

$$0 \geq (\eta/2)|x^\perp|^2_{H^1} - \beta(|x^\perp|_{H^1} + |x^0|_{H^1}), \tag{3.11}$$

where β depends only on σ and e (but not on x or λ).

So, taking $\theta = \beta(\eta)^{-1}$, one has

$$|x^\perp|_{H^1} \leq \theta + (\theta^2 + 2\theta|x^0|_{H^1})^{1/2}. \tag{3.12}$$

Now, we claim that there exists a constant $\rho > 0$ such that

$$|x|_{H^1} < \rho$$

for any solution $x \in W^{2,1}_{2\pi}(0,2\pi)$ to Eq.(3.10) (ρ independent of x and λ).

Assume the claim does not hold. Then, there will be a sequence (λ_n) in (0,1) and a sequence (x_n) in $W^{2,1}_{2\pi}(0,2\pi)$ with $|x_n|_{H^1} \to \infty$ such that

$$x_n''(t) + m^2 x_n(t) + (1-\lambda_n)(\eta/2)x_n(t) + \lambda_n g(t, x_n(t)) = \lambda_n e(t). \tag{3.13}$$

From inequality (3.12) it follows that

$$|x_n^0|_{H^1} \to \infty \quad \text{and} \quad |x_n^\perp|_{H^1}(|x_n^0|_{H^1})^{-1} \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Thus, the sequence $x_n(|x_n^0|_{H^1})^{-1}$ is bounded in $H^1(0,2\pi)$.

By using the compact imbedding of $H^1(0,2\pi)$ into $C([0,2\pi])$, one can assume, by taking a subsequence if necessary, that there exists $v \in \text{Span}\{\cos mt, \sin mt\}$ such that

$$\begin{aligned} x_n(|x_n^0|_{H^1})^{-1} &\to v \quad \text{in} \quad C([0,2\pi]), \\ x_n(|x_n^0|_{H^1})^{-1} &\rightharpoonup v \quad \text{in} \quad H^1(0,2\pi), \\ x_n^0(|x_n^0|_{H^1})^{-1} &\to v \quad \text{in} \quad C([0,2\pi]). \end{aligned}$$

Let us set

$$v_n = x_n^0(|x_n^0|_{H^1})^{-1}.$$

Multiplying Eq.(3.13) by $v_n\lambda_n^{-1}$ and using integration by parts, we obtain

$$0 \le (1-\lambda_n)\lambda_n^{-1}(m^2+1)^{-1}(\eta/2)|x_n^0|_{H^1}$$
$$= (2\pi)^{-1}\int_0^{2\pi}[e(t)-g(t,x_n(t))]v_n(t)dt.$$

So that by taking the liminf as $n\to\infty$, we have

$$\int_0^{2\pi} e(t)v(t)dt \ge \liminf_{n\to\infty}\int_{v>0} g(t,x_n(t))v_n(t)dt + \liminf_{n\to\infty}\int_{v<0} g(t,x_n(t))v_n(t)dt.$$

Let $I^+ = \{t\in[0,2\pi] : v(t)>0\}$ and $I^- = \{t\in[0,2\pi] : v(t)<0\}$. Then, for each $t\in I^+$ there exists an integer $\nu(t)\in\mathbb{N}$ such that for all $n\ge\nu(t)$, one has

$$|x_n^\perp|_C(|x_n^0|_{H^1})^{-1} < \frac{1}{4}v(t)$$

and

$$|x_n^0(t)(|x_n^0|_{H^1})^{-1} - v(t)| < \frac{1}{4}v(t).$$

Therefore, for all $n\ge\nu(t)$, one has

$$x_n(t)(|x_n^0|_{H^1})^{-1} \ge (x_n^0(t)-|x_n^\perp|_C)(|x_n^0|_{H^1})^{-1} \ge \frac{1}{2}v(t).$$

It follows that for each $t\in I^+$, there exists an integer $\nu(t)\in\mathbb{N}$ such that for all $n\ge\nu(t)$,

$$v_n(t)>0 \quad\text{and}\quad x_n(t)\ge\frac{1}{2}v(t)|x_n^0|_{H^1}\ge R \qquad (\text{since } |x_n^0|_{H^1}\to+\infty).$$

On the other hand, for each $t\in I^-$, there exists an integer $\mu(t)\in\mathbb{N}$ such that for all $n\ge\mu(t)$, one has

$$x_n(t)(|x_n^0|_{H^1})^{-1} \le (x_n^0(t)+|x_n^\perp|_C)(|x_n^0|_{H^1})^{-1} \le \frac{1}{2}v(t).$$

So, for $n\ge\mu(t)$, $x_n(t)\le\frac{1}{2}v(t)|x_n^0|_{H^1}\to-\infty$.

Now, in order to apply Fatou's Lemma, we need to show that there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$g(t, x_n(t))v_n(t) \geq f(t)$$

for some $f \in L^1(0, 2\pi)$. Indeed, from inequality (3.11), one gets

$$|x_n^\perp|_{H^1}^2(|x_n^0|_{H^1})^{-1} \leq 2\theta |x_n^\perp|_{H^1}(|x_n^0|_{H^1})^{-1} + 2\theta.$$

So, by the second relation in (3.14), one has that for $n \geq n_0$, $|x_n^\perp|_{H^1}^2(|x_n^0|_{H^1})^{-1} \leq 4\theta$.

Since $\tilde{\gamma}(t, x_n(t)) \geq 0$ (see (3.8)), one has that for $n \geq n_0$,

$$\begin{aligned}
\tilde{\gamma}(t, x_n(t))x_n(t)v_n(t) &= \tilde{\gamma}(t, x_n(t))x_n(t)x_n^0(t)(|x_n^0|_{H^1})^{-1} \\
&= \frac{1}{2}(|x_n^0|_{H^1})^{-1}[(x_n(t))^2 + (x_n^0(t))^2 \\
&\quad - (x_n(t) - x_n^0(t))^2] \cdot \tilde{\gamma}(t, x_n(t)) \\
&\geq -\frac{1}{2}\tilde{\gamma}(t, x_n(t))(x_n^\perp(t))^2(|x_n^0|_{H^1})^{-1} \\
&\geq -2\theta\beta_1\tilde{\gamma}(t, x_n(t))
\end{aligned}$$

for some $\beta_1 > 0$.

So, for $n \geq n_0$,

$$\tilde{\gamma}(t, x_n(t))x_n(t)v_n(t) \geq -2\theta\beta_1(\Gamma(t) + (\eta/2)) \quad \text{(see (3.8))}.$$

Thus, using the decomposition of g in (3.9), one has that for $n \geq n_0$,

$$\begin{aligned}
g(t, x_n(t))v_n(t) &= \tilde{\gamma}(t, x_n(t))x_n(t)v_n(t) + h(t, x_n(t))v_n(t) \\
&\geq -2\theta\beta_1(\Gamma(t) + (\eta/2)) - \sigma(t)K_1 = f(t)
\end{aligned}$$

since $\sup_{[0,2\pi]} |v_n(t)| < K_1$ for some constant $K_1 > 0$.

So, by Fatou's Lemma and the properties of liminf, one has

$$\liminf_{n\to\infty} \int_{v>0} g(t, x_n(t))v_n(t)dt \geq \int_{v>0} g_+(t)v(t)dt$$

$$\liminf_{n\to\infty} \int_{v<0} g(t, x_n(t))v_n(t)dt \geq \int_{v<0} g_-(t)v(t)dt$$

Therefore, we have

$$\int_0^{2\pi} e(t)v(t)dt \geq \int_{v>0} g_+(t)v(t)dt + \int_{v<0} g_-(t)v(t)dt.$$

This is a contradiction with assumption (3.6). Thus, the claim holds.

Furthermore, by the compact imbedding of $H^1(0,2\pi)$ into $C([0,2\pi])$, one has that there exists $K_0 > 0$ such that $|x|_C < K_0$ for any solution to Eq.(3.10). The proof is complete.

REMARK 1. Assumptions (3.3) and (3.4) are, in particular, satisfied if there exist functions $c, d \in L^1(0,2\pi)$ such that

$$xg(t,x) \geq -c(t)|x| - d(t) \tag{3.15}$$

for a.e. $t \in [0,2\pi]$ and all $x \in \mathbb{R}$.

Indeed, it follows from (3.15) that $d(t) \geq 0$ for a.e. $t \in [0,2\pi]$. Therefore, taking $R = -r = 1$ and $a(t) = -A(t) = c(t) + d(t)$, we are done.

COROLLARY 1. *Assume inequalities*

$$0 \leq \liminf_{|x|\to\infty} x^{-1}g(t,x) \leq \limsup_{|x|\to\infty} x^{-1}g(t,x) \leq \Gamma(t) \tag{3.16}$$

hold uniformly a.e. on $[0,2\pi]$, *where* $\Gamma \in L^1(0,2\pi)$ *satisfies conditions in Theorem 1.*

Moreover, suppose there exist functions $a, A \in L^1(0,2\pi)$ *and constants* $r, R \in \mathbb{R}$ *satisfying assumptions* (3.3) *and* (3.4).

Then, Eq. (3.5) *has at least one solution for each* $e \in L^1(0,2\pi)$ *provided condition* (3.6) *is fulfilled.*

PROOF. It suffices to show that assumption (3.16) implies (3.1). Indeed, it follows from (3.16) that for all $\varepsilon > 0$ there exists $B(\varepsilon) = B > 0$ such that

$$-\varepsilon \leq x^{-1}g(t,x) \leq \Gamma(t) + \varepsilon$$

for a.e. $t \in [0,2\pi]$ and all $x \in \mathbb{R}$ with $|x| \geq B$. Choosing $b_\varepsilon = 0$. The proof is complete.

COROLLARY 2. *Assume the inequality*

$$\limsup_{|x|\to\infty} x^{-1}g(t,x) \le \Gamma(t) \tag{3.17}$$

holds uniformly a.e. on $[0,2\pi]$, *where* $\Gamma \in L^1(0,2\pi)$ *satisfies conditions in Theorem 1.*

Furthermore, suppose there exists a constant $R > 0$ *such that*

$$\gamma(t) \le x^{-1}g(t,x) \tag{3.18}$$

for a.e. $t \in [0,2\pi]$ *and all* $x \in \mathbb{R}$ *with* $|x| \ge R$, *where*

$$0 \le \gamma(t) \tag{3.19}$$

for a.e. $t \in [0,2\pi]$ *with strict inequality on a subset of positive measure.*

Then, Eq. (3.5) *has at least one solution for every* $e \in L^1(0,2\pi)$.

PROOF. Assumptions (3.18), (3.19) imply that (3.1)–(3.4) are fulfilled with $b_\varepsilon = 0$, $A = a = 0$, and $R = -r$.

Now, let $I^+ = \{t \in [0,2\pi] : \gamma(t) > 0\}$. Then, one has

$$g_+(t) = \liminf_{x\to\infty} g(t,x) = \infty \quad \text{and} \quad \limsup_{x\to-\infty} g(t,x) = -\infty$$

a.e. on I^+.

Since I^+ has a positive measure, it follows that

$$\int_{v>0} g_+(t)v(t)dt + \int_{v<0} g_-(t)v(t)dt = \infty.$$

The existence of a solution to Eq.(3.5), for every $e \in L^1(0,2\pi)$, follows from Theorem 1. The proof is complete.

To illustrate our results, we conclude this paper with a very elementary example in the linear nonautonomous case.

EXAMPLE 1. Let us consider the (linear in x) function $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(t, x) = \Gamma(t)x \tag{3.20}$$

with

$$\Gamma(t) = \begin{cases} 0 & \text{for } t = 0 \\ ct^{-1/2} & \text{for } 0 < t \leq 2\pi \end{cases} \tag{3.21}$$

where c is a constant such that

$$0 < \frac{c\alpha}{\sqrt{2\pi}} < \frac{1}{2}. \tag{3.22}$$

(Recall that $\alpha = \pi^2/3$.)

Then, Eq.(1.1) has a (unique) solution for every $e \in L^1(0, 2\pi)$.

Indeed, it is easy to check that all conditions in Corollary 2 are satisfied with $\Gamma_0 \equiv \Gamma_\infty \equiv 0$, and $\gamma \equiv \Gamma_1 \equiv \Gamma$ where Γ is given in (3.21), (3.22).

Obviously, the function Γ given in (3.21) provides for crossing of infinitely many eigenvalues of Eq.(1.3) on subsets of $[0, 2\pi]$ of positive measure.

REFERENCES

[1] H. BRÉZIS: *Analyse fonctionnelle: Théorie et applications*, Masson, Paris 1983.

[2] P. DRÁBEK: *Landesman-Lazer condition for nonlinear problems with jumping nonlinearities*, J. Differential Equations 85 (1990), 186-199.

[3] C.P. GUPTA: *A two-point boundary value problem of Dirichlet type with resonance at infinitely many eigenvalues*, Jour. Math. Anal. Appl. 146 (1990), 501-511.

[4] C.P. GUPTA - J. MAWHIN: *Asymptotic conditions at the two first eigenvalues for the periodic solutions of Liénard differential equations and an inequality of E. Schmidt*, Z. Anal. Anw. 3 (1984), 33-42.

[5] R. IANNACCI - M.N. NKASHAMA: *Unbounded perturbations of forced second order ordinary differential equations at resonance*, J. Differential Equations **69** (1987), 289-309.

[6] A.C. LAZER - D. E. LEACH: *Bounded pertubations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. **82** (1969), 49-68.

[7] J. MAWHIN: *Compacité, Monotonie et Convexité dans L'étude des Problèmes aux Limites Semi-linéaires*, Séminaire d'Analyse Moderne no. 19, Université de Sherbrooke, Québec 1981.

[8] J. MAWHIN - J.R. WARD: *Periodic solutions of some forced Liénard differential equations at resonance*, Arch. Math. (Basel) 41 (1983), 337-351.

[9] R. REISSIG: *Extension of some results concerning the generalized Liénard equations*, Ann. Mat. Pura Appl. **104** (1975) 269-281.

[10] R. REISSIG: *Continua of periodic solutions of the Liénard equations*, in "Constructive Methods for Nonlinear Boundary Value Problems and Nonlinear Oscillations," ISNM, Birkhaüser, Basel **48** (1979) 126-133.

[11] N. ROUCHE - J. MAWHIN: *Ordinary Differential Equations: Stability and Periodic Solutions*, Pitman, Boston 1980.

Lavoro pervenuto alla redazione il 16 ottobre 1991
ed accettato per la pubblicazione il 5 dicembre 1991
su parere favorevole di G. Dell'Antonio e di R. Conti

INDIRIZZO DEGLI AUTORI:

C.P. Gupta - Department of Mathematical Sciences - Northern Illinois University - DeKalb - Illinois 60115 - USA.

M.N. Nkashama - Department of Mathematics - University of Alabama at Birmingham - Birmingham - Alabama 35294 - USA