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A necessary and sufficient condition for oscillation of neutral type hyperbolic equations

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RIASSUNTO – *Si ottiene una condizione necessaria e sufficiente per l'oscillazione delle soluzioni delle equazioni differenziali iperboliche lineari di tipo neutro della forma* (*) *dove* Ω *è un dominio limitato di* $\mathbb{R}^n$ *con una frontiera regolare e* $\Delta u(x,t) = \sum_{i=1}^{n} u_{x_i x_i}(x,t)$.

ABSTRACT – *In the present paper a necessary and sufficient condition is obtained for oscillation of the solutions of neutral type linear hyperbolic differential equations of the form*

$$(*) \qquad \begin{aligned} &u_{tt}(x,t) + \sum_{I} a_i u_{tt}(x,t-\tau_i) - \Big[\Delta u(x,t) + \sum_{J} b_j(t)\Delta u(x,\rho_j(t))\Big] + \\ &+ cu(x,t) + \sum_{K} c_k u(x,t-\sigma_k) = 0\,, \quad (x,t) \in \Omega \times (0,\infty) \equiv G\,, \end{aligned}$$

where Ω *is a bounded domain in* $\mathbb{R}^n$ *with a piecewise smooth boundary and* $\Delta u(x,t) = \sum_{i=1}^{n} u_{x_i x_i}(x,t)$.

KEY WORDS – *Oscillation - Neutral type hyperbolic equation.*

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1 – Introduction

In the recent few years the fundamental theory of partial differential equations with a deviating argument has been developed intensively. However the qualitative theory of these classes of partial differential equations, important for the applications, is still in an initial stage of its development. Thus, for instance, to the oscillation theory for this class of equations only a small number of papers in the period since 1984. Sufficient conditions for oscillation of the solutions of hyperbolic differential equations with delay were obtained in the paper of GEORGIOU and KREITH [2]. Conditions for oscillation of the solutions of neutral typer hyperbolic differential equations were obtained by MISHEV and BAINOV [3], [4] and YOSHIDA [7].

2 – Statement of the problem

In the present paper a necessary and sufficient condition for oscillation of the solutions of neutral type linear hyperbolic equations of the form

$$\begin{aligned} &u_{tt}(x,t)+\sum_I a_i u_{tt}(x,t-\tau_i)-\Big[\Delta u(x,t)+\sum_J b_j(t)\Delta u(x,\rho_j(t))\Big]+ \\ &+cu(x,t)+\sum_K c_k u(x,t-\sigma_k)=0\,, \quad (x,t)\in\Omega\times(0,\infty)\equiv G\,, \end{aligned} \tag{1}$$

is obtained, where Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary, $\Delta u(x,t)=\sum_{i=1}^{n} u_{x_ix_i}(x,t)$, I, J, K are finite sets of successive positive integers containing the number 1.

Consider boundary conditions of the form

$$\frac{\partial u}{\partial n}=0\,, \quad (x,t)\in\partial\Omega\times[0,\infty) \tag{2}$$

and

$$u=0\,, \quad (x,t)\in\partial\Omega\times[0,\infty)\,. \tag{3}$$

We shall say that conditions (H) are met if the following conditions hold:

H1. $b_j(t)\in C([0,\infty);\mathbb{R})$ for $j\in J$.

H2. $\rho_j(t) \in C([0,\infty) : \mathbb{R})$ and $\lim_{t\to\infty} \rho_j(t) = \infty$ for $j \in J$.
H3. $c, a_i, c_k \in \mathbb{R}$ for $i \in I, k \in K$.
H4. $\tau_i = \text{const} \geq 0, \sigma_k = \text{const} \geq 0$ for $i \in I, k \in K$.

DEFINITION 1. *The solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ of problem* (1), (2) ((1), (3)) *is said to oscillate in the domain G if for any positive number α there exists a point $(x_0, t_0) \in \Omega \times (\alpha, \infty)$ such that the equality $u(x_0, t_0) = 0$ should hold.*

3 – Main results

In the subsequent theorems a necessary and sufficient condition for oscillation of the solutions of problems (1), (2) and (1), (3) in the domain G is obtained.

With each solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ of problem (1), (2) we associate the function

$$v(t) = \int_\Omega u(x,t)dx, \quad t \geq 0. \tag{4}$$

LEMMA 1. *Let conditions (H) hold and let $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ be a solution of problem* (1), (2). *The the functional $v(t)$ defined by* (4) *satisfies the differential equation*

$$v''(t) + \sum_{I} a_i v''(t-\tau_i) + cv(t) + \sum_{K} c_k v(t-\sigma_k) = 0, \quad t \geq t_0, \tag{5}$$

where t_0 is a sufficiently large positive number.

PROOF. Let $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ be a solution of problem (1), (2). From condition H2 it follows that there exists a number $\mu > 0$ such that $\rho_j(t) \geq 0$ for $t \geq \mu, j \in J$. Introduce the notation

$$t_0 = \max\{\mu, \tau_i, \sigma_k; i \in I, k \in K\}.$$

Integrate both sides of equation (1) with respect to x over the domain Ω and for $t \geq t_0$ obtain

$$\begin{aligned}(6)\qquad &\frac{d^2}{dt^2}\Big[\int\limits_\Omega u(x,t)dx + \sum_I a_i \int\limits_\Omega u(x,t-\tau_i)dx\Big] - \\ &-\Big[\int\limits_\Omega \Delta u(x,t)dx + \sum_J b_j(t) \int\limits_\Omega \Delta u(x,\rho_j(t))dx\Big] + \\ &+c\int\limits_\Omega u(x,t)dx + \sum_K c_k \int\limits_\Omega u(x,t-\sigma_k)dx = 0\,.\end{aligned}$$

From Green's formula it follows that

$$(7)\qquad \int\limits_\Omega \Delta u(x,t)dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0\,,$$

$$(8)\qquad \int\limits_\Omega \Delta u(x,\rho_j(t))dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}(x,\rho_j(t))dS = 0\,, \quad j \in J$$

Using (7) and (8), from (6) we obtain that

$$v''(t) + \sum_I a_i v''(t-\tau_i) + cv(t) + \sum_K c_k v(t-\sigma_k) = 0$$

which was to be proved. □

In the domain ω consider the Dirichlet problem

$$(9)\qquad \Delta U(x) + \alpha U(x) = 0\,, \quad x \in \Omega\,,$$

$$(10)\qquad U(x) = 0\,, \quad x \in \partial\Omega\,,$$

where $\alpha =$ const. It is well known [5] that the smallest eigenvalue α_0 of problem (9), (10) is positive and the corresponding eigenfunction $\varphi(x)$ can be chosen so that $\varphi(x) > 0$ for $x \in \Omega$.

Assume the following two additional conditions fulfilled:

H5. $b_j(t) \equiv b_j \in \mathbb{R}$ for $j \in J$.

H6. $\rho_j(t) = t - \mu_j$ for $j \in J$, where $\mu_j =$ const ≥ 0.

With each solution $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ of problem (1), (3) we associate the function

$$\omega(t) = \int_\Omega u(x,t)\varphi(x)dx\,, \quad t \geq 0\,. \tag{11}$$

We shall note that such averaging was first used by YOSHIDA [6].

LEMMA 2. *Let conditions H3-H6 hold and let* $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ *be a solution of problem* (1), (3). *Then the funtion* $\omega(t)$ *defined by* (11) *satisfies the differential equation*

$$\begin{aligned} &\omega''(t) + \sum_I a_i\omega''(t-\tau_i) + \alpha_0\Big[\omega(t) + \sum_J b_j\omega(t-\mu_j)\Big] \\ &+c\omega(t) + \sum_K c_k\omega(t-\sigma_k) = 0\,, \quad t \geq t_0\,, \end{aligned} \tag{12}$$

where t_0 *is a sufficiently large positive number.*

PROOF. Let $u(x,t) \in C^2(G) \cap C^1(\overline{G})$ be a solution of problem (1), (3). Introduce the notation

$$t_0 = \max\{\tau_i, \mu_j, \sigma_k : i \in I, j \in J, k \in K\}\,.$$

Multiply both sides of equation (1) by the eigenfunction $\varphi(x)$ and integrate with respect to x over the domain Ω. For $t \geq t_0$ we obtain

$$\begin{aligned} &\frac{d^2}{dt^2}\Big[\int_\Omega u(x,t)\varphi(x)dx + \sum_I a_i \int_\Omega u(x,t-\tau_i)\varphi(x)dx\Big] + \\ &-\Big[\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_J b_j \int_\Omega \Delta u(x,t-\mu_j)\varphi(x)dx\Big] + \\ &+c\int_\Omega u(x,t)\varphi(x)dx + \sum_K c_k \int_\Omega u(x,t-\sigma_k)\varphi(x)dx = 0\,. \end{aligned} \tag{13}$$

From Green's formula it follows that

$$\begin{aligned} &\int_\Omega \Delta u(x,t)\varphi(x)dx = \int_\Omega u(x,t)\Delta\varphi(x)dx = \\ &= -\alpha_0 \int_\Omega u(x,t)\varphi(x)dx = -\alpha_0\omega(t)\,, \end{aligned} \tag{14}$$

$$\begin{aligned}(15)\qquad &\int_\Omega \Delta u(x,t-\mu_j)\varphi(x)dx = \int_\Omega u(x,t-\mu_j)\Delta\varphi(x)dx = \\ &= -\alpha_0 \int_\Omega u(x,t-\mu_j)\varphi(x)dx = -\alpha_0\omega(t-\mu_j)\,.\end{aligned}$$

Using (14) and (15), from (13) we obtain

$$\begin{aligned}&\omega''(t) + \sum_I a_i\omega''(t-\tau_i) + \alpha_0\Big[\omega(t) + \sum_J b_j\omega(t-\mu_j)\Big] \\ &+\alpha\omega(t) + \sum_K c_k\omega(t-\sigma_k) = 0\end{aligned}$$

which was to be proved. □

From the lemmas proved above it follows that the finding of conditions for oscillation of the solutions of equation (1) in the domain G is reduced to the investigation of the oscillatory properties of neutral type ordinary differential equations of the form

$$(16)\qquad \frac{d^2}{dt^2}\Big[x(t) + \sum_I p_i x(t-\tau_i)\Big] + \sum_K q_k x(t-\sigma_k) = 0\,, \quad t \geq t_0\,.$$

Assume that the following conditions are fulfilled:

H7. $p_i \in \mathbb{R}, \tau_i = \text{const} \geq 0$ for $i \in I$.

H8. $q_k \in \mathbb{R}, \sigma = \text{const} \geq 0$ for $k \in K$.

DEFINITION 2. *The solution $x(t)$ of the differential* (16) *is said to oscillate if the function $x(t)$ has a sequence of zeros tending to $+\infty$. Otherwise the solution is said to be nonoscillating.*

In the proof of the subsequent theorems we shall use the following result of ARINO and GYÖRI [1].

THEOREM 1 [1]. *Let conditions H7-H8 hold. A necessary and sufficient condition for equation* (16) *to have a nonoscillating solution is that the corresponding characteristic equation*

$$(17)\qquad \lambda^2\Big[1 + \sum_I p_i e^{-\lambda\tau_i}\Big] + \sum_K q_k e^{-\lambda\sigma_k} = 0$$

should have a real root.

A corollary of Lemma 1 and Theorem 1 is the following necessary and sufficient condition for oscillation of the solutions of problem (1), (2).

THEOREM 2. *Let condition (H) hold. A necessary and sufficient condition for all solutions of problem* (1), (2) *to oscillate in the domain G is that the equation*

$$f(\lambda) \equiv \lambda^2\Big[1+\sum_I a_i e^{-\lambda\tau_i}\Big] + c + \sum_K c_k e^{-\lambda\sigma_k} = 0 \tag{18}$$

should have no real roots.

PROOF. Necessity. If $\lambda_0 \in \mathbb{R}$ is a root of equation (18), then the function $u(x,t) = e^{\lambda_0 t}$ is a nonoscillating positive solution of problem (1), (2) in the domain $\Omega \times [t_0, \infty)$.

Sufficiency. Suppose that the assertion is not true and let $u(x,t)$ be a nonoscillating solution of problem (1), (2). Let $u(x,t) > 0$ for $(x,t) \in G$. (The case when $u(x,t) < 0$ for $(x,t) \in G$ is considered analogously. From Lemma 1 it follows that the function $v(t)$ defined by (4) is a positive solution of the differential equation (5). Then from Theorem 1 applied to (5) it follows that equation (18) has a real root which contradicts the condition of the theorem. □

A corollary of Lemma 2 and Theorem 1 is the following necessary and sufficient condition for oscillation of the solutions of problem (1), (3).

THEOREM 3. *Let conditions $H3 - H6$ hold. Then a necessary and sufficient condition for all solutions of problem* (1), (3) *to oscillate in the domain G is that the equation*

$$g(\lambda) \equiv \lambda^2\Big[1+\sum_I a_i e^{-\lambda\tau_i}\Big] + \alpha_0\Big[1+\sum_J b_j e^{-\lambda\mu_j}\Big] + c + \sum_K c_k e^{-\lambda\sigma_k} = 0 \tag{19}$$

should have no real roots.

PROOF. Necessity. If $\lambda_0 \in \mathbb{R}$ is a root of equation (19), then the function $u(x,t) = e^{\lambda_0 t}\varphi(x)$ is a nonoscillating positive solution of problem (1), (3) in the domain $\Omega \times [t_0, \infty)$.

Sufficiency. Suppose that the assertion is not true and let $u(x,t)$ be a nonoscillating solution of problem (1), (3). Let $u(x,t) > 0$ for $(x,t) \in G$.

(The case when $u(x,t)<0$ for $(x,t)\in G$ is considered analogously.) From Lemma 2 it follows that the function $\omega(t)$ defined by (11) is a positive solution of the differential equation (12). Then from Theorem 1 applied to (12) it follows that equation (19) has a real root which contradicts the condition of the theorem. □

EXAMPLE 1. Consider the equation

$$
\begin{aligned}
&u_{tt}(x,t)+a_1u_{tt}(x,t-\tau_1)-\Big[\Delta u(x,t)+b_1(t)\Delta u(x,\rho_1(t))\Big]+\\
&+cu(x,t)+c_1u(x,t-\sigma_1)=0\,,\quad (x,t)\in\Omega\times(0,\infty)\equiv G\,,
\end{aligned}
\tag{20}
$$

with boundary condition (2). Assume that the coefficients of (20) satisfy the conditions:

$$
\begin{aligned}
&b_1(t),\rho_1(t)\in C([0,\infty);\mathbb{R}),\ \lim_{t\to\infty}\rho_1(t)=\infty\,,\\
&a_1,c,c_1\in\mathbb{R},\tau_1,\sigma_1=\mathrm{const}>0,c+c_1<0\,.
\end{aligned}
$$

In this case equation (18) takes the form

$$
f(\lambda)\equiv\lambda^2\Big(1+a_1e^{-\lambda\tau_1}\Big)+c+c_1e^{-\lambda\sigma_1}=0\,.
$$

A straightforward verification yields $f(0)f(+\infty)<0$. Hence the equation has a real root and Theorem 2 implies that problem (20), (2) has a nonoscillating solution.

EXAMPLE 2. Consider the equation

$$
\begin{aligned}
&u_{tt}(x,t)+a_1u_{tt}(x,t-\tau_1)-\Big[\Delta u(x,t)+b_1\Delta u(x,t-\mu_1)\Big]+\\
&+cu(x,t)+c_1u(x,t-\sigma_1)=0\,,\quad (x,t)\in\Omega\times(0,\infty)\equiv G\,,
\end{aligned}
\tag{21}
$$

with boundary condition (3). Assume that the coefficients of (21) satisfy the conditions:

$$
\begin{aligned}
&a_1,b_1,c,c_1\in\mathbb{R}\,,\\
&\tau_1,\mu_1,\sigma_1=\mathrm{const}>0\,,\\
&\alpha_0(1+b_1)+c+c_1<0\,.
\end{aligned}
$$

In this equation (19) takes the form

$$
g(\lambda)\equiv\lambda^2\Big(1+a_1e^{-\lambda\tau_1}\Big)+\alpha_0\Big(1+b_1e^{-\lambda\mu_1}\Big)+c+c_1e^{-\lambda\sigma_1}=0\,,
$$

where $\alpha_0 = \text{const} > 0$ is the eigenvalue of problem (9), (10). A straightforward verification yields $g(0)g(+\infty) < 0$. Hence the equation ha a real root and Theorem 3 implies that problem (21), (3) has a nonoscillating solution.

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