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The spectra and numerical ranges of nonlinear operators in reflexive spaces

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RIASSUNTO – *Lo scopo di questo lavoro è quello di dimostrare che lo spettro di ogni applicazione semicontinua da uno spazio di Banach reale e riflessivo al suo duale è contenuto nella chiusura del suo range numerico. Si ottiene inoltre un teorema di esistenza ed unicità per una classe di equazioni funzionali non lineari.*

ABSTRACT – *The aim of this paper is to prove that the spectrum of a demicontinuous function from a reflexive real Banach space to its dual is contained in the closure of its numerical range. As a by-product of this, we obtain an existence theorem for the solvability of nonlinear functional equations in Banach spaces.*

KEY WORDS – *Nonlinear operators - Spectra - Numerical ranges.*

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1 – Introduction

BROWDER [1-2] and MINTY [5] have established a number of results in a series of papers for the nonlinear equations involving monotone operators in Banach spaces. Browder has applied these results to find the general existence and uniqueness theorems for the solutions of nonlinear boundary value problems for different settings of nonlinear partial differential equations. Later ZARANTONELLO [9] has proved a related theorem for continuous Hilbert space operators. He has also shown that the closure of the numerical range contains the spectrum.

The main aim of this paper is to prove that the spectrum of a demicontinuous function from a reflexive real Banach space to its dual is contained in the closure of its numerical range. As a by-product of this, we obtain an existence theorem for the solvability of nonlinear functional equations in Banach spaces.

– Definitions and notations

Let X denote a reflexive real Banach space and X^* its dual. A function $A\colon D(A) \subset X \longrightarrow X^*$ is said to be *demicontinuous* at a point $x \in D(A)$ if for $\{x_n\} \subset D(A)$, $x_n \to x$ implies $Ax_n \xrightarrow{\omega} Ax$. The symbol "$\longrightarrow$"("$\xrightarrow{\omega}$") denotes strong (weak) convergence.

1.1 – Duality mapping

We recall that a continuous fucntion $\mu\colon \mathbb{R}^+ = \{t\colon t \geq 0\} \to \mathbb{R}^+$ is called a gauge function if $\mu(0) = 0$, and μ is strictly increasing. Let X be a reflexive real Banach space and X^* its dual. We denote by $[\cdot,\cdot]$ the duality pairing between X^* and X. A mapping $J\colon X \to X^*$ is said to be a duality mapping between X and X^* with respect to gauge function μ if

$$(C1) \qquad [Jx, x] = \mu(\|x\|)\|x\|, \quad \text{and}$$

$$(C2) \qquad \|Jx\| = \mu(\|x\|) \quad \text{for} \quad x \in X.$$

We note that if $\mu(t) = t$, J is said to be a normalized duality mapping. If X^* is strictly convex, then J is uniquely determined by μ, and if X is also reflexive, then J is a single-valued demicontinuous mapping X onto X^*, which is bounded and positively homogeneous. Furthermore, J, is monotone and satisfies the property

$$(C3) \qquad [Jx - Jy, x - y] = [Jx, x - y] - [Jy, x - y] \geq$$
$$\geq |\mu(\|x\|) - \mu(\|y\|)|\|x - y\|$$

for all $x, y \in X$.

When J is normalized duality we have

$$(C4) \qquad [Jx - Jy, x - y] \geq |\,\|x\| - \|y\|\,|\|x - y\|.$$

1.2 – Spectrum

For $A\colon X \to X^*$, we define the spectrum of A, denoted by $\sigma(A)$, as the set

$$\sigma(A) = \{\lambda \in \mathbb{C}\colon A - \lambda J \quad \text{is not invertible}\},$$

where J is normalized duality. That means, $A - \lambda J$ is invertible if $A - \lambda J$ is bijective and $(A - \lambda J)^{-1}\colon X^* \to X$ is continuous.

1.3 – Numerical range

The numerical range of $A\colon X \to X^*$, denoted by $V[A]$ – a generalization of the Zarantonello numerical range to the case of the reflexive Banach space operator - is the set

$$V[A] = \left\{\frac{[Ax - Ay, x - y]}{[Jx - Jy, x - y]}\colon x, y \in X, x \neq y\right\},$$

where J is the strictly monotone normalized duality. If X is a Hilbert space and $J = I$, $V[A]$ coincides with the Zarantonello numerical range [9], defined by

$$N[A] = \left\{\frac{\langle Ax - Ay, x - y\rangle}{\|x - y\|^2}\colon x, y \in X, x \neq y\right\},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on X.

2 – Spectra and numerical ranges

In this section, the first theorem deals with elementary properties, similar to the Zarantonello numerical range, of the numerical range $V[A]$, and the second one connects the spectrum and numerical range.

THEOREM 2.1. *Suppose $A, B\colon X \longrightarrow X^*$ are mapping from a reflexive real Banach space X to its dual X^*, and $\lambda \in \mathbb{K}$(field). Then*

(i) $V[\lambda A] = \lambda V[A]$;
(ii) $V[A + B] \subseteq V[A] + V[B]$; *and*
(iii) $V[A - \lambda J] = V[A] - \{\lambda\}$,

where $J\colon X \longrightarrow X^*$ is strictly monotone normalized duality.

PROOF. Assertions (i) and (ii) follow directly from the definition. To prove (iii), if $x, y \in D(A+J) = D(A) \cap D(J) \neq \Phi$ with $x \neq y$, we have

$$\frac{[(A-\lambda J)x-(A-\lambda J)y, x-y]}{[Jx-Jy, x-y]} =$$
$$= \frac{[Ax-Ay, x-y]-\lambda[Jx-Jy, x-y]}{[Jx-Jy, x-y]} = V[A]-\{\lambda\}.$$

THEOREM 2.2. *Let X be a reflexive and locally uniformly convex real Banach space and its dual X^* strictly convex. If $A\colon X \longrightarrow X^*$ is demicontinuous, then*

$$\sigma(A) \subseteq \overline{\text{co}}(V[A]).$$

PROOF. Let us assume $\lambda \notin \overline{\text{co}}(V[A])$. Then, for some $x, y \in X$ with $x \neq y$, and strictly monotone normalized duality $J\colon X \longrightarrow X^*$, we have

$$(C1) \qquad |[(A-\lambda J)x-(A-\lambda J)y, x-y]| =$$
$$= |[Ax-Ay, x-y]-\lambda[Jx-Jy, x-y]| =$$
$$= \left|\frac{[Ax-Ay, x-y]}{[Jx-Jy, x-y]}-\lambda\right|[Jx-Jy, x-y] \geq$$
$$\geq d[Jx-Jy, x-y] \geq d\big|\|x\|-\|y\|\big|\,\|x-y\|.$$

This implies that

$$\|(A-\lambda J)x-(A-\lambda J)y\| \geq d|\,\|x\|-\|y\|\,|$$

for all $x, y \in X$. It follows that $A-\lambda J$ is one-to-one.

For $x \in X$, and $c(r) = dr - \|(A-\lambda J)(0)\|$, a continuous real-valued function on $\mathbb{R}$ with $c(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, we have

$$\|(A-\lambda J)x\|\,\|x\| \geq |[(A-\lambda J)x-(A-\lambda J)(0), x]| -$$
$$-|[(A-\lambda J)(0), x]| \geq d\|x\|^2 - \|(A-\lambda J)(0)\|\,\|x\| =$$
$$= c(\|x\|)\|x\|,$$

and so $\|(A-\lambda J)x\| \geq c(\|x\|)$ for $x \neq 0$. For each $M > 0$, therefore, there exists $k(M)$ such that if $\|(A-\lambda J)x\| \leq M$, then $\|x\| \leq k(M)$. As a result, $(A-\lambda J)^{-1}$ caries bounded subsets of $R(A-\lambda J)$ into bounded subsets of X, and is continuous from $R(A-\lambda J)$ to X, for if $v_m = (A-\lambda J)x_m \longrightarrow v$, then $\|x_m\| \leq N$ for constant N, and for some $x_0 \in X$, as $m \longrightarrow \infty$,

$$(A-\lambda J)x_m - (A-\lambda J)x_0 \longrightarrow v - (A-\lambda J)x_0.$$

Since X is reflexive, there exists some subsequence, again denoted by (x_m), such that $x_m \xrightarrow{\omega} x_0$ as $m \longrightarrow \infty$. Thus, by Condition $(C1)$ and the above arguments, as $m \longrightarrow \infty$, we have

$$d\big|\|x_m\| - \|x_0\|\big|\|x_m - x_0\| \leq \big|[(A-\lambda J)x_m - (A-\lambda J)x_0, x_m - x_0]\big| \longrightarrow 0.$$

This implies that

$$\|x_m\| \longrightarrow \|x_0\| \quad \text{as} \quad m \longrightarrow \infty.$$

Since X is locally uniformly convex, $x_m \xrightarrow{\omega} x_0$ and $\|x_m\| \longrightarrow \|x_0\|$, this implies that $x_m \longrightarrow x_0$. It follows that $(A-\lambda J)^{-1}v_m \longrightarrow x_0$, and by the demicontinuity of A and J (and hence $A-\lambda J$), we find that $(A-\lambda J)x_0 = v$.

Now we only need to show that the null element 0 of X^* is in $R(A-\lambda J)$. Indeed, if$v$ is an arbitrary element of X^*, then $Sx = (A-\lambda J)x - v$ will satisfy the hypotheses of Theorem 2.2 whenever $A-\lambda J$ does, and $Sx = 0$ iff $(A-\lambda J)x = v$.

Let Λ be a directed set of all finite dimensional subspaces of X ordered by inclusion, and $\{x_G \colon G \in \Lambda\}$ be a function from Λ to X or X^*. For $G \varepsilon \Lambda$, let $p_G \colon G \longrightarrow X$ be the injective mapping from G into X, and $p_{G^*} \colon X^* \longrightarrow G^*$ be the dual map projecting X^* onto G^*. We define a continuous map$(A-\lambda J)_G \colon G \longrightarrow G^*$ by $(A-\lambda J)_G = p_{G^*}A(A-\lambda J)p_G$, represented by a diagram

$$\begin{array}{ccc} X & \xrightarrow{\;A-\lambda J\;} & X^* \\ {\scriptstyle p_G}\uparrow & & \downarrow{\scriptstyle p_{G^*}} \\ G & \xrightarrow{\;(A-\lambda J)_G\;} & G^* \end{array}$$

For $x \in G$, we have

$$\left|[(A-\lambda J)_G x, x]\right| = \left|[p_{G^*}(A-\lambda J)x, x]\right| = \\ = \left|[(A-\lambda J)x, x]\right| \geq c(\|x\|)\|x\|,$$

and for $x, y \in G$, we find

$$\left|[(A-\lambda J)_G x - (A-\lambda J)_G y, x-y]\right| = \\ = \left|[(A-\lambda J)x - (A-\lambda J)y, x-y]\right| \geq \\ \geq d\|x-y\|\big|\|x\| - \|y\|\big|.$$

This implies that Condition $(C1)$ holds for $(A-\lambda J)_G$ and, hence, $(A-\lambda J)_G$ is injective. By the Brouwer theorem [11, Theorem 16C] on the invariance of domain, $(A-\lambda J)_G$ is an open mapping. Thus, there exists a unique element x_G of G such that $(A-\lambda J)_G x_G = 0$. For this element, we find

$$0 = \left|[(A-\lambda J)_G x_G, x_G]\right| \geq c(\|x_G\|)\|x_G\|,$$

so that there exists a constant L independent of G such that $\|x_G\| \leq L$ for all G in Λ. Since X is a reflexive space, each closed ball in X is weakly compact. Therefore, $\{x_G \colon G \in \Lambda\}$ has at least one limit point x_0 in the weak topology of X.

Note that $(A-\lambda J)x_G \xrightarrow{\omega} 0$, for if x is any given element of X and if we take G_0 to be the one-dimensional subspace of X spanned by x, then for G in Λ with $G_0 \subset G$, we find that $x \in G$ and

$$[(A-\lambda J)x_G, x] = [(A-\lambda J)_G x_G, x] = 0.$$

Next, let $G_1 \in \Lambda$ and consider $G \subset \Lambda$ with $G_1 \subset G$. Then

$$d\|x_G - x_{G_1}\|\big|\|x_G\| - \|x_{G_1}\|\big| \leq \left|[(A-\lambda J)x_G - (A-\lambda J)x_{G_1}, x_G - x_{G_1}]\right| \leq \\ \leq \left|[(A-\lambda J)x_{G_1}, x_G]\right|.$$

Since X_0 lies in the weak closure of the set $\{x_G \colon G_1 \subset G, G \in \Lambda\}$, this implies that

$$\|x_0 - x_{G_1}\|\big|\|x_0\| - \|x_{G_1}\|\big| \leq \left\{d^{-1}\left|[(A-\lambda J)x_{G_1}, x_0]\right|\right\}, G_1 \in \Lambda.$$

Since $|[(A-\lambda J)x_{G_1}, x_0]| \longrightarrow 0$ as a function of G_1 on the directed set $\Lambda, x_{G_1} \longrightarrow x_0$. Then the demicontinuity of A and J (and hence $A - \lambda J$) implies that $(A-\lambda J)x_{G_1} \xrightarrow{\omega} (A-\lambda J)x_0$. Since we also have $(A - \lambda J)x_{G_1} \xrightarrow{\omega} 0$, it follows that $(A-\lambda J)x_0 = 0$.

Thus, $A - \lambda J$ is bijective and $(A-\lambda J)^{-1}$ is continuous. This, in turn, implies that $\lambda \notin \sigma(A)$. This proves the theorem.

Note that as a byproduct of Theorem 2.2, we obtain an existence theorem for the solvability of nonlinear functional equations in reflexive Banach spaces. This is an analog of ZARANTONELLO's result [9] for the case of the Hilbert spaces.

THEOREM 2.3. *Suppose $A: X \longrightarrow X^*$ is demicontinuous from a reflexive and locally uniformly convex real Banach space X to its strictly convex dual X^*, and $\lambda \notin V[A]$. If $d = \inf\{|\lambda - \mu| : \mu \in V[A]\} > 0$, and $J: X \longrightarrow X^*$ is normalized duality, then the equation*

$$Ax - \lambda Jx = u$$

has a unique solution for every $u \in X^*$.

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