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On functional equations of the form $A(x) - A(\tau(x)) = \varphi(x)$ and Goursat problem for the equation $\frac{\partial^2 z(x,y)}{\partial x \partial y} = f(x,y)$

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RIASSUNTO – *Si studiano le equazioni funzionali del tipo* $A(x) - A(\tau(x)) = \varphi(x)$, *associate, se Fix* $\tau = \{p\}$, *alla condizione* $A(p) = 0$ *e, più in generale, alla condizione* $A(p') = 0$ *dove* p' *è un punto di Fix* τ. *Inoltre, si forniscono condizioni sufficienti per l'esistenza e l'unicità della soluzione e della soluzione debole del seguente problema di Goursat:*

$$\begin{aligned} \frac{\partial^2 z(x,y)}{\partial x \partial y} &= f(x,y) \qquad && (x,y) \in I \times J \\ z(x, a(x)) &= \varphi_1(x) && x \in I \\ z(\beta(y), y) &= \varphi_2(y) && y \in J . \end{aligned}$$

ABSTRACT – *We consider the functional equations of the kind* $A(x) - A(\tau(x)) = \varphi(x)$, *associated, if Fix* $\tau = \{p\}$, *with the condition* $A(p) = 0$ *and, in a more general way, with the condition* $A(p') = 0$ *where* p' *is a point of Fix* τ. *Furthermore, sufficient conditions are given for the existence and uniqueness of solution and weak solution to the following Goursat problem:*

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$$\begin{aligned} \frac{\partial^2 z(x,y)}{\partial x \partial y} &= f(x,y) & (x,y) \in I \times J \\ z\big(x, \alpha(x)\big) &= \varphi_1(x) & x \in I \\ z\big(\beta(y), y\big) &= \varphi_2(y) & y \in J\,. \end{aligned}$$

KEY WORDS - *Goursat problem* - *Weak solution.*
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1 – Introduction

Let I be an interval of $\mathbb{R}$, $\tau(x)$ and $\varphi(x)$ be functions belonging, resp., to I^I and $\mathbb{R}^I$ such that[(1)]:

$$\operatorname{Fix}\tau = \{p\}\,, \qquad \varphi(p) = 0\,.$$

Let us consider the following functional problem:

$$\text{(P)} \qquad \begin{aligned} & A(x) - A(\tau(x)) = \varphi(x) \qquad x \in I \\ & A(p) = 0\,. \end{aligned}$$

E. GOURSAT in [10] prove (see, section 9) that, if $I = [0,a]$, $\tau(x) \in C^1(I,I)$, $\varphi(x) \in C^1(I,\mathbb{R})$, $p = 0$ and, moreover, $\tau(x)$ satisfies the following condition:

A) $\qquad \tau'(x) > 0\ \forall x \in]0,a]$ *and* $\exists c \in]0,1[: \tau(x) < cx\ \forall x \in]0,a]$,

then there exists a solution of problem (P) belonging to $C^1(I,\mathbb{R})$.

G. FICHERA in [4] demonstrate (see, theorems I, II and III) that, if $I = [0,a]$, $\tau(x)$ and $\varphi(x)$ are C^n functions [resp., C^∞ functions (resp., real analytic functions)], $p = 0$ and, moreover, $\tau(x)$ satisfies the following condition:

B) $\qquad \tau'(x) > 0$, $\tau(x) < x\ \forall x \in]0,a]$ *and* $\tau'(0) < 1$,

then there exists a unique solution of problem (P) belonging to $C^n(I,\mathbb{R})$ [resp., $C^\infty(I,\mathbb{R})$ (resp., the class of the real analytic functions from I into $\mathbb{R}$)].

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(1) Whatever be the function $h(x)$, the symbol $\operatorname{Fix} h$ denotes the set of all fixed points of $h(x)$.

The Fichera's results imply that one of Goursat (indeed: A) $\Longrightarrow$ B)).

Other Authors have also studied the functional problem (P) (see, for example, [3, 5, 6, 12]).

In section 2 of this paper we consider the following statements:

i) *there exists* $n \in \mathbb{N}$ *such that* $\tau(x) \in C^n(I, I)$, $\varphi(x) \in C^n(I, \mathbb{R})$ *and*, $\forall m \in \{0, \dots, n\}$, *the following functions:*

$$\frac{d^m \tau(x)}{dx^m}, \qquad \frac{d^m \varphi(x)}{dx^m}$$

are bounded on I,

i′) $\tau(x) \in C^\infty(I, I)$, $\varphi(x) \in C^\infty(I, \mathbb{R})$ *and*, $\forall n \in \mathbb{N}_0$, *the following functions:*

$$\frac{d^n \tau(x)}{dx^n}, \qquad \frac{d^n \varphi(x)}{dx^n}$$

are bounded on I,

i″) $\tau\colon I \to I$, $\varphi\colon I \to \mathbb{R}$ *are real analytic functions such that, for each* $n \in \mathbb{N}_0$, *the following functions:*

$$\frac{d^n \tau(x)}{dx^n}, \qquad \frac{d^n \varphi(x)}{dx^n}$$

are bounded on I,

ii) $(\tau^k)_{k \in \mathbb{N}}$ *converges uniformly on* I,[(2)]

iii) $|\tau'(p)| < 1$.

We prove, see theorem 2.1 [resp., theorem 2.2 (resp., theorem 2.3)], that: if the statements i) [resp., i′) (resp., i″))], ii) and iii) hold, then the functional problem (P) has a unique solution belonging to $C^n(I, \mathbb{R})$ [resp., $C^\infty(I, \mathbb{R})$ (resp., the class of the real analytic functions from I into $\mathbb{R}$)].

In the particular case $I = [0, a]$, obviously:

$$\text{B)} \Longrightarrow \Big(\text{ii) and iii)}\Big),$$

so, the above results generalize the theorems of Goursat and Fichera quoted before.

(2) Whatever be $n \in \mathbb{N}$, the symbol τ^n denotes the *nth* iterate of $\tau(x)$.

Still in the case $I = [0, a]$, the following condition (which appear in A) and B)):

$$\tau(x) < 0 \quad \forall x \in]0, a],$$

implies the condition ii) (see [5], p. 239); so the hypothesis:

$$\tau'(x) > 0 \quad \forall x \in]0, a],$$

which appears in A) and B) is non-essential (see theorems 2.1, 2.2 and 2.3).

Now, let I and J be intervals of $\mathbb{R}$, $y_0 \in J$, $f(x,y) \in C^0(I \times J, \mathbb{R})$, $\varphi_1(x) \in C^0(I, \mathbb{R})$, $\varphi_2(y) \in C^0(J, \mathbb{R})$, $\alpha(x) \in C^0(I, J)$, $\beta(y) \in C^0(J, I)$.

Let us set:

$$\tau(x) = \beta(\alpha(x)) \quad \forall x \in I.$$

In section 3 we suppose $\mathrm{Fix}\,\tau \neq \emptyset$ and consider the following functional problem:

$$(\mathrm{P}') \qquad \begin{cases} A(x) - A(\tau(x)) = \varphi_1(x) - \varphi_2(\alpha(x)) + \int_x^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)\, dv \quad x \in I \\ A(p') = 0, \end{cases}$$

being p' a point of $\mathrm{Fix}\,\tau$.

We give (see theorem 3.1 and theorem 3.3) two sufficient conditions for the existence of a solution of problem (P′) and (see theorem 3.2 and theorem 3.4) a sufficient condition for the existence of a continuous solution of problem (P′) and two sufficient conditions for the existence and uniqueness of continuous solution of problem (P′).

[A generalization of the theorems of section 3 is given in the Appendix.]

The results of sections 2 and 3 are used in section 4 to study the *Goursat problem* for the partial differential equation of hyperbolic type:

$$(\mathrm{G}) \qquad \frac{\partial^2 z(x,y)}{\partial x \partial y} = f(x,y) \qquad (x,y) \in I \times J,$$

i.e., the problem consists in finding solutions $z(x,y)$ of (G) satisfying the following conditions:

$$(\mathrm{G}') \qquad \begin{aligned} z(z,\alpha(x)) &= \varphi_1(x) \qquad x \in I \\ z(\beta(y),y) &= \varphi_2(y) \qquad y \in J\,. \end{aligned}$$

In the classical *Goursat problem*, which also deal with equations more general than (G), the intervals I and J are compact and the functions $\alpha(x)$ and $\beta(y)$ satisfy the following condition:

j) *$\alpha(x)$ and $\beta(y)$ belong to class C^1 and there exists $c \in]0,1[$ such that:*

$$|\tau(x)| \le c|x| \qquad \forall x \in I\,,$$

(see, for example, [1, 2, 3, 6, 8, 9, 10, 11, 15, 16, 17, 18]).

In section 4 of this paper, first of all we prove (see proposition 4.1) that: *whatever be $A(x) \in \mathbb{R}^I$ and $B(y) \in \mathbb{R}^J$, the function $z(x,y) = A(x) + B(y) + F(x,y)$ is a solution (resp., weak solution) of problem* (G) (G′) *if and only if $A(x)$ belongs to $C^1(I,\mathbb{R})$ (resp., $C^0(I,\mathbb{R})$), satisfies the functional equation:*

$$A(x) - A(\tau(x)) = \varphi_1(x) - \varphi_2(a(x)) + \int_x^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \quad x \in I\,,$$

and furthermore:

$$B(y) = \varphi_2(y) - F(\beta(y),y) - A(\beta(y)) \quad \forall y \in J\,.$$

Then we give (see theorem 4.4) a sufficient condition for the existence of a weak solution of problem (G) (G′) which, when $\mathrm{Fix}\,\tau$ is countable, is also sufficient for uniqueness.

Moreover, in the case in which $\mathrm{Fix}\,\tau = \{p\}$, we give (see theorem 4.5) a sufficient condition for existence and uniqueness of weak solution of problem (G) (G′) and we prove (see theorem 4.6) that: if the following statements are verified:

i‴) *$\varphi_1(p) = \varphi_2(\alpha(p))$ and the functions $\tau(x)$, $\tau'(x)$, $\pi(x)$, $\pi'(x)$ are bounded on I,*

ii) $(\tau^k)_{k\in\mathbb{N}}$ *converges uniformly on* I,

iii) $|\tau'(p)| < 1$,

then there exists a unique solution of problem (G) (G′) given by:

$$z(x,y) = \sum_{k=0}^{\infty} \pi(\tau^k(x)) + \varphi_2(y) - \sum_{k=0}^{\infty} \pi\Big(\tau^k(\beta(y))\Big) - F(\beta(y),y) + F(x,y)\,,$$

being

$$\pi(x) = \varphi_1(x) - \varphi_2(\alpha(x)) + \int_x^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(x,y)dv$$

and

$$F(x,y) = \int_{x_0}^{x} du \int_{y_0}^{y} f(x,y)dv\,.$$

Let us conclude this section observing that the condition j) above considered implies the following two statements:

jj) $\mathrm{Fix}\,\tau = \{0\}$, $(\tau^k)_{k\in\mathbb{N}}$ *converges uniformly on* I *to function vanishing on* I,

jjj) $|\tau'(0)| < 1$;

so, if j) is true also the conditions ii) and iii) are satisfied.

2 – Existence and uniqueness of solution for the functional problem (P)

Throughout this section I is an interval of $\mathbb{R}$, $\tau(x)$ and $\varphi(x)$ are functions belonging, resp., to I^I and $\mathbb{R}^I$ such that:

$$\mathrm{Fix}\,\tau = \{p\}\,, \qquad \varphi(p) = 0\,.$$

Considered the functional problem (P) quoted in Introduction, let us prove that:

THEOREM 2.1. *If the following statements are verified:*

i) *there exists* $n \in \mathbb{N}$ *such that* $\tau(x) \in C^n(I, I)$, $\varphi(x) \in C^n(I, \mathbb{R})$ *and,* $\forall m \in \{0, \dots, n\}$, *the following functions:*

$$\frac{d^m \tau(x)}{dx^m}, \qquad \frac{d^m \varphi(x)}{dx^m}$$

are bounded on I,

ii) $(\tau^k)_{k \in \mathbb{N}}$ *converges uniformly on* I,

iii) $|\tau'(p)| < 1$,

then the series of real functions on I:

$$\sum_{k=0}^{\infty} \varphi(\tau^k(x)) \tag{2.1}$$

converges on I *to a function that is the unique solution of problem* (P) *belonging to* $C^n(I, \mathbb{R})$.

PROOF. Let us note that, being $\operatorname{Fix} \tau = \{p\}$, according to ii) and continuity of $\tau(x)$ we have:

$$\lim_k \tau^k(x) = p \qquad \forall x \in I\,, \tag{2.2}$$

consequently, $\tau(x)$ has no periodic point of period two[3].

Then, according to theorem 3.6 of [14], to prove theorem 2.1 it is enough to show that the series (2.1) converges on I to a function belonging to $C^n(I, \mathbb{R})$.

Now, from iii), being $\tau'(x)$ continuous on p there exist $\delta > 0$ and $\sigma \in]0, 1[$ such that:

$$(y \in I \quad \text{and} \quad |y - p| < \delta) \implies |\tau'(y)| < \sigma\,;$$

[3] A point x of I is said *periodic point of* $\tau(x)$ *of period two if:*

$$\tau(x) \neq x = \tau^2(x)\,.$$

consequently, from ii) and (2.2), there exists $\nu \in \mathbb{N}$ such that:

$$(2.3) \qquad \left|\tau'(\tau^k(x))\right| < \sigma \qquad \forall k \geq \nu \quad \forall x \in I .$$

On the other hand, it results:

$$D\Big(\varphi(\tau^{\nu+k}(x))\Big) = \varphi'(\tau^{\nu+k}(x))\tau'(\tau^{\nu+k-1}(x))\dots\tau'(\tau^{\nu}(x))\dots\tau'(x) \quad \forall k \in \mathbb{N}_0 ,$$

so, from i) and (2.3), there exists a positive real number c such that:

$$(2.4) \qquad \left|D\Big(\varphi(\tau^{\nu+k}(x))\Big)\right| \leq c\sigma^k \qquad \forall k \in \mathbb{N}_0 \quad \forall x \in I .$$

Thus, it results:

$$\begin{aligned}\sum_{k=0}^{\infty}\left|D\Big(\varphi(\tau^k(x))\Big)\right| &\leq \sum_{k=0}^{\nu-1}\left|D\Big(\varphi(\tau^k(x))\Big)\right| + \sum_{k=\nu}^{\infty} c\sigma^k \\ &= \sum_{k=0}^{\nu-1}\left|D\Big(\varphi(\tau^k(x))\Big)\right| + \frac{c\sigma^\nu}{1-\sigma}\end{aligned}$$

so, the series:

$$\sum_{k=0}^{\infty} D\Big(\varphi(\tau^k(x))\Big)$$

converges uniformly on I.

Then, since:

$$\sum_{k=0}^{\infty}\varphi(\tau^k(p)) = \sum_{k=0}^{\infty}\varphi(p) = 0 ,$$

by applying a standard theorem on the term-by-term differentiation of infinite series, it follows that the series (2.1) converges on I to a function belonging to $C^1(I, \mathbb{R})$.

The theorem is so proved in the case in which $n = 1$.

Now, the proof of theorem 2.1 can be easily completed bearing in mind the inequality (2.4) and proof of theorem I of [4].

From theorem 2.1 trivially follows that:

THEOREM 2.2. *If the following statements hold:*

i′) $\tau(x) \in C^\infty(I, I)$, $\varphi(x) \in C^\infty(I, \mathbb{R})$ *and,* $\forall n \in \mathbb{N}_0$, *the following functions:*

$$\frac{d^n\tau(x)}{dx^n}, \qquad \frac{d^n\varphi(x)}{dx^n}$$

are bounded on I,

ii) $(\tau^k)_{k\in\mathbb{N}}$ *converges uniformly on* I,

iii) $|\tau'(p)| < 1$,

then the series (2.1) *converges on* I *to a function that is the unique solution of problem* (P) *belonging to* $C^\infty(I, \mathbb{R})$.

Moreover:

THEOREM 2.3. *If the following statements hold:*

i″) $\tau\colon I \to I$, $\varphi\colon I \to \mathbb{R}$ *are real analytic functions such that, for each* $n \in \mathbb{N}_0$, *the following functions:*

$$\frac{d^n\tau(x)}{dx^n}, \qquad \frac{d^n\varphi(x)}{dx^n}$$

are bounded on I,

ii) $(\tau^k)_{k\in\mathbb{N}}$ *converges uniformly on* I,

iii) $|\tau'(p)| < 1$,

then the series (2.1) *converges on* I *to a function that is the unique solution of problem* (P) *belonging to the class of real analytic functions on* I.

PROOF. Theorem 2.2 implies the convergence on I of the series (2.1) to a function $g(x)$ belonging to $C^\infty(I, \mathbb{R})$.

So, we have only to prove that $g(x)$ is a real analytic function on I.

To this aim we can always suppose $p = 0$ and observe that ii) implies that:

$$\forall \eta > 0 \; \exists q = q(\eta)\colon |\tau^{n+1}(x)| < \eta \quad \forall x \in I \quad \forall n \geq q\,.$$

Then, the same proof of [4], p. 11, theorem III, shows that $g(x)$ is a real analytic function in each compact interval $J \subset I$. Consequently $g(x)$ is a real analytic function on I.

The assertion is so proved.

As we said in Introduction, theorems 2.1, 2.2 and 2.3 generalize, resp., theorems I, II and III of [4].

3 – Existence and uniqueness of solution for the functional problem (P′)

Let I and J be intervals of $\mathbb{R}$, $y_0 \in J$, $f(x,y) \in C^0(I \times J, \mathbb{R})$, $\alpha(x) \in C^0(I,J)$, $\beta(y) \in C^0(J,I)$, $\varphi_1(x) \in C^0(I,\mathbb{R})$, $\varphi_2(y) \in C^0(J,\mathbb{R})$.

In this section, we set:

$$\tau(x) = \beta(\alpha(x)) \qquad \forall x \in I$$

and suppose $\operatorname{Fix}\tau \neq \emptyset$.

We deal with functional problem (P′) quoted in Introduction.

Let us now consider the following statements which will be used in the next four theorems:

1) *$\alpha(x)$ is bounded on I and $f(x,y)$ is bounded on $I \times J$.*
2) *There exists $\mu > 0$ such that:*

$$|\varphi_1(x) - \varphi_2(\alpha(x))| \leq \mu|\tau(x) - x| \qquad \forall x \in I\,.$$

3) *$\tau(x)$ is bounded on I and has no periodic point of period two.*

3′) *$\tau(x)$ is bounded on I and $(\tau^k)_{k\in\mathbb{N}}$ converges uniformly on I.*

4) *Whatever be $p \in \operatorname{Fix}\tau$ there exist a neighbourhood U of p and $\sigma \in]0,1[$ such that:*

$$|\tau(x) - p| \leq \sigma|x - p| \qquad \forall x \in U\,.$$

Now, set:

$$\pi(x) = \varphi_1(x) - \varphi_2(\alpha(x)) + \int_x^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \qquad \forall x \in I\,,$$

let us prove the following

THEOREM 3.1. *Let the statements* 1) *and* 2) *be satisfied. If the series*

$$\sum_{k=0}^{\infty} |\tau^{k+1}(x) - \tau^k(x)| \tag{3.1}$$

converges on I, then the series:

$$\sum_{k=0}^{\infty} \pi(\tau^k(x)) \tag{3.2}$$

converges absolutely on I to a function that is solution of problem (P′).

PROOF. Let us start observing that:

P_1) *If series* (3.2) *converges on I to a function $A(x)$, then it results:*

$$A(p) = 0 \qquad \forall p \in \operatorname{Fix}\tau .$$

In fact, bearing in mind 2), for each $p \in \operatorname{Fix}\tau$ we have:

$$\pi(p) = \varphi_1(p) - \varphi_2(\alpha(p)) + \int_p^{\tau(p)} du \int_{y_0}^{\alpha(p)} f(u,v)dv = 0 ,$$

then, being $\tau^k(p) = p \quad \forall k \in \mathbb{N}$, it follows:

$$A(p) = 0 .$$

P_1) is so proved.

Let M_1 and M_2 be real numbers such that:

$$|f(x,y)| \le M_1 \quad \forall (x,y) \in I \times J , \qquad |\alpha(x)| \le M_2 \quad \forall x \in I .$$

Being:

$$\pi(\tau^k(x)) = \varphi_1(\tau^k(x)) - \varphi_2\Big(\alpha(\tau^k(x))\Big) + \int_{\tau^k(x)}^{\tau^{k+1}(x)} du \int_{y_0}^{\alpha(\tau^k(x))} f(u,v)dv ,$$

bearing in mind 2), we have:

$$\begin{aligned} |\pi(\tau^k(x))| &\le \mu|\tau^{k+1}(x) - \tau^k(x)| + M_1M_2|\tau^{k+1}(x) - \tau^k(x)| \\ &= (\mu + M_1M_2)|\tau^{k+1}(x) - \tau^k(x)| . \end{aligned} \tag{3.3}$$

Then, since the series (3.1) converges on I, series (3.2) converges absolutely on I to a function that we denote by $A(x)$.

Being:

$$A(\tau(x)) = \sum_{k=1}^{\infty} \pi(\tau^k(x)),$$

it results:

$$A(x) - A(\tau(x)) = \pi(x).$$

The theorem is so proved.

THEOREM 3.2. *Let the statements* 1) *and* 2) *be verified. If series* (3.1) *converges uniformly on I, then the series* (3.2) *converges uniformly on I to a function $A(x)$ that is a continuous solution of problem* (P').

Moreover, if $\operatorname{Fix}\tau$ *is countable, then $A(x)$ is the unique solution of problem* (P') *belonging to* $C^0(I, \mathbb{R})$.

PROOF. If series (3.1) converges uniformly on I, according to inequality (3.3) contained in proof of theorem 3.1, it follows that series (3.2) converges uniformly on I to a function $A(x)$.

So, from theorem 3.1, $A(x)$ is a solution of problem (P') and, since $\tau(x)$ and $\pi(x)$ are continuous on I, $A(x)$ is continuous on I.

Furthermore, if $\operatorname{Fix}\tau$ is countable, from theorem 3.6 of [14] follows that $A(x)$ is the unique continuous solution of problem (P').

The theorem is so proved.

THEOREM 3.3. *Let the statements* 1), 2), 3), 4) *be verified. Then the series* (3.2) *converges absolutely on I to a function $A(x)$ that is solution of problem* (P').

PROOF. According to theorem 3.1 it is enough to prove that:

P_2) *The series* (3.1) *converges on I.*

We shall prove P_2) utilizing only the hypotheses 3) and 4).

In accordance with theorem 4.2 of [13] and condition 3) there exists a function $g(x)$ from I onto $\operatorname{Fix}\tau$ such that:

$$g(x) = \lim_k \tau^k(x) \qquad \forall x \in I.$$

Then, whatever be $\bar{x} \in I$, considered the point $g(\bar{x})$ of $\operatorname{Fix}\tau$ we have that:

$$\forall H(g(\bar{x})) \quad \exists \nu_{\bar{x}}^{H} \in \mathbb{N}: \tau^{n}(\bar{x}) \in H \quad \forall n \geq \nu_{\bar{x}}^{H}. \tag{3.5}$$

On the other hand, from 4), relatively to $g(\bar{x})$ there exist a neighbourhood U of $g(\bar{x})$ and $\sigma \in]0,1[$ such that:

$$|\tau(x) - g(\bar{x})| \leq \sigma |x - g(\bar{x})| \qquad \forall x \in U. \tag{3.6}$$

Then, from (3.5) and (3.6) it follows that:

$$\exists \nu \in \mathbb{N}: |\tau(\tau^{n}(\bar{x})) - g(\bar{x})| \leq \sigma |\tau^{n}(\bar{x}) - g(\bar{x})| \qquad \forall n \geq \nu.$$

Consequently:

$$|\tau^{\nu+1}(\bar{x}) - g(\bar{x})| \leq \sigma |\tau^{\nu}(\bar{x}) - g(\bar{x})|$$
$$|\tau^{\nu+2}(\bar{x}) - g(\bar{x})| \leq \sigma^{2} |\tau^{\nu}(\bar{x}) - g(\bar{x})|$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$|\tau^{\nu+h}(\bar{x}) - g(\bar{x})| \leq \sigma^{h} |\tau^{\nu}(\bar{x}) - g(\bar{x})|$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Then whatever be $h \in \mathbb{N}_0$ it results:

$$\begin{aligned} |\tau^{\nu+h+1}(\bar{x}) - \tau^{\nu+h}(\bar{x})| &\leq |\tau^{\nu+h+1}(\bar{x}) - g(\bar{x})| + |\tau^{\nu+h}(\bar{x}) - g(\bar{x})| \\ &\leq \sigma^{h+1} |\tau^{\nu}(\bar{x}) - g(\bar{x})| + \sigma^{h} |\tau^{\nu}(\bar{x}) - g(\bar{x})| \\ &= \sigma^{h}(\sigma + 1) |\tau^{\nu}(\bar{x}) - g(\bar{x})|. \end{aligned}$$

Consequently, it is trivial to prove that series (3.1) converges on $\bar{x}$. Since $\bar{x}$ is a generic point of I, series (3.1) converges on I. Proposition P_2) is so proved.

THEOREM 3.4. *Let the statements* 1), 2), 3′), 4) *be verified. Then* $\operatorname{Fix}\tau$ *is a singleton and the series* (3.2) *converges uniformly on* I *to a function* $A(x)$ *that is the unique solution of problem* (P′) *belonging to* $C^{0}(I, \mathbb{R})$.

PROOF. According to theorem 3.2 it is enough to prove that:

P$_3$) $\mathrm{Fix}\,\tau$ *is a singleton and the series* (3.1) *converges uniformly on* I.

We shall prove P$_3$) utilizing only the hypotheses 3′) and 4).

From 3′), being $\tau(x)$ continuous, it easily follows that there exists a continuous function $g(x)$ from I onto $\mathrm{Fix}\,\tau$ such that:

$$g(x) = \lim_k \tau^k(x) \qquad \forall x \in I .$$

Then, since I is an interval, the set $\mathrm{Fix}\,\tau$ is a singleton or an interval; so, from 4) it follows that $\mathrm{Fix}\,\tau$ is a singleton.

Said p the unique point of $\mathrm{Fix}\,\tau$, it results:

$$\lim_k \tau^k(x) = p \qquad \forall x \in I .$$

Then, from 3′) and 4) it easily follows that:

$$\exists \nu \in \mathbb{N} \colon \left|\tau(\tau^n(x)) - p\right| \le \sigma\left|\tau^n(x) - p\right| \qquad \forall n \ge \nu \quad \forall x \in I ,$$

so, we have:

$$\left|\tau^{\nu+h}(x)) - p\right| \le \sigma^h\left|\tau^\nu(x) - p\right| \qquad \forall h \in \mathbb{N} \quad \forall x \in I .$$

Consequently, bearing in mind the proof of theorem 3.3, it follows that series (3.1) converges uniformly on I.

Proposition P$_3$) is so proved.

4 – Goursat problem

Let I and J be intervals of $\mathbb{R}$, $(x_0, y_0) \in I \times J$, $f(x,y) \in C^0(I \times J, \mathbb{R})$, $\alpha(x) \in J^I$, $\beta(y) \in I^J$, $\varphi_1(x) \in \mathbb{R}^I$, $\varphi_2(y) \in \mathbb{R}^J$.

Considered the equation of hyperbolic type (G) according to [7, 14] we said *solution* (resp., *weak solution*) of (G) a function of the form:

$$z(x,y) = A(x) + B(y) + \int_{x_0}^{x} du \int_{y_0}^{y} f(u,v)dv ,$$

being $A(x) \in C^1(I, \mathbb{R})$ (resp., $A(x) \in C^0(I, \mathbb{R})$) and $B(y) \in C^1(J, \mathbb{R})$ (resp., $B(y) \in C^0(J, \mathbb{R})$).

This section is devoted to problem of the existence and uniqueness (resp., existence and existence and uniqueness) of solution (resp., weak solution) of (G) satisfying the condition (G′) where $\alpha(x)$, $\beta(y)$, $\varphi_1(x)$, $\varphi_2(y)$ belong to class C^1 (resp., C^0).

Set:

$$\tau(x) = \beta(\alpha(x)) \quad \forall x \in I, \qquad F(x,y) = \int_{x_0}^{x} du \int_{y_0}^{y} f(u,v)dv \quad \forall (x,y) \in I \times J,$$

let us prove that:

PROPOSITION 4.1. *Whatever be the functions $A(x) \in \mathbb{R}^I$ and $B(y) \in \mathbb{R}^J$ the following statements are equivalent:*

1) *The function $z(x,y) = A(x) + B(y) + F(x,y)$ is a solution (resp., weak solution) of problem* (G) (G′).
2) *$A(x)$ belongs to $C^1(I, \mathbb{R})$ (resp., $C^0(I, \mathbb{R})$) and satisfies the functional equation:*

$$A(x) - A(\tau(x)) = \varphi_1(x) - \varphi_2(\alpha(x)) + \int_{x}^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \qquad x \in I,$$

furthermore:

$$\text{(h)} \qquad B(y) = \varphi_2(y) - F(\beta(y), y) - A(\beta(y)) \qquad \forall y \in J.$$

PROOF. 1) $\Longrightarrow$ 2). Since $z(x,y)$ is a solution (resp., weak solution) of problem (G) (G′), we have:

$$\text{(4.1)} \qquad \varphi_1(x) = z(x, \alpha(x)) = A(x) + B(\alpha(x)) + F(x, \alpha(x))$$

$$\text{(4.2)} \qquad \varphi_2(y) = z(\beta(y), y) = A(\beta(y)) + B(y) + F(\beta(y), y),$$

from which, replacing y by $\alpha(x)$ in (4.2), we have:

$$\text{(4.3)} \qquad \varphi_2(\alpha(x)) = A(\tau(x)) + B(\alpha(x)) + F(\tau(x), \alpha(x)),$$

thus, subtracting (4.3) from (4.1), it follows:

$$\begin{aligned} A(x) - A(\tau(x)) &= \varphi_1(x) - \varphi_2(\alpha(x)) + F(\tau(x), \alpha(x)) - F(x, \alpha(x)) \\ &= \varphi_1(x) - \varphi_2(\alpha(x)) + \int_{x_0}^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \\ &\quad - \int_{x_0}^{x} du \int_{y_0}^{\alpha(x)} f(u,v)dv \\ &= \varphi_1(x) - \varphi_2(\alpha(x)) + \int_{x}^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \,. \end{aligned}$$

On the other hand, from (4.2) it follows (h).

2) $\Longrightarrow$ 1). It is clear that the function:

$$z(x,y) = A(x) + \varphi_2(y) - A(\beta(y)) - F(\beta(y), y) + F(x,y)$$

is a solution (resp., weak solution) of (G).

On the other hand, let us note that:

$$\begin{aligned} z(x, \alpha(x)) &= A(x) + \varphi_2(\alpha(x)) - A(\tau(x)) - F(\tau(x), \alpha(x)) + F(x, \alpha(x)) \\ &= A(x) - A(\tau(x)) + \varphi_2(\alpha(x)) - \int_{x}^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \\ &= \varphi_1(x)\,, \end{aligned}$$

moreover:

$$\begin{aligned} z(\beta(y), y) &= A(\beta(y)) + \varphi_2(y) - A(\beta(y)) - F(\beta(y), y) + F(\beta(y), y) \\ &= \varphi_2(y)\,. \end{aligned}$$

So, the condition (G′) is also satisfied.

The theorem is thus proved.

Now, supposed $\mathrm{Fix}\,\tau \neq \emptyset$ and fixed a point $p' \in \mathrm{Fix}\,\tau$, we consider the functional problem (P′) studied in section 3.

Let us prove that:

PROPOSITION 4.2. *Problem* (G) (G′) *has a solution (resp., weak solution) if and only if functional problem* (P′) *has a solution belonging to* $C^1(I, \mathbb{R})$ *(resp.,* $C^0(I, \mathbb{R})$*).*

PROOF. Let $z(x,y) = A(x) + B(y) + F(x,y)$ be solution (resp., weak solution) of problem (G)(G′). Put $A^*(x) = A(x) - A(p')$ and $B^*(y) = B(y) + A(p')$, we have that: $z(x,y) = A^*(x) + B^*(y) + F(x,y)$, $A^*(x) \in C^1(I, \mathbb{R})$ (resp., $C^0(I, \mathbb{R})$) and $A^*(p') = 0$.

Thus, according to proposition 4.1, it results that $A^*(x)$ is a solution of functional problem (P′).

Conversely, let $A^*(x)$ be a solution of functional problem (P′) belonging to $C^1(I, \mathbb{R})$ (resp., $C^0(I, \mathbb{R})$).

According to proposition 4.1, set $B^*(y) = \varphi_2(y) - F(\beta(y), y) - A^*(\beta(y))$, we have that:

$$z(x,y) = A^*(x) + B^*(y) + F(x,y)$$

is a solution (resp., weak solution) of problem (G)(G′).

The theorem is so proved.

From propositions 4.1 and 4.2, bearing in mind the proof of theorem 4.3 of [14], it can be easily proved that:

PROPOSITION 4.3. *The following statements are equivalent:*

1) *Problem* (G) (G′) *has a unique solution (resp., weak solution).*
2) *Functional problem* (P′) *has a unique solution belonging to* $C^1(I, \mathbb{R})$ *(resp.,* $C^0(I, \mathbb{R})$*).*

Now, set:

$$\pi(x) = \varphi_1(x) - \varphi_2(\alpha(x)) + \int_x^{\tau(x)} du \int_{y_0}^{\alpha(x)} f(u,v)dv \qquad \forall x \in I\,,$$

let us observe that, according to theorems 3.2, 4.1, 4.2 and 4.3, it is easy to prove that:

THEOREM 4.4. *Let the following statements be verified:*

1) $\alpha(x)$ *is bounded on* I *and* $f(x,y)$ *is bounded on* $I \times J$.
2) *There exists* $\mu > 0$ *such that:*

$$|\varphi_1(x) - \varphi_2(\alpha(x))| \le \mu|\tau(x) - x| \qquad \forall x \in I\,.$$

If series:

$$\sum_{k=0}^{\infty} |\tau^{k+1}(x) - \tau^k(x)|$$

converges uniformly on I*, then problem* (G) (G′) *has a weak solution given by:*

$$z(x,y) = \sum_{k=0}^{\infty} \pi(\tau^k(x)) + \varphi_2(y) - \sum_{k=0}^{\infty} \pi(\tau^k(\beta(y))) - F(\beta(y),y) + F(x,y)\,.$$

Moreover, if $\operatorname{Fix}\tau$ *is countable, then* $z(x,y)$ *is the unique weak solution of problem* (G) (G′).

Next two theorems are referred to case in which:

$$\operatorname{Fix}\tau = \{p\}\,.$$

From theorems 3.4, 4.1 and 4.3 it easily follows that:

THEOREM 4.5. *Let the following statements be verified:*

1) $\alpha(x)$ *is bounded on* I *and* $f(x,y)$ *is bounded on* $I \times J$.
2) *There exists* $\mu > 0$ *such that:*

$$|\varphi_1(x) - \varphi_2(\alpha(x))| \le \mu|\tau(x) - x| \qquad \forall x \in I\,.$$

3′) $\tau(x)$ *is bounded on* I *and* $(\tau^k)_{k\in\mathbb{N}}$ *converges uniformly on* I.

4′) *There exist a neighbourhood U of p and $\sigma \in]0,1[$ such that:*

$$|\tau(x) - p| \leq \sigma |x - p| \qquad \forall x \in U\,.$$

Then problem (G) (G′) *has a unique weak solution given by:*

$$z(x,y) = \sum_{k=0}^{\infty} \pi(\tau^k(x)) + \varphi_2(y) - \sum_{k=0}^{\infty} \pi(\tau^k(\beta(y)))$$
$$- F(\beta(y),y) + F(x,y)\,.$$

Now, let us prove that:

THEOREM 4.6. *If the following statements hold:*

i‴) $\varphi_1(p) = \varphi_2(\alpha(p))$ *and the functions* $\tau(x)$, $\tau'(x)$, $\pi(x)$, $\pi'(x)$ *are bounded on* I,

ii) $(\tau^k)_{k\in\mathbb{N}}$ *converges uniformly on* I,

iii) $|\tau'(p)| < 1$,

then there exists a unique solution of problem (G) (G′) *given by:*

$$z(x,y) = \sum_{k=0}^{\infty} \pi(\tau^k(x)) + \varphi_2(y) - \sum_{k=0}^{\infty} \pi(\tau^k(\beta(y)))$$
$$- F(\beta(y),y) + F(x,y)\,.$$

PROOF. According to theorem 2.1 the series:

$$\sum_{k=0}^{\infty} \pi(\tau^k(x))$$

converges to a function that is the unique solution of problem (P′) belonging to $C^1(I,\mathbb{R})$.

So, from proposition 4.3 there exists a unique solution of problem (G) (G′); thus, in accordance with theorem 4.1 it follows the assertion. The theorem is so proved.

The following two remarks conclude this section:

O_1) If $\tau(x) \in C^1([0,a],[0,a])$ and there exists $\sigma \in]0,1[$ such that:

$$\tau(x) < \sigma x \qquad \forall x \in]0,a]\,,$$

then the conditions i‴), ii) and iii) are satisfied; thus, the theorem on existence and uniqueness of a solution to problem (G) (G′) proved by Goursat in section 9 of [10] is generalized by theorem 4.6.

O_2) Bearing in mind Lemma 3 of [3] we have that: if the hypotheses of theorem 3 of [3] are satisfied, then i‴), ii) and iii) hold; thus, relative to the question of existence and uniqueness of solution to problem (G) (G′), theorem 4.6 generalizes theorem 3 of [3].

5 – Appendix

Throughout this section I is an interval of $\mathbb{R}$, $\tau(x)$ and $\varphi(x)$ are functions belonging, resp., to $C^0(I,I)$ and $C^0(I,\mathbb{R})$ such that:

$$\mathrm{Fix}\,\tau \neq \emptyset \quad \text{and} \quad \varphi(p) = 0 \qquad \forall p \in \mathrm{Fix}\,\tau\,.$$

We consider the following functional problem:

$$(\mathrm{P}'') \qquad \begin{aligned} &A(x) - A(\tau(x)) = \varphi(x) \qquad x \in I \\ &A(p') = 0\,. \end{aligned}$$

where p' is a point of $\mathrm{Fix}\,\tau$.

The problem (P″), which has already been considered in [7, 14], includes both the problem (P) considered in section 2 and the problem (P′) considered in section 3.

The main aim of this section is to point out the possibility of obtaining theorems relative to problem (P″) which include the ones demonstrated in section 3 relative to problem (P′).

In fact, considered the following statements:

3) *$\tau(x)$ is bounded on I and has no periodic point of period two.*

3′) *$\tau(x)$ is bounded on I and $(\tau^k)_{k\in\mathbb{N}}$ converges uniformly on I.*

4) *Whatever be* $p \in \operatorname{Fix}\tau$ *there exist a neighbouhood* U *of* p *and* $\sigma \in]0,1[$ *such that:*

$$|\tau(x) - p| \leq \sigma |x - p| \qquad \forall x \in U\,.$$

5) *There exists* $c > 0$ *such that:*

$$|\varphi(x)| \leq c|\tau(x) - x| \qquad \forall x \in I\,,$$

bearing in mind the proofs of theorems of section 3 it is easily proved that:

PROPOSITION 5.1. *Let the statement* 5) *be satisfied. If the series:*

$$\sum_{k=0}^{\infty} |\tau^{k+1}(x) - \tau^k(x)| \tag{5.1}$$

converges on I, *then the series:*

$$\sum_{k=0}^{\infty} \varphi(\tau^k(x)) \tag{5.2}$$

converges absolutely on I *to a function that is solution of problem* (P″).

PROPOSITION 5.2. *Let the statement* 5) *be verified. If series* (5.1) *converges uniformly on* I, *then the series* (5.2) *converges uniformly on* I *to a function* $A(x)$ *that is a continuous solution of problem* (P″).

Moreover, if $\operatorname{Fix}\tau$ *is countable, then* $A(x)$ *is the unique solution of problem* (P″) *belonging to* $C^0(I,\mathbb{R})$.

PROPOSITION 5.3. *Let the statements* 3), 4), 5) *be verified. Then the series* (5.2) *converges absolutely on* I *to a function* $A(x)$ *that is solution of problem* (P″).

PROPOSITION 5.4. *Let the statements* 3′), 4), 5) *be verified. Then* $\operatorname{Fix}\tau$ *is a singleton and the series* (5.2) *converges uniformly on* I *to a function* $A(x)$ *that is the unique solution of problem* (P″) *belonging to* $C^0(I,\mathbb{R})$.

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