Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 583-596

Uniform majorization of ultradistributions and decomposition theorem

S-Y. CHUNG - D. KIM

RIASSUNTO – *Si dimostra che ogni ω-ultradistribuzione con supporto compatto K può essere maggiorata uniformemente da funzioni di prova solo sul supporto K. Applicando questo risultato si ottiene una condizione necessaria e sufficiente affinché ogni ω-ultradistribuzione con supporto nell'unione $X \cup Y$ di due insiemi compatti, si possa decomporre nella somma di due ω-ultradistribuzioni con supporti appartenenti a X e Y rispettivamente.*

ABSTRACT – *We prove that every ω-ultradistribution with compact support K can be uniformly majorized by the behavior of test functions only on the support K. Also applying this result we give a necessary and sufficient condition that every ω-ultradistribution with support in the union $X \cup Y$ of two compact sets can be decomposed as the sum of two ω-ultradistributions whose supports belong to X and Y respectively.*

KEY WORDS – *Weight function - Ultradistribution - Decomposition.*

A.M.S. CLASSIFICATION: 46F05 - 46F15

– Introduction

The space of $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ of nonquasianalytic ultradifferentiable functions has been introduced by Beurling [1] and Björck [2]. But, their definition is, more or less, inconvenient because the growth condition is not given on its derivatives, but on the Fourier transform. Recently, MEISE and TAYLOR [3], [11] gave the clear characterization of the class as follows; let Ω be an open set in $\mathbb{R}^n$ and φ^* the Young conjugate of

$\varphi(t) = \omega(e^t)$ where ω is the weight function satisfying the appropriate conditions. Then

$$\mathcal{E}_{(\omega)}(\Omega) = \{f \in C^\infty(\Omega) | \text{for each } \lambda \in \mathbb{N} \text{ and each compact set } K \subset \Omega,$$
$$|f|_{\omega,K,\lambda} = \sup_{\substack{x\in K\\ \alpha\in N^n}} \frac{|\partial^\alpha f(x)|}{\exp[\lambda\varphi^*(\frac{|\alpha|}{\lambda})]} < +\infty\}$$

and

$$\mathcal{E}_{\{\omega\}}(\Omega) = \{f \in C^\infty(\Omega) | \text{for each compact set } K \subset \Omega,$$
$$|f|_{\omega,K,\frac{1}{\lambda}} < +\infty \text{ for some } \lambda \in \mathbb{N}\}.$$

The space of ultradistributions of Beurling type (resp. of Roumieu type) is defined to be the strong dual of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$).

In the present paper we study the decomposition theorem: Any ultradistribution u with support in $K_1 \cup K_2$, say, $u \in \mathcal{E}'_*(K_1 \cup K_2)$, $* = (\omega)$ or $\{\omega\}$, be decomosed as $u = u_1 + u_2$ where u_j is an ultradistribution with support in K_j, for $j = 1, 2$. This is always possible for the Sato hyperfunctions and for the quasianalytic ultradistributions as shown in HÖRMANDER [7]. But, in general this is not true even for the Schwartz distributions. LOJASIEWICZ [9] showed that, so called, the regularly-situated condition on K_1 and K_2 is necessary and sufficient for the decomposition of the Schwartz distributions. For the space $\mathcal{E}'_{M_p}$ of ultradistributions given by the defining sequence $(M_p)_{p\in\mathbb{N}_0}$ (see KOMATSU [8] or ROUMIEU [12], [13]), Hörmander gave an example showing that the decomposition is in general not possible. However, CHUNG-KIM [4] could give the necessary and sufficient condition for the decoposability in $\mathcal{E}'_{M_p}(K_1 \cup K_2)$. Now, the main result of this paper states that the following assertions are equivalent:

(1) Either $X \cap Y = \phi$ or X and Y are $*$-regularly situated.

(2) Any $u \in \mathcal{E}'_*(X \cup Y)$ can be decomposed as $u_1 + u_2$ where $u_1 \in \mathcal{E}'_*(X)$ and $u_2 \in \mathcal{E}'_*(Y)$,

where X and Y are compact subsets of $\mathbb{R}^n$ and $*$ denotes (ω) or $\{\omega\}$. In the above statement the "(ω)-regularly situated" means that for any $\lambda \in \mathbb{N}$ there exists $\lambda' \in \mathbb{N}$ and $C > 0$ such that for any $m \in \mathbb{N}$,

$$\frac{d(x, X\cap Y)^p}{p!} \exp[\lambda'\varphi^*(\frac{p}{\lambda'})] \le C\frac{d(x,Y)^m}{m!} \exp[\lambda\varphi^*(\frac{m}{\lambda})]$$

for some $p \in \mathbb{N}$ and for all $x \in X$.

Before proving the above results we characterize the ultradistributions with compact support. We show that every ultradistribution with compact support can be uniformly majorized by the behavior of test functions only on the its support with a counterexample showing that it cannot be completely majorized by the usual seminorms on $\mathcal{E}_{(\omega)}(\Omega)$. These are essential for the proof of the decomposition theorem. In proving the decomposition theorem we approach along the context of MALGRANGE [10] and MEISE and TAYLOR [11]. In this paper we only consider the case of Beurling type, but the case of Roumieu type can be obtained with the slight variations.

For standard notations not explained in the text we refer to CHUNG-KIM-KIM [5].

1 – ω-ultradistributions with compact support

We introduce the weight functions and give technical results relating to the Young conjugates which are needed later as in [5], and refer the proofs to [5].

DEFINITION 1.1. *Let $\omega : \mathbb{R} \to [0, \infty)$ be a continuous even function which is increasing on $[0, \infty)$ and satisfies $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$. It is called a weight function, if it has the following properties:*
(α) $0 = \omega(0) \leq \omega(s+t) \leq \omega(s) + \omega(t)$ for all $s, t \in \mathbb{R}$;
(β) $\int_{-\infty}^{\infty} \omega(t)/1 + t^2 \, dt < +\infty$;
(γ) $\lim_{t\to\infty} \log t/\omega(t) = 0$;
(δ) $\varphi : t \longrightarrow \omega(e^t)$ is convex on $\mathbb{R}$;
(ϵ) there exists $C > 0$ with $\int_1^{\infty} \omega(yt)/t^2/, dt \leq C\omega(y) + C$ for all $y \geq 0$;
(ζ) there exists $H \geq 1$ with $2\omega(t) \leq \omega(Ht) + H$ for all $t \geq 0$.

From the property (δ) we can define the Young conjugate $\phi^* : [0, \infty) \to [0, \infty)$ by

$$\phi^*(x) = \sup_{t\geq 0}\{xt - \phi(t)\}$$

LEMMA 1.2. *Let $\varphi(t)$ be the given one in Definition 1.1(δ). Then* *for each* $\lambda > 0$ *and* $p \in \mathbb{N}$ *it follows that*

$$\exp[\lambda\varphi^*(\frac{p}{\lambda})] = \sup_{t\geq 0}\left[\frac{t^p}{\exp\lambda\omega(t)}\right]. \tag{1.1}$$

THEOREM 1.3. *Let* $H \geq 1$, $\lambda > 0$ *and* $0 < \epsilon < 1$. *Then we obtain that for each* $p \in \mathbb{N}$

$$H^p \exp[\lambda\varphi^*(p/\lambda)] \leq \exp[\lambda/H\varphi^*(pH/\lambda)] \tag{1.2}$$

and

$$\epsilon^p \exp[\lambda\varphi^*(p/\lambda)] \geq \exp[\lambda/\epsilon\varphi^*(p\epsilon/\lambda)]. \tag{1.3}$$

THEOREM 1.4. *Let* $\lambda > 0$ *and* $p, q \in \mathbb{N}$. *Then*

$$\exp[\lambda\varphi^*(\frac{p}{\lambda})]\exp[\lambda\varphi^*(\frac{q}{\lambda})] \leq \exp[\lambda\varphi^*(\frac{p+q}{\lambda})] \tag{1.4}$$

and

$$\exp[\lambda\varphi^*(\frac{p+q}{\lambda})] \leq e^{\lambda H}H^{p+q}\exp[\lambda\varphi^*(\frac{p}{\lambda})]\exp[\lambda\varphi^*(\frac{q}{\lambda})] \tag{1.5}$$

where H *is the constant in 1.1*(ζ).

Now we will give another characterization of ω-ultradistributions. The following lemma is the Whitney extension theorem for ω-ultradifferentiable functions:

LEMMA 1.5 (Chung-Kim-Kim [5]). *Let* $\omega(t)$ *be a weight function satisfying the conditions* $(\alpha) \sim (\zeta)$. *Then the restriction map*

$$\rho_K : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \longrightarrow \mathcal{E}_{(\omega)}(K) \tag{1.6}$$

is surjective, where $\rho_K(f) = (\partial^\alpha|_K)_{\alpha\in\mathbb{N}^n}$.

Lemma 1.5 allows us to identify the space $\mathcal{E}_{(\omega)}(K)$ with the set of functions in $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ if necessary. Under this point of view we have the following.

LEMMA 1.6. *Let K be a compact convex set in $\mathbb{R}^n$. Then the topologies on $\mathcal{E}_{(\omega)}(K)$ given by the norm $\|\cdot\|_{\omega,K,\lambda}$ and the norm $|\cdot|_{\omega,K,\lambda}$ are equivalent.*

PROOF. The definition of the norms gives

$$|F|_{\omega,K,\lambda} \le \|F\|_{\omega,K,\lambda}. \tag{1.7}$$

On the other hand, Taylor's formula implies that there exists $z \in K$ such that for $F = (f^\alpha)_{\alpha\in\mathbf{N}^n}$

$$\begin{aligned}|(R_y^m F)^k(x)| &= \left|\sum_{|l|=m-|k|+1} f^{k+l}(z)\frac{(y-x)^l}{l!}\right| \\ &\le |F|_{\omega,K,\lambda}\exp\left[\lambda\varphi^*(\frac{m+1}{\lambda})\right]\frac{(n|x-y|)^{m-|k|+1}}{(m-|k|+1)!} \\ &\le |F|_{\omega,K,\lambda}\exp\left[\frac{\lambda}{n}\varphi^*(\frac{n}{\lambda}(m+1))\right]\frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!}.\end{aligned}$$

Here n denotes the dimension of space and the last inequality follows from Theorem 1.3. Thus we have

$$\|F\|_{\omega,K,\lambda} \le 2|F|_{\omega,K,n\lambda}, \tag{1.8}$$

which completes the proof.

Now we define an ideal of $\mathcal{E}_{(\omega)}(\Omega)$ as follows:

$$J_{(\omega)}(K:\Omega) = \{f \in \mathcal{E}_{(\omega)}(\Omega) \,|\, \partial^\alpha f|_K = 0 \text{ for all } \alpha\}$$

where K is a compact subset of Ω. Then we can easily obtain the following Lemma:

LEMMA 1.7. *$J_{(\omega)}(K:\Omega)$ is the closure of the set of all functions in $\mathcal{E}_{(\omega)}(\Omega)$ vanishing in a neighborhood of K.*

We note that the restriction map $\rho_K : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \to \mathcal{E}_{(\omega)}(K)$ in (1.6) is continuous, since

$$\|\rho_K(f)\|_{\omega,K,\lambda} \leq 2|f|_{\omega,\hat{K},n\lambda}$$

where $\hat{K}$ is the convex hull of K. Combining this fact and Lemma 1.7 we have the following theorem.

THEOREM 1.8. *The sequence*

$$0 \to J_{(\omega)}(K : \mathbb{R}^n) \overset{i}{\hookrightarrow} \mathcal{E}_{(\omega)}(\mathbb{R}^n) \overset{\rho_K}{\to} \mathcal{E}_{(\omega)}(K) \to 0$$

is topologically exact where i is the inclusion map.

REMARK 1.9. It is clear by Theorem 1.8 that the space $\mathcal{E}_{(\omega)}(K)$ is topologically isomorphic to $\mathcal{E}_{(\omega)}(\mathbb{R}^n)/J_{(\omega)}(K : \mathbb{R}^n)$. Thus $u \in \mathcal{E}'_{(\omega)}(K)$ i.e. a continuous linear functional u on $\mathcal{E}_{(\omega)}(K)$ is a ω-ultradistribution which is orthogonal to $J_{(\omega)}(K : \Omega)$, so that it defines an ω-ultradistribution with support in K. Hence it follows from Lemma 1.7 that the space $\mathcal{E}'_{(\omega)}(K)$ can be considered as the set of all ultradistributions with support in K. Thus the following is another characterization of $\mathcal{E}'_{(\omega)}(K)$.

COROLLARY 1.10. *The following two properties are equivalent :*

(i) $u \in \mathcal{E}_{(\omega)}(K)$.

(ii) there exist $\lambda \geq 1$ and $C > 0$ such that

(1.9)

$$|u(\phi)| \leq C \left[\sup_{\substack{x\in K \\ \alpha\in\mathbb{N}^n}} \frac{|\partial^\alpha \phi(x)|}{\exp[\lambda\varphi^*(|\alpha|/\lambda]} + \sup_{\substack{x,y\in K \\ x\neq y \\ |k|\leq m \\ m\in\mathbb{N}}} \frac{(m-|k|+1)! \left|\partial^k f(x) - \sum_{|l|\leq m-|k|} \partial^{k+l} f(y)(x-y)^l/l!\right|}{|x-y|^{m-|k|+1}\exp[\lambda\varphi^*(m+1/\lambda)]} \right]$$

for $\phi \in \mathcal{E}_\omega(\mathbb{R}^n)$.

COROLLARY 1.11. *If $u \in \mathcal{E}'_{(\omega)}(K)$ and $\partial^\alpha\phi(x) = 0$ for all α and $x \in K$ then $u(\phi) = 0$.*

PROOF. The proof can be easily obtained from (1.9).

The following example shows that the right-hand side of (1.9) cannot be replaced by the usual norm $\sup_{\substack{x\in K\\ \alpha\in\mathbf{N}^n}} \frac{|\partial^\alpha\phi(x)|}{\exp[\lambda\varphi^*(|\alpha|/\lambda)]}$.

EXAMPLE 1.12. Let (x_j) be a sequence in $\mathbb{R}^n$ such that

$$|x_1| > |x_2| > \cdots \to 0$$

and $K = \{x_1, x_2, \cdots\} \cup \{0\}$. Define

$$u(\phi) = \sum_{j=1}^{\infty} m_j[\phi(x_j) - \phi(0)]$$

where (m_j) is a sequence of positive numbers such that

$$\sum_{j=1}^{\infty} m_j|x_j| = 1, \qquad \sum_{j=1}^{\infty} m_j = \infty.$$

Such a sequence always exists, since $\lim_{j\to\infty} |x_j| = 0$. Then u defines an ultradistribution with support K. In fact it defines a distribution of order 1. Now we suppose that for some $\lambda \geq 1$ and $C > 0$

$$|u(\phi)| \leq C|\phi|_{\omega,K,\lambda}, \qquad \phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^n).$$

Choose $\phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ which is equal to 1 in a neighborhood of $\{x_1, \cdots, x_j\}$ and 0 near $\{x_{j+1}, x_{j+2}, \cdots,\} \cup \{0\}$. Then it is clear that

$$\sum_{i\leq j} m_j \leq C,$$

which leads to a contradiction when j goes to ∞.

On the other hand, even though the second term of the right-hand side in (1.9) can not be omitted, we can give an optimal condition for some compact set as follows:

THEOREM 1.13. *Let K be a compact set in $\mathbb{R}^n$ with finitely many connected components such that any two points x, y in the same component can be joined by a rectifiable curve in K of length $\leq B|x-y|$. If u is* *an ultradistribution with support in K, it follows that there exists $\lambda \geq 1$ and constant $C > 0$ such that*

$$|u(\phi)| \leq C \sup_{\substack{x \in K \\ \alpha \in \mathbf{N}^n}} \frac{|\partial^\alpha \phi(x)|}{\exp[\lambda \varphi^*(|\alpha|/\lambda)]}, \qquad \phi \in \mathcal{E}_{(\omega)}(\mathbb{R}^n). \tag{1.10}$$

PROOF. Let $s \to x(s)$ be a curve in K with $x(0) = y$ and arc length s. Then

$$|F_k(s)| \leq \frac{Cs^{m-|k|+1}}{(m-|k|+1)!} \sum_{|l|=m+1} \sup_{x \in K} |\partial^l \phi(x)| \tag{1.11}$$

if

$$F_k(s) = \partial^k \phi(x(s)) - \sum_{|l| \leq m-|k|} \partial^{k+l} \phi(y) \frac{(x(s)-y)^l}{l!}.$$

This is obvious when $|k| = m$. If $|k| < m$ and (1.11) is already proved for derivatives of higher order, we conclude by induction that

$$\left|\frac{dF_k(s)}{ds}\right| \leq \frac{Cs^{m-|k|}}{(m-|k|)!} \sum_{|l|=m+1} \sup_{x \in K} |\partial^l \phi|.$$

Since $F_k(0) = 0$ we obtain (1.11) with C replaced by nC. If $d(x,y)$ is the infimum of the curves from x to y in K then (1.11) gives

$$\begin{aligned}
&\left|\partial^k \phi(x) - \sum_{|l| \leq m-|k|} \partial^{k+l} \phi(y) \frac{(x-y)^l}{l!}\right| \\
&\leq \frac{Cd(x,y)^{m-|k|+1}}{(m-|k|+1)!} \sum_{|l|=m+1} \sup_{x \in K} |\partial^l \phi(x)| \\
&\leq C|\phi|_{\omega,K,\lambda} \exp\left[\frac{\lambda}{nB} \varphi^*\left(\frac{nB(m+1)}{\lambda}\right)\right] \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!}
\end{aligned}$$

Then it follows that

$$\|\phi\|_{\omega,K,\lambda} \leq 2|\phi|_{\omega,K,nB\lambda} \tag{1.12}$$

which completes the proof.

The compact sets satisfying the hypothesis in Theorem 1.13 are sometimes said to be regular in the sense of Whitney.

COROLLARY 1.14. *Let K be regular compact set in the sense of Whitney and ϕ a function in $C^\infty(K)$ with $|\phi|_{\omega,K,\lambda} < +\infty$. Then ϕ can be extended to a function in $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$.*

PROOF. By (1.12) and Lemma 1.5 it is obvious.

2 – Decomposition theorem

Let X and Y be compact subsets of $\mathbb{R}^n$ throughout this section. LOJASIEWICZ [9] proved that $u \in \mathcal{E}'(X \cup Y)$ can be decomposed as a sum $u_1 + u_2$ where $u_1 \in \mathcal{E}'(X)$ and $u_2 \in \mathcal{E}'(Y)$ if X and Y satisfy some geometric conditions. But in 1985 HÖRMANDER [7] showed that the decomposition as above always holds for the quasianalytic ultradistributions, but not for the nonquasianalytic ultradistributions. Recently, CHUNG-KIM [4] gave a necessary and sufficient condition for X and Y so that the decomposition is possible for the (nonquasianalytic) ultradistributions $\mathcal{E}'_{(M_p)}$ defined by sequences. We give here a necessary and sufficient condition for the case of $\mathcal{E}'_{(\omega)}$.

DEFINITION 2.1. *Two compacts sets X and Y are (ω)-regularly situated if for each $\lambda \geq 1$ there exist $\lambda' \geq 1$ and $C > 0$ such that for any $m \in \mathbb{N}$*

$$\frac{d(x, X \cap Y)^p}{p!} \exp[\lambda' \varphi^*(\frac{p}{\lambda'})] \leq C \frac{d(x,y))^m}{m!} \exp[\lambda \varphi^*(\frac{m}{\lambda})] \tag{2.1}$$

for some $p \in \mathbb{N}$ and for all $x \in X$.

Now let δ be the diagonal mapping

$$\delta : \mathcal{E}_{(\omega)}(X \cup Y) \to \mathcal{E}_{(\omega)}(X) \times \mathcal{E}_{(\omega)}(Y)$$

defined by $\delta(F) = (F|_X, F|_Y)$. Then δ is a continuous injection. Let π be the mapping

$$\pi : \mathcal{E}_{(\omega)}(X) \times \mathcal{E}_{(\omega)}(Y) \to \mathcal{E}_{(\omega)}(X \cap Y)$$

defined by $\pi(F,G) = F|_{X\cap Y} - G|_{X\cap Y}$. Then Lemma 1.5 implies π is a continuous surjection and $\pi\circ\delta = 0$. Now we can obtain the dual mappings

$$\delta' : \mathcal{E}'_{(\omega)}(X) \oplus \mathcal{E}'_{(\omega)}(Y) \to \mathcal{E}'_{(\omega)}(X \times Y)$$

and

$$\pi' : \mathcal{E}'_{(\omega)}(X \cap Y) \to \mathcal{E}'_{(\omega)}(X) \oplus \mathcal{E}'_{(\omega)}(Y)$$

with $\delta'(u,v) = u+v$ and $\pi'(u) = (u,-u)$.

Now we are in a position to prove the main result of this paper.

THEOREM 2.2. *Under the above assumption the following statements are equivalent :*

(i) Either $X \cap Y = \phi$ or X and Y are (ω)-regularly situated.

(ii) The sequence

$$0 \to \mathcal{E}_{(\omega)}(X \cup Y) \xrightarrow{\delta} \mathcal{E}_{(\omega)}(X) \times \mathcal{E}_{(\omega)}(Y) \xrightarrow{\pi} \mathcal{E}_{(\omega)}(X \cap Y) \to 0$$

is exact.

(iii) The sequence

$$0 \to \mathcal{E}'_{(\omega)}(X \cap Y) \xrightarrow{\pi'} \mathcal{E}'_{(\omega)}(X) \oplus \mathcal{E}'_{(\omega)}(Y) \xrightarrow{\delta'} \mathcal{E}'_{(\omega)}(X \cup Y) \to 0$$

is exact.

(iv) Every ultradistribution $u \in \mathcal{E}'_{(\omega)}(X \cup Y)$ can be decomposed as $u_1 + u_2$, where $u_1 \in \mathcal{E}'_{(\omega)}(X)$ and $u_2 \in \mathcal{E}'_{(\omega)}(Y)$.

PROOF. The equivalences of (ii), (iii) and (iv) are easily obtained from the the theory of the duality in F-space. Now we will prove the equivalence of (i) and (ii).

The proof of (i) $\Longrightarrow$ (ii) : In proving this we have only to show that $\mathrm{Ker}\pi \subset \mathrm{Im}\delta$. Let $F = (f^k) \in \mathcal{E}_{(\omega)}(X)$ and $G = (g^k) \in \mathcal{E}_{(\omega)}(Y)$ with $f^k = g^k$ on $X \cap Y$ for all $k \in \mathbb{N}^n$. Define a jet $H = (h^k)$ by

$$h^k(x) = \begin{cases} f^k(x) & , \quad x \in X \\ g^k(x) & , \quad x \in Y. \end{cases}$$

Then it suffices to show that H is a Whitney jet in $\mathcal{E}_{(\omega)}(X \cup Y)$, which implies that for each $\lambda \geq 1$ there exists $C > 0$ such that

$$(2.2) \qquad |(R_y^m H)^k(x)| \leq C \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!} \exp[\lambda \varphi^*(\frac{m+1}{\lambda})]$$

for all $m \in \mathbb{N}$, $|k| \leq m$ and $x, y \in X \cup Y$. The case that x and y both belong to X or both belong to Y is clear. Hence we may assume that $x \in X$ and $y \in Y$. If we extend G to $\tilde{G} \in \mathcal{E}_{(\omega)}(X \cup Y)$ by using Lemma 1.5 and replace F by $F - \tilde{G}$, then it reduces to the case when $G = 0$ and consequently $F|_{X \cap Y} = 0$. In this case our inequality (2.2) can be written simply as

$$(2.3) \qquad |f^k(x)| \leq C \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!} \exp[\lambda \varphi^*(\frac{m+1}{\lambda})]$$

for all $m \in \mathbb{N}$, $|k| \leq m$, $x \in X$ and $y \in Y$. Choose $z \in X \cap Y$ satisfying $|x - z| = d(x, X \cap Y)$. Then it follows from the hypothesis (i) that there exist $\lambda' \geq 1$ and $C_1 > 0$ such that for each $m' \in \mathbb{N}$

$$(2.4) \qquad \frac{|x-z|^p}{p!} \exp[\lambda' \varphi^*(\frac{p}{\lambda'})] \leq C_1 \frac{|x-y|^{m'}}{m'!} \exp[\lambda \varphi^*(\frac{m'}{\lambda})]$$

for some $p \in \mathbb{N}$. Here we may assume $\lambda' \geq \lambda$, since $\exp[\lambda \varphi^*(\frac{p}{\lambda})]$ is a decreasing function of λ by (1.1). Since F belongs to $\mathcal{E}_{(\omega)}(X)$ there exist $C_2 > 0$ such that

$$(2.5) \qquad |f^k(x)| \leq C_2 \frac{|x-z|^{m''-|k|+1}}{(m''-|k|+1)!} \exp[\lambda' H \varphi^*(\frac{m''+1}{\lambda' H})]$$

for all $m'' \in \mathbb{N}$, $|k| \leq m''$, $x \in X$ and $z \in X \cap Y$ where H is the constant in Definition 1.1(ζ).

To show the inequality (2.3) let $m \in \mathbb{N}$ and $|k| \leq m$. If we take $m' = m - |k| + 1$ and $m'' = p + |k| - 1$ in (2.4) and (2.5) respectively it follows from (1.2) and (1.5) that for some $C_3 > 0$

$$\begin{aligned}|f^k(x)| &\le C_2 \frac{|x-z|^p}{p!} \exp[\lambda' H \varphi^*(\frac{p+|k|}{\lambda' H})] \\ &\le C_3 \frac{|x-z|^p}{p!} \exp[\lambda' \varphi^*(\frac{p}{\lambda'})] \exp[\lambda' \varphi^*(\frac{|k|}{\lambda'})] \\ &\le C_1 C_3 \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!} \exp[\lambda \varphi^*(\frac{m-|k|+1}{\lambda})] \exp[\lambda' \varphi^*(\frac{|k|}{\lambda'})]\end{aligned}$$

Since $\lambda' \ge \lambda$ it follows from (1.1) and (1.4) that there exists a constant $C > 0$ such that

$$|f^k(x)| \le C \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!} \exp[\lambda \varphi^*(\frac{m+1}{\lambda})]$$

which is required.

The proof of (ii) $\Longrightarrow$ (i) : We assume that Kerπ =Imδ. Then Imδ is closed. Hence the open mapping theorem for F-space implies that δ is a homomorphism. Thus for each $\lambda \ge 1$ there exist $\lambda_1, \lambda_2 \in \mathbb{N}$ and $C > 0$ such that

$$\|F\|_{\omega, X\cup Y, \lambda} \le C[\|F\|_{\omega, X, \lambda_1} + \|F\|_{\omega, Y, \lambda_2}]$$

for all $F \in \mathcal{E}_{(\omega)}(X \cup Y)$. In particular, if $F = 0$ on Y then

$$|f^k(x)| \le C\|F\|_{\omega, X, \lambda_1} \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!} \exp[\lambda \varphi^*(\frac{m+1}{\lambda})] \tag{2.6}$$

for all $m \in \mathbb{N}$, $|k| \le m$, $x \in X$ and $y \in Y$.

Let $f \in \mathcal{E}_{(\omega)}(\mathbb{R}^n)$ with supp$f$ in $\mathbb{R}^n \backslash Y$. Considering f as a jet $(\partial^k f)$ it follows from (1.8) that

$$|\partial^k f(x)| \le 2C|f|_{\omega, \hat{X}, \lambda'} \frac{|x-y|^{m-|k|+1}}{(m-|k|+1)!} \exp[\lambda \varphi^*(\frac{m+1}{\lambda})] \tag{2.7}$$

where $\hat{X}$ is the convex hull of X and $\lambda' = n\lambda_1$.

Let f_0 be an ultradifferentiable function with $f_0(0) = 1$ and supp$f_0 \subset \{x | \, |x| \le 1\}$ and $f(x) = f_0(\frac{x-x_0}{\epsilon})$ where $x_0 \in X$ and $\epsilon = d(x_0, X \cap Y)$.

Then supp$f$ does not meet Y. Applying (2.7) with $k = (0, \cdots, 0)$ to $f(x)$ we have

$$1 \leq 2C|f|_{\omega,\dot{X},\lambda} \frac{|x_0 - y|^{m+1}}{(m+1)} \exp[\lambda\varphi^*(\frac{m+1}{\lambda})]. \tag{2.8}$$

On the other hand, since $|f|_{\omega,\dot{X},\lambda'}$ is positive there exists $p \in \mathbb{N}^n$ such that

$$|f|_{\omega,\dot{X},\lambda'} \leq 2 \frac{|\partial^p f_0(x)|}{\epsilon^{|p|} \exp[\lambda'\varphi^*(|p|/\lambda')]}. \tag{2.9}$$

Therefore, it follows from (2.8) and (2.9) that for some $C > 0$

$$\frac{\epsilon^{|p|}}{|p|!} \exp[\lambda'\varphi^*(\frac{|p|}{\lambda'})] \leq C \frac{|x_0 - y|^{m+1}}{(m+1)!} \exp[\lambda\varphi^*(\frac{m+1}{\lambda})] \tag{2.10}$$

Since $x_0 \in X$ and $y \in Y$ are arbitrary and $\epsilon = d(x_0, X \cap Y)$, (2.10) implies that X and Y are (ω)-regularly situated.

On the other hand, if X and Y are disjoint then all things considered

above are immediate. Therefore, the theorem is proved.

As stated in the introduction we only consider the decomposition theorem for the space $\mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ of ω-ultradistributions of Beurling type. But the case of Roumieu type $\mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$ can be proved by slight variations under the condition "$\{\omega\}$-regularly situated", which means that for each $\lambda' \geq 1$ there exist $\lambda \geq 1$ and $C > 0$ such that

$$\frac{d(x, X \cap Y)^p}{p!} \exp[\frac{1}{\lambda'}\varphi^*(\lambda' p)] \leq C \frac{d(x, Y)^m}{m!} \exp[\frac{1}{\lambda}\varphi^*(\lambda m)]$$

for all $m \in \mathbb{N}$, $|k| \leq m$, and $x \in X$.

REFERENCES

[1] A. BEURLING: *Quasi-analyticity and general distributions*, Lecture 4 and 5, AMS Summer Institute, Stanford 1961.

[2] G. BJÖRCK: *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1985), 351-407.

[3] R.W. BRAUN - R. MEISE - B.A. TAYLOR: *Ultradifferentiable functions and Fourier analysis*, preprint.

[4] S.Y CHUNG - D. KIM: *Characterizations of ultradistributions with compact support and decomposition by support*, Proc. Amer. Math. Soc. **112** (1990), 65-74.

[5] S.Y. CHUNG - D. KIM - S.K. KIM: *Equivalence of the spaces of ultradifferentiable functions and its application to the Whitney extension theorem*, Rend. Mat. VII **12** (1992), 365-380.

[6] A. GROTHENDIECK: *Topological vector spaces*, Gordon and Breach, Science Publishers, Inc., New York 1973.

[7] L. HÖRMANDER: *Between distributions and hyperfunctions*, Astérisque **131** (1985), 89-106.

[8] H. KOMATSU: *Ultradistributions I*, J. Fac. Sci. Univ. Tokyo Sect. IA **20** (1973), 25-105.

[9] S. LOJASIEWICZ: *Sur le probléme de la division*, Studia Math. **8** (1959), 87-136.

[10] B. MALGRANGE: *Ideals of differentiable functions*, Oxford University Press 1966.

[11] R. MEISE - B.A. TAYLOR: *Whitney's extension theorem for ultradifferentiable functions of Beurling type*, Ark. Mat. **26** (1988), 265-287.

[12] C. ROUMIEU: *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. Paris, 3 Sér. **77** (1960), 41-121.

[13] C. ROUMIEU: *Ultradistributions définis sur* $\mathbb{R}^n$ *et sur certaines classes de variété differentiables*, J. Analyse Math. **10** (1962), 153-192.

Lavoro pervenuto alla redazione il 22 febbraio 1991
ed accettato per la pubblicazione il 7 giugno 1992
su parere favorevole di A. Avantaggiati e di L. Rodino

INDIRIZZO DEGLI AUTORI:

Soon-Yeong Chung[1)] - Department of Mathematics - Duksung Women's University - Seoul 132-714

Dohan Kim[2)] - Department of Mathematics - Seoul National University - Seoul 151-742, Korea

This work has been partially supported by the GARC-KOSEF[1)] and by the Ministry of Education[2)].