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Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems

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RIASSUNTO – *Si costruisce una teoria del grado topologico per le applicazioni della forma $F = L + S$, dove L è un operatore monotono massimale ed S è un'applicazione non lineare di classe (S_+) nel dominio di L. La teoria è applicata allo studio di una classe di problemi parabolici non lineari ai valori iniziali.*

ABSTRACT – *We construct a topological degree for a class of mappings of the form $F = L + S$ where L is closed densely defined maximal monotone operator and S is a nonlinear map of class (S_+) with respect to the domain of L. The degree theory is then applied in the study of a class of nonlinear parabolic initial-boundary value problems.*

KEY WORDS – *Topological degree - Nonlinear parabolic problems - Mappings of monotone type.*

A.M.S. CLASSIFICATION: 47H17 - 35K30

1 – Introduction

The topological degree theory of mappings has been one of the most important tools in the study of nonlinear functional equations. The classical degree for continuous mappings from a bounded open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ was introduced by Brouwer in 1912. In the celebrated paper by Leray and Schauder in 1934 the degree was constructed for mappings in infinite dimensional Banach spaces of the form $F = I + C$, where I

is the identity map and C is compact. Since then a number of further extensions have been introduced.

Important recent contributions are due to Browder in the framework of studying nonlinear mappings of monotone type from a real reflexive Banach space X to its dual space X^* ([3,4], see also [1,2]). The present note provides a further contribution in this direction. We shall construct an approximative degree theory for a class of mappings of the form $F = L+S$ from the domain $D(L)$ in X to X^*, where L is a closed densely defined maximal monotone operator and S is a nonlinear map of class (S_+) with respect to $D(L)$. Our construction is based on suitable approximations of $L+S$ by a family of mappings of class (S_+) with respect to the graph norm topology of $D(L)$, as indicated in the previous work by Browder [4,5] (cf. also [8]). The degree theory obtained makes it possible to use continuation methods in the study of nonlinear equations

$$Lu + S(u) = h, \quad u \in D(L) \tag{1.1}$$

where S may be pseudomonotone or quasimonotone with respect to $D(L)$. As a specific example of the equation (1.1) we deal with nonlinear parabolic initial-boundary value problem of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) = h & \text{in } \Omega \times [0,T] \\ u(x,0) = 0 & \text{in } \Omega \\ D^\alpha u(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \text{ for all } |\alpha| \leq m-1, \end{cases} \tag{1.2}$$

where Ω is a bounded open subset in $\mathbb{R}^N$, h is a given function defined in $Q = \Omega \times [0,T]$ and A is a divergence operator of order $2m$,

$$Au(x,t) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,t,u(x,t),\ldots,D^m u(x,t))$$

satisfying some growth, monotonicity and coercivity conditions. In fact, our results on the existence of weak solutions for (1.2) are based on the systematic study of various monotonicity properties of A and we get some refinements to the classical results obtained before by different methods.

2 – Prerequisites

Let X be a real reflexive Banach space and let X^* stand for its dual space with respect to the continuous pairing $\langle\cdot,\cdot\rangle$. We may assume without loss of generality that X and X^* are locally uniformly convex (see [6], for example). The norm convergence in X and X^* is denoted by $\to$, and the weak convergence by $\rightharpoonup$. We shall be dealing with mappings T acting from a subset $D(T)$ in X to X^*. T is said to be *bounded*, if it takes bounded sets of X to bounded sets of X^*, and *demicontinuous*, if $u_n \to u$ implies $T(u_n) \rightharpoonup T(u)$ in X^*. We also need the following classes of mappings of monotone type. A mapping $T : D(T) \to X^*$ is called

-*monotone* (we denote $T \in (MON)$) if $\langle T(u) - T(v), u - v\rangle \geq 0$ for all $u, v \in D(T)$.

-*quasimonotone* ($T \in (QM)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \rightharpoonup u$ we have $\limsup\langle T(u_n), u_n - u\rangle \geq 0$.

-*pseudomonotone* ($T \in (PM)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \rightharpoonup u$ and $\limsup\langle T(u_n), u_n - u\rangle \leq 0$, we have $\lim\langle T(u_n), u - u_n\rangle = 0$, and if $u \in D(T)$, then $T(u_n) \rightharpoonup T(u)$.

-*of class* (S_+) ($T \in (S_+)$) if for any sequence $\{u_n\}$ in $D(T)$ with $u_n \rightharpoonup u$ and $\limsup\langle T(u_n), u_n - u\rangle \leq 0$, we have $u_n \to u$.

If we assume that all mappings are demicontinuous and defined in the whole space X, then $(S_+) \subset (PM) \subset (QM)$ and $(MON) \subset (PM)$. It is also important to observe that $(S_+) + (QM)$ is contained in (S_+). A monotone map $T : D(T) \to X^*$ is called *maximal monotone* ($T \in (MM)$) if its graph

$$G(T) = \{(u, T(u)) \in X \times X^* \mid u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$. If L is a linear densely defined monotone map from $D(L)$ to X^*, then a necessary and sufficient condition for $L \in (MM)$ is that $G(L)$ is a closed subspace of $X \times X^*$ and L^* is monotone (see [6], for example).

In our study we deal with mappings of the form $F = L + S$ where L is a given linear densely defined maximal monotone map from $D(L) \subset X$ to X^* and S is a bounded demicontinuous map of monotone type from X to X^* satisfying one of the monotonicity conditions with respect to the graph norm topology of $D(L)$. Thus, for instance, we call S *pseudomonotone with respect to* $D(L)$, if for any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$,

$Lu_n \rightharpoonup Lu$ and $\limsup \langle S(u_n), u_n - u\rangle \le 0$, we have $\lim\langle S(u_n), u_n - u\rangle = 0$ and $S(u_n) \rightharpoonup S(u)$. Analogous definitions apply for mappings *of class* (S_+) and *quasimonotone mappings with respect to* $D(L)$.

It is well-known that the conditions

$$\|J(u)\| = \|u\|, \ \langle J(u), u\rangle = \|u\|^2 \text{ for all } u \in X$$

determine a unique map J from X to X^*, which is called the *duality map*. In our case it is bijective bicontinuous strictly monotone and of class (S_+). Since J^{-1} can be identified with the duality map from X^* to X^{**}, it is also of class (S_+). Using the duality map one can show that a map T is maximal monotone if and only if the range of $T + \lambda J$ is the whole space X^* for every $\lambda > 0$. For more details and proofs we refer to [6].

3 – Construction of a degree function

Let X be a real reflexive Banach space. We assume that X and its dual space X^* are locally uniformly convex. Let L be a closed linear maximal monotone map from $D(L) \subset X$ to X^* such that $D(L)$ is dense in X. Since the graph of L is a closed set in $X \times X^*$, $Y = D(L)$ equipped with the graph norm

$$\|u\|_Y = \|u\|_X + \|Lu\|_{X^*}, \quad u \in Y,$$

becomes a real reflexive Banach space. We shall assume that Y and its dual space Y^* are also locally uniformly convex.

Let j stand for the natural embedding of Y to X and j^* for its adjoint from X^* to Y^*. For each open and bounded subset G of X we denote

$$\mathcal{F}_G(L; S_+) = \{L + S : \bar{G} \cap D(L) \to X^* \mid S \text{ is a bounded demicontinuous map of class } (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\}$$

and

$$\mathcal{H}_G(L; S_+) = \{L + S(t) : \bar{G} \cap D(L) \to X^* \mid S(t)\ (0 \le t \le 1) \text{ is a bounded homotopy of class } (S_+) \text{ with respect to } D(L) \text{ from } \bar{G} \text{ to } X^*\}.$$

Recall that $S(t)$ with $0 \leq t \leq 1$ is called a homotopy of class (S_+) with respect to $D(L)$, if the conditions $u_n \rightharpoonup u$, $Lu_n \rightharpoonup Lu$, $t_n \to t$ and $\limsup\langle S(t_n)u_n, u_n - u\rangle \leq 0$ imply $u_n \to u$ and $S(t_n)(u_n) \rightharpoonup S(t)(u)$. Note that the class $\mathcal{H}_G(L; S_+)$ includes all affine homotopies $L+(1-t)S_1+tS_2$ with $S_1, S_2 \in \mathcal{F}_G(L; S_+)$. In order to find suitable approximations for mappings $F \in \mathcal{F}_G(L; S_+)$ we denote

$$\hat{L} = j^* \circ L \circ j,$$

which obviously is a bounded linear monotone map from Y to Y^*. Similarly we denote

$$\hat{S}(t) = j^* \circ S(t) \circ j : j^{-1}(\bar{G}) \to Y^*$$

whenever $S(t)$ is a homotopy from $\bar{G}$ to X^*. Since j is continuous from Y to X, $j^{-1}(\bar{G}) = \bar{G} \cap D(L)$ is closed and $j^{-1}(G) = G \cap D(L)$ is open in Y. It is easy to check that

$$\overline{j^{-1}(G)} \subset j^{-1}(\bar{G}); \quad \partial(j^{-1}(G)) \subset j^{-1}(\partial G). \tag{3.1}$$

Note that we have used the same notation for closures and boundaries in both X and Y. In what follows we also need the map $M : Y \to Y^*$ defined by

$$(M(u), v) = \langle Lv, J^{-1}(Lu)\rangle, \quad u, v \in Y, \tag{3.2}$$

where $(\cdot,\cdot)$ denotes the pairing between Y and Y^*, and J^{-1} is the inverse of the duality map $J : X \to X^*$. In fact, for all those $u \in Y$ for which $M(u) \in j^*(X^*)$, we have $J^{-1}(Lu) \in D(L^*)$ and by (3.2)

$$M(u) = j^* L^* J^{-1}(Lu).$$

We shall need this representation later in proving Lemma 2. For each admissible map $F \in \mathcal{F}_G(L; S_+)$ or homotopy $F(t) \in \mathcal{H}_G(L; S_+)$ and for each $\varepsilon > 0$ we define

$$\hat{F}_\varepsilon = \hat{L} + \hat{S} + \varepsilon M \text{ and } \hat{F}_\varepsilon(t) = \hat{L} + \hat{S}(t) + \varepsilon M.$$

Then we have

LEMMA 1. *If $F(t) \in \mathcal{H}_G(L; S_+)$ and $\varepsilon > 0$, then $\hat{F}_\varepsilon(t)$ is a bounded homotopy of class (S_+) from $j^{-1}(\bar{G}) \subset Y$ to Y^*. In particular, for each $\varepsilon > 0$, $\hat{F}_\varepsilon$ is a bounded demicontinuous map of class (S_+) from $j^{-1}(\bar{G}) \subset Y$ to Y^*.*

PROOF. Assume $F(t) \in \mathcal{H}_G(L; S_+)$ and $\varepsilon > 0$. Let $\{u_n\} \subset \bar{G} \cap D(L)$ with $u_n \rightharpoonup u$ in Y, $t_n \to t$ and $\limsup(\hat{F}_\varepsilon(t)(u_n), u_n - u) \leq 0$. Then $u_n \rightharpoonup u$ in X, $Lu_n \rightharpoonup Lu$ in X^* and

$$\limsup\{\langle Lu_n - Lu, u_n - u\rangle + \langle S(t_n)(u_n), u_n - u\rangle + \varepsilon\langle Lu_n - Lu, J^{-1}(Lu_n) - J^{-1}(Lu)\rangle\} \leq 0.$$

Since L is monotone and J^{-1} is strictly monotone we conclude that

$$\limsup\langle S(t_n)(u_n), u_n - u\rangle \leq 0.$$

By the (S_+)-property of $S(t)$ we obtain $u_n \to u$ in X and $S(t_n)(u_n) \rightharpoonup S(t)(u)$ in X^*. Therefore also

$$\lim\langle Lu_n - Lu, J^{-1}(Lu_n) - J^{-1}(Lu)\rangle = 0$$

implying by the (S_+)-property of J^{-1} that $Lu_n \to Lu$ in X^* and the assertion follows. □

Let $F(t) \in \mathcal{H}_G(L; S_+)$ and let $\{h(t) | 0 \leq t \leq 1\}$ be a continuous curve in X^*. We denote

$$K = \{u \in j^{-1}(\bar{G}) | \hat{F}_\varepsilon(t)(u) = j^* h(t) \text{ for some } \varepsilon > 0 \text{ and } 0 \leq t \leq 1\}.$$

Note that $j(K) \subset \overline{G}$ implying that K is bounded in X. The fact that K is bounded also in Y follows from

LEMMA 2. *There exists a constant $R > 0$, independent of ε and t, such that $K \subset B_R(Y) = \{v \in Y \mid \|v\|_Y < R\}$.*

PROOF. Without loss of generality we may assume that $h(t) \equiv 0$. Let $u \in K$ be arbitrary. Then for some $\varepsilon > 0$ and $0 \le t \le 1$,

$$(3.3) \qquad \langle Lu, v\rangle + \langle S(t)(u), v\rangle + \varepsilon\langle L^* J^{-1}(Lu), v\rangle = 0$$

for all $v \in D(L)$. Observe that $J^{-1}(Lu) \in D(L^*)$ since $M(u) \in j^*(X^*)$. Since $D(L)$ is dense in X, the equation (3.3) holds for all $v \in X$. Hence we can insert $v = J^{-1}(Lu)$ to get

$$\langle Lu, J^{-1}(Lu)\rangle + \langle S(t)(u), J^{-1}(Lu)\rangle + \varepsilon\langle L^* J^{-1}(Lu), J^{-1}(Lu)\rangle = 0.$$

Recalling that L^* is monotone we obtain

$$\|Lu\|^2_{X^*} \le \|S(t)(u)\|_{X^*}\|J^{-1}(Lu)\|_X.$$

Since $\|J^{-1}(Lu)\|_X = \|Lu\|_{X^*}$ and since $S(t)$ is a bounded homotopy from a bounded set $\bar{G}$ to X^* we conclude that

$$\|Lu\|_{X^*} \le c_1$$

for some positive constant c_1 independent of $\varepsilon > 0$ and $t \in [0,1]$, completing the proof. □

The relationship between $F(t) \in \mathcal{H}_G(L; S_+)$ and its approximation $\hat{F}_\varepsilon(t)$ is shown by

LEMMA 3. *Let $A \subset \bar{G}$ be a closed set, $F(t) \in \mathcal{H}_G(L; S_+)$ an admissible homotopy and $h(t)$ a continuous curve in X^* such that*

$$h(t) \notin F(t)(A \cap D(L)) \quad \textit{for all } t \in [0,1].$$

Then there exists $\varepsilon_0 > 0$ such that

$$j^*h(t) \notin \hat{F}_\varepsilon(t)(j^{-1}(A)) \quad \textit{for all } t \in [0,1] \textit{ and } 0 < \varepsilon < \varepsilon_0.$$

PROOF. We may assume again that $h(t) \equiv 0$. We shall argue by contradiction. Let us assume that there exist sequences $\{\varepsilon_n\}$, $\{t_n\}$ and $\{u_n\} \subset j^{-1}(A)$ such that $\varepsilon_n \to 0+$, $t_n \to t \in [0,1]$ and

$$\hat{L}u_n + \hat{S}(t_n)(u_n) + \varepsilon_n M(u_n) = 0 \tag{3.4}$$

for all $n \in \mathbb{N}$. By Lemma 2 the sequence $\{u_n\}$ is bounded in Y implying that $u_n \rightharpoonup u$ in X and $Lu_n \rightharpoonup Lu$ in X^* with $u \in D(L)$, for a subsequence. Using the fact that L and J^{-1} are monotone we get from (3.4)

$$\begin{aligned}
&\limsup \langle S(t_n)(u_n), u_n - u\rangle = \\
&= \limsup\{-\langle Lu_n - Lu, u_n - u\rangle - \varepsilon_n \langle Lu_n - Lu, J^{-1}(Lu_n) - J^{-1}(Lu)\rangle\} \\
&\le 0.
\end{aligned}$$

Since $S(t)$ is in $\mathcal{H}_G(L; S_+)$, $u_n \to u$ in X and $S(t_n)(u_n) \rightharpoonup S(t)(u)$ with $u \in A$. By (3.4)

$$(\hat{L}u_n, v) + (\hat{S}(t_n)(u_n), v) + \varepsilon_n (M(u_n), v) = 0$$

for all $v \in Y$ and $n \in \mathbb{N}$. Letting $n \to \infty$ we then have

$$\langle Lu, v\rangle + \langle S(t)(u), v\rangle = 0 \text{ for all } v \in D(L).$$

Since $D(L)$ is dense in X,

$$Lu + S(t)(u) = 0 \text{ with } u \in A \cap D(L)$$

contradicting our assumption. Hence the proof is complete. □

If we choose $A = \partial G$, $S(t) = S \in \mathcal{F}_G(L; S_+)$ and $h(t) = h \in X^*$ in Lemma 3, then the condition $h \notin (L + S)(\partial G \cap D(L))$ implies that there exists $\varepsilon_0 > 0$ such that

$$j^* h \notin (\hat{L} + \hat{S} + \varepsilon M)(j^{-1}(\partial G))$$

for all ε with $0 < \varepsilon < \varepsilon_0$. Recalling (3.1) we also have

$$j^* h \notin (\hat{L} + \hat{S} + \varepsilon M)(\partial(j^{-1}(G)))$$

for all $0 < \varepsilon < \varepsilon_0$. Moreover, by Lemma 2 there exists a constant $R > 0$, independent of ε, such that

(3.5) $\quad j^*h \neq \hat{L}u + \hat{S}(u) + \varepsilon M(u)$ for all $u \in \bar{G}$, $\|u\|_Y \geq R$ and $\varepsilon > 0$.

Denoting $G_R(Y) = j^{-1}(G) \cap B_R(Y)$ we therefore have

$$j^*h \notin (\hat{L} + \hat{S} + \varepsilon M)(\partial G_R(Y))$$

for all $0 < \varepsilon < \varepsilon_0$. Since $\hat{F} = \hat{L} + \hat{S} + \varepsilon M$ is a map of class (S_+) from $j^{-1}(\bar{G})$ to Y^* by Lemma 1, the value of the unique topological S_+-degree (see [3], [1])

$$d_{S_+}(\hat{L} + \hat{S} + \varepsilon M, G_R(Y), j^*h)$$

is well-defined for all $0 < \varepsilon < \varepsilon_0$ and independent of R provided (3.5) holds. Moreover, the value of the degree remains stable for all $0 < \varepsilon < \varepsilon_0$. Indeed, if ε_1 and ε_2 are arbitrary with $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, we have

$$(1-t)(\hat{L} + \hat{S} + \varepsilon_1 M) + t(\hat{L} + \hat{S} + \varepsilon_2 M) = \hat{L} + \hat{S} + \varepsilon(t)M,$$

where $\varepsilon(t) = (1-t)\varepsilon_1 + t\varepsilon_2$ satisfies $0 < \varepsilon_1 \leq \varepsilon(t) \leq \varepsilon_2$. Hence

$$j^*h \notin (\hat{L} + \hat{S} + \varepsilon(t)M)(\partial G_R(Y)) \text{ for all } 0 \leq t \leq 1$$

and the invariance under admissible homotopies of the S_+-degree implies that

$$d_{S_+}(\hat{L} + \hat{S} + \varepsilon_1 M, G_R(Y), j^*h) = d_{S_+}(\hat{L} + \hat{S} + \varepsilon_2 M, G_R(Y), j^*h).$$

Consequently, it is relevant to define a function d_L by

(3.6) $$d_L(L + S, G, h) = \lim_{\varepsilon \to 0+} d_{S_+}(\hat{L} + \hat{S} + \varepsilon M, G_R(Y), j^*h).$$

whenever $h \notin (L + S)(\partial G \cap D(L))$ and R is sufficiently large. We shall show in the next section that (3.6) defines a classical topological degree for $\mathcal{F}_G(L; S_+)$ with respect to the class $\mathcal{H}_G(L; S_+)$ of admissible homotopies and the normalizing map $L + J$. In the sequel we use the constant R in various situations and we always assume R to satisfy Lemma 2.

4 – Properties of the degree

For each open bounded subset G of X, $F = L + S \in \mathcal{F}_G(L; S_+)$ and $h \in X^*\backslash F(\partial G \cap D(L))$ the formula (3.5) defines the value of the function d_L. In order to check that d_L is in fact a good degree for the class $\mathcal{F}_G(L; S_+)$ we recall first the classical axioms of the topological degree function to be verified.

(a) If $d_L(L + S, G, h) \neq 0$, then there exists $u \in G \cap D(L)$ such that $Lu + S(u) = h$.

(b) (Additivity of domain) If G^1 and G^2 are open disjoint subsets of G and $h \notin (L + S)[(\bar{G}\backslash(G^1 \cup G^2)) \cap D(L)]$, then

$$d_L(L + S, G, h) = d_L(L + S, G^1, h) + d_L(L + S, G^2, h).$$

(c) (Invariance under admissible homotopies) If $F(t) \in \mathcal{H}_G(L; S_+)$ and $h(t) \notin F(t)(\partial G \cap D(L))$ for all $t \in [0, 1]$ where $h(t)$ is a continuous curve in X^*, then

$$d_L(F(t), G, h(t)) \text{ is constant for all } t \in [0, 1].$$

(d) $L + J$ is the normalizing map, i.e.,

$$d_L(L + J, G, h) = 1 \text{ whenever } h \in (L + J)(G \cap D(L)).$$

We shall verify the axioms (a) to (d) by using the corresponding axioms for the S_+-degree satisfied by the related approximations.

(a) Assume $d_L(L + S, G, h) \neq 0$. If $h \notin (L + S)(G \cap D(L))$, then also $h \notin (L + S)(\bar{G} \cap D(L))$ and Lemma 3 implies $j^*h \notin \hat{F}_\varepsilon(j^{-1}(\bar{G}))$ for all $\varepsilon > 0$ small enough. A contradiction follows from (3.6).

(b) Let G^1 and G^2 be open subsets of G with $G^1 \cap G^2 = \emptyset$. Assume that $h \notin (L+S)[(\bar{G}\backslash(G^1 \cup G^2)) \cap D(L)]$. By Lemmata 2 and 3 there exist constants $\varepsilon_0 > 0$ and $R > 0$ such that

$$\hat{L}u + \hat{S}u + \varepsilon M(u) \neq j^*h$$

for all $u \in j^{-1}(\bar{G})$ with $\|u\|_Y \geq R$, $\varepsilon > 0$, and

$$j^*h \notin (\hat{L} + \hat{S} + \varepsilon M)(j^{-1}(\bar{G}\backslash(G^1 \cup G^2)))$$

for all $0 < \varepsilon < \varepsilon_0$. Note that $j^{-1}(\bar{G}\backslash(G^1 \cup G^2)) = j^{-1}(\bar{G})\backslash(j^{-1}(G^1) \cup j^{-1}(G^2))$. Then we also have

$$(4.1) \qquad j^*h \notin (\hat{L} + \hat{S} + \varepsilon M)(\overline{G_R(Y)}\backslash(G_R^1(Y) \cup G_R^2(Y)))$$

for all $0 < \varepsilon < \varepsilon_0$. Hence by (3.5) and the properties of the S_+-degree

$$\begin{aligned} &d_L(L + S, G, h) = d_{S_+}(\hat{L} + \hat{S} + \varepsilon M, G_R(Y), j^*h) \\ =&d_{S_+}(\hat{L} + \hat{S} + \varepsilon M, G_R^1(Y), j^*h) + d_{S_+}(\hat{L} + \hat{S} + \varepsilon M, G_R^2(Y), j^*h) \\ =&d_L(L + S, G^1, h) + d_L(L + S, G^2, h), \end{aligned}$$

where $0 < \varepsilon < \varepsilon_0$.

(c) Let $F(t) = L + S(t) \in \mathcal{H}_G(L; S_+)$ and $h(t)$ a continuous curve in X^* with $h(t) \notin F(t)(\partial G \cap D(L))$ for all $t \in [0,1]$. By Lemma 3 there exists $\varepsilon_0 > 0$ such that

$$j^*h(t) \notin \hat{F}_\varepsilon(t)(j^{-1}(\partial G)) \text{ for all } t \in [0,1] \text{ and } 0 < \varepsilon < \varepsilon_0.$$

By Lemma 2 there exists $R > 0$ such that

$$\hat{L}u + \hat{S}(t)(u) + \varepsilon M(u) \neq j^*h$$

for all $t \in [0,1]$, $\varepsilon > 0$ and $u \in j^{-1}(\bar{G})$ with $\|u\|_Y \geq R$. Hence $j^*h(t) \notin \hat{F}_\varepsilon(t)(\partial G_R(Y))$ for all $t \in [0,1]$, $0 < \varepsilon < \varepsilon_0$, and the invariance under S_+-homotopies of the S_+-degree gives

$$\begin{aligned} d_L(F(t), G, h(t)) &= \lim_{\varepsilon \to 0+} d_{S_+}(\hat{F}_\varepsilon(t), G_R(Y), j^*h(t)) \\ &= \text{constant} \quad \text{for all } t \in [0,1]. \end{aligned}$$

(d) We must check that $L + J$ admits the property of a normalizing map. It is well-known that $L + J$ from $D(L)$ to X^* is one-to-one and onto. Let G be an open bounded subset in X, $h \in (L + J)(G \cap D(L))$ and let $B_r(X) = \{v \in X \mid \|v\|_X < r\}$ contain G. Using (b) we get

$$d_L(L + J, G, h) = d_L(L + J, B_r(X), h).$$

We consider the solutions of the equation

$$(4.2) \qquad Lu + J(u) = th, \quad u \in D(L),\ 0 \le t \le 1.$$

Then (4.2) implies

$$\begin{aligned}\|u\|_X^2 &= -\langle Lu, u\rangle + t\langle h, u\rangle \\ &\le \|h\|_{X^*}\|u\|_X.\end{aligned}$$

Therefore the solutions of (4.2) satisfy $\|u\|_X \le \|h\|_{X^*}$. Thus, by choosing $r > \|h\|_{X^*}$,

$$Lu + J(u) \ne th \text{ for all } t \in [0,1],\ u \in D(L),\ \|u\|_X = r.$$

Using (c) we have

$$d_L(L + J, B_r(X), h) = d_L(L + J, B_r(X), 0)$$

and by (3.5)

$$\begin{aligned}d_L(L + J, B_r(X), 0) &= \lim_{\varepsilon\to 0+} d_{S_+}(\hat{L} + \hat{J} + \varepsilon M, j^{-1}(B_r(X)) \cap B_R(Y), 0) \\ &= \lim_{\varepsilon\to 0+} d_{S_+}(\hat{L} + \hat{J} + \varepsilon M, B_R(Y), 0).\end{aligned}$$

Let J_Y denote the duality map from Y to Y^*. It is easy to see that

$$(1 - t)J_Y(u) + t(\hat{L}u + \hat{J}(u) + \varepsilon M(u)) \ne 0$$

for all $t \in [0,1]$, $0 < \varepsilon < \varepsilon_0$, $\|u\|_Y = R$. Hence

$$\lim_{\varepsilon\to 0+} d_{S_+}(\hat{L} + \hat{J} + \varepsilon M, B_R(Y), 0) = d_{S_+}(J_Y, B_R(Y), 0) = 1.$$

Collecting the results above we get the desired result

$$d_L(L + J, G, h) = 1.$$

Thus we can conclude the following

THEOREM 1. *Let X be a real reflexive Banach space, L a closed linear maximal monotone densely defined map from $D(L) \subset X$ to X^*, G an open bounded subset in X and $\mathcal{F}_G(L; S_+)$ the class of admissible mappings. Then there exists a topological degree function d_L satisfying the properties (a) to (d) with respect to the class $\mathcal{H}_G(L; S_+)$ of admissible homotopies and normalizing map $L + J$.*

5 – Existence theorems

We describe some standard results which can be derived by continuation methods as soon as a classical degree theory is available. Let X be a real reflexive Banach space, L a closed linear maximal monotone map: $D(L) \to X^*$ with $D(L)$ dense in X and G an open bounded subset in X. If $F = L + S \in \mathcal{F}_G(L; S_+)$ and $h \in X^*$ is given, we are interested in the solvability of the equation

$$Lu + S(u) = h, \quad u \in \bar{G} \cap D(L). \tag{5.1}$$

More generally, denoting

$$\mathcal{F}_G(L; PM) = \{F = L + S | S : \bar{G} \to X^* \text{ is pseudomonotone with respect to } D(L)\}$$

and

$$\mathcal{F}_G(L; QM) = \{F = L + S | S : \bar{G} \to X^* \text{ is quasimonotone with respect to } D(L)\}$$

we can use the fact that the mappings $F = L + S \in \mathcal{F}_G(L; QM)$ admit good approximations $\{F_\varepsilon = L + S + \varepsilon J | \varepsilon > 0\}$ in $\mathcal{F}_G(L; S_+)$, and homotopy arguments can be applied to the broader classes $\mathcal{F}_G(L; QM)$ and $\mathcal{F}_G(L; PM)$. Our basic existence theorem is the following

THEOREM 2. *Let G be an open bounded subset in X with $0 \in G$ and let $F = L + S \in \mathcal{F}_G(L; QM)$. If*

$$Lu + (1-t)J(u) + tS(u) \neq 0 \text{ for all } u \in \partial G \cap D(L) \text{ and } 0 \leq t < 1, \tag{5.2}$$

then $0 \in \overline{(L + S)(\bar{G} \cap D(L))}$.

PROOF. We may assume that $0 \notin \overline{(L+S)(\partial G \cap D(L))}$; otherwise there is nothing to prove. In order to employ the homotopy argument of the d_L-degree we show the existence of $\varepsilon_0 > 0$ such that

$$(5.3) \qquad (1-t)(Lu + J(u)) + t(Lu + S(u) + \varepsilon J(u)) \neq 0$$

for all $u \in \partial G \cap D(L)$, $0 \leq t \leq 1$ and $0 < \varepsilon < \varepsilon_0$. Indeed, assume the contrary, i.e., there are sequences $\{u_n\}$ in $\partial G \cap D(L)$, $\{t_n\}$ in $[0,1]$ and $\{\varepsilon_n\}$ with $\varepsilon_n \to 0+$ such that

$$(5.4) \qquad Lu_n + (1-t_n)J(u_n) + t_n(S(u_n) + \varepsilon_n J(u_n)) = 0, \quad n \in \mathbb{N}.$$

Taking subsequences, if necessary, we have $u_n \rightharpoonup u$ in X, $t_n \to t$, $J(u_n) \rightharpoonup w$ in X^* and $S(u_n) \rightharpoonup z$ in X^*. By (5.4), $Lu_n \rightharpoonup -(1-t)w - tz$. Since the graph of L is weakly closed, $u \in D(L)$ and $Lu = -(1-t)w - tz$.

The case $t_n \to t = 1$ is excluded because (5.4) gives

$$Lu_n + S(u_n) = -(1 - t_n + t_n\varepsilon_n)J(u_n) + (1-t_n)S(u_n) \to 0$$

implying $0 \in \overline{(L+S)(\partial G \cap D(L))}$. Hence we can assume that $0 \leq t < 1$. By (5.4), monotonicity of L and the (QM)-property of S we have

$$\begin{aligned}
&(1-t)\limsup\langle J(u_n), u_n - u\rangle \\
=&\limsup\langle (1 - t_n + \varepsilon_n t_n)J(u_n), u_n - u\rangle \\
=&\limsup\langle -Lu_n - t_n S(u_n), u_n - u\rangle \\
=&-\liminf\{\langle Lu_n, u_n - u\rangle + t_n\langle S(u_n), u_n - u\rangle\} \\
\leq&-\liminf\langle Lu_n - Lu, u_n - u\rangle - t\liminf\langle S(u_n), u_n - u\rangle \leq 0
\end{aligned}$$

Hence $\limsup\langle J(u_n), u_n - u\rangle \leq 0$ and the (S_+)-property of J implies $u_n \to u$ in X with $u \in \partial G \cap D(L)$. Letting $n \to \infty$ in (5.4) we get

$$Lu + (1-t)J(u) + tS(u) = 0 \text{ with } u \in \partial G \cap D(L) \text{ and } 0 \leq t < 1,$$

which contradicts (5.2).

Consequently, there exists $\varepsilon_0 > 0$ such that (5.3) holds. Since $L+S+\varepsilon J \in \mathcal{F}_G(L; S_+)$ and $L + J$ is the normalizing map for d_L, the conditions

(5.3) and (c) imply that for each ε with $0 < \varepsilon < \varepsilon_0$ there exists $u_\varepsilon \in G \cap D(L)$ such that

$$Lu_\varepsilon + S(u_\varepsilon) + \varepsilon J(u_\varepsilon) = 0. \tag{5.5}$$

Letting $\varepsilon \to 0+$ in (5.5) we get $Lu_\varepsilon + S(u_\varepsilon) \to 0$ implying that $0 \in \overline{(L+S)(G \cap D(L))} \subset \overline{(L+S)(\bar{G} \cap D(L))}$, completing the proof. □

For mappings $F = L + S$ in the classes $\mathcal{F}_G(L; S_+)$ or $\mathcal{F}_G(L; PM)$ we have better results. In fact, if $L + S \in \mathcal{F}_G(L; S_+)$ we can derive directly from the properties (a), (c), and (d) of the d_L-degree the following

Corollary 1. *Let G be an open bounded subset in X with $0 \in G$ and let $F = L + S \in \mathcal{F}_G(L; S_+)$. If the condition (5.2) holds, then $0 \in (L+S)(\bar{G} \cap D(L))$. Moreover, if (5.2) holds also for $t = 1$, then $d_L(L+S, G, 0) = 1$.*

For mappings $F = L + S$ in $\mathcal{F}_G(L; PM)$ we respectively have

Corollary 2. *Let G be a convex open bounded subset in X with $0 \in G$ and let $F = L + S \in \mathcal{F}_G(L; PM)$. If the condition (5.2) holds, then $0 \in (L+S)(\bar{G} \cap D(L))$.*

Proof. In view of Theorem 2 it will be sufficient to show that $(L + S)(\bar{G} \cap D(L))$ is closed in X^*. Indeed, let $\{y_n\}$ be a sequence in X^* such that $y_n = Lu_n + S(u_n)$ with $\{u_n\} \subset \bar{G} \cap D(L)$ and $y_n \to y$ in X^*. Since G is bounded and convex, we can assume that $u_n \rightharpoonup u$ in X with $u \in \bar{G}$ and $S(u_n) \rightharpoonup z$ in X^*. Hence $Lu_n \rightharpoonup y - z$. Since also the graph of L is weakly closed, $u \in \bar{G} \cap D(L)$ and $Lu = y - z$. Consequently,

$$\begin{aligned}\limsup \langle S(u_n), u_n - u\rangle &= \limsup \langle y_n - Lu_n, u_n - u\rangle \\ &= -\liminf \langle Lu_n - Lu, u_n - u\rangle \leq 0.\end{aligned}$$

Since S is pseudomonotone with respect to $D(L)$ we obtain $S(u_n) \rightharpoonup S(u) = z$ and $\langle S(u_n), u_n\rangle \to \langle S(u), u\rangle$. Thus $Lu + S(u) = y \in (L + S)(\bar{G} \cap D(L))$. □

Next we produce a version of Borsuk's theorem.

THEOREM 3. *Let G be an open bounded subset in X such that $0 \in G$ and G is symmetric with respect to the origin. Assume that $S : \bar{G} \to X^*$ is a mapping satisfying the condition*

$$S(-u) = -S(u) \text{ for all } u \in \partial G. \tag{5.6}$$

Then the following assertions hold

(A) *If $L+S \in \mathcal{F}_G(L; S_+)$, then $0 \in (L+S)(\bar{G} \cap D(L))$ and $d_L(L+S, G, 0)$ is odd whenever defined.*
(B) *If $L+S \in \mathcal{F}_G(L; PM)$ and G is convex, then $0 \in (L+S)(\bar{G} \cap D(L))$*
(C) *If $L+S \in \mathcal{F}_G(L; QM)$, then $0 \in \overline{(L+S)(\bar{G} \cap D(L))}$.*

PROOF. Borsuk's theorem holds for the mappings of class (S_+). On the other hand, if S satisfies (5.6), then $S + \varepsilon J$, $L + S$ and $\hat{L} + \tilde{S} + \varepsilon M$ satisfy the corresponding condition. Thus the proof is analogous to the proofs of Theorem 2 and Corollaries 1 and 2. □

We close this section by some surjectivity results.

THEOREM 4. *Let $L+S \in \mathcal{F}_X(L; PM)$ $(L+S \in \mathcal{F}_X(L; QM))$ and assume that S satisfies the condition*

(i) *if $Lu_n + S(u_n) \to w$ in X^*, then $\{u_n\}$ is bounded in X,*

and one of the conditions

(ii) *there exists $R > 0$ such that $\langle S(u), u\rangle > 0$ for all $\|u\| \geq R$,*
(iii) *there exists $R > 0$ such that $S(-u) = -S(u)$ for all $\|u\| \geq R$.*

Then $(L+S)(D(L)) = X^*$ ($(L+S)(D(L))$ is dense in X^*, respectively).

PROOF. We deal with the case $L + S \in \mathcal{F}_X(L; PM)$ where S satifies the conditions (i) and (ii). All other cases are shown analogously. Let $h \in X^*$ be given. By the condition (i) there exist constants $R' \geq R$ and $\delta > 0$ such that
$\|Lu + S(u) + \varepsilon J(u) - th\| \geq \delta$ for all $u \in D(L)$, $\|u\| = R'$, $0 \leq t \leq 1$ and $0 \leq \varepsilon < \frac{\delta}{R'}$. Thus we obtain

$$d_L(L + S + \varepsilon J, B_{R'}(X), h) = d_L(L + S + \varepsilon J, B_{R'}(X), 0)$$

whenever $0 < \varepsilon < \frac{\delta}{R'}$. Denote

$$F(t)(u) = Lu + (1-t)J(u) + t(S(u) + \varepsilon J(u)), \quad 0 \le t \le 1.$$

If $F(t)(u) = 0$ for some $t \in [0,1]$, $\|u\| = R'$ and $0 < \varepsilon < \frac{\delta}{R'}$, then by (ii)

$$0 = \langle F(t)(u), u\rangle = \langle Lu, u\rangle + (1 - t + \varepsilon t){R'}^2 + t\langle S(u), u\rangle > 0,$$

a contradiction. Therefore, by the invariance under homotopies, we have for all $0 < \varepsilon < \frac{\delta}{R'}$,

$$d_L(L + S + \varepsilon J, B_{R'}(X), 0) = d_L(L + J, B_{R'}(X), 0) = 1.$$

Hence there exists $u_\varepsilon \in D(L)$ such that $Lu_\varepsilon + S(u_\varepsilon) + \varepsilon J(u_\varepsilon) = h$. Letting $\varepsilon \to 0+$ we have $Lu + S(u) = h$ for some $u \in D(L)$. □

Remark. We note that both of the conditions (i) and (ii) are met, if S satisfies the strong coercivity condition

$$(i)_S \qquad \frac{\langle S(u), u\rangle}{\|u\|} \to \infty \text{ as } \|u\| \to \infty.$$

6 – Applications to parabolic initial-boundary value problems

We shall consider initial-boundary value problems for differential operators of the form

$$(6.1) \quad \frac{\partial u(x,t)}{\partial t} + \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u(x,t), Du(x,t), \dots, D^m u(x,t))$$

in $Q = \Omega \times [0, T]$, where Ω is an open bounded subset in $\mathbb{R}^N$, $m \ge 1$ and the coefficents A_α are functions of $(x,t) \in Q$ and of $\xi = (\eta, \zeta) \in \mathbb{R}^{N_0}$ with $\eta = \{\eta_\beta \mid |\beta| \le m-1\} \in \mathbb{R}^{N_1}$, $\zeta = \{\zeta_\beta \mid |\beta| = m\} \in \mathbb{R}^{N_2}$ and $N_1 + N_2 = N_0$. We assume that each $A_\alpha(x,t,\xi)$ is a Carathéodory function, i.e., measurable in (x,t) for fixed $\xi \in \mathbb{R}^{N_0}$ and continuous in ξ for almost all $(x,t) \in Q$. Then the familiar growth condition

(A_1) There exist $p > 1$, $c_1 > 0$ and $k_1 \in L^{p'}(Q)$, $p' = \frac{p}{p-1}$, such that

$$|A_\alpha(x,t,\eta,\zeta)| \le c_1(|\zeta|^{p-1} + |\eta|^{p-1} + k_1(x,t))$$

for all $(x,t) \in Q$, $\xi = (\eta, \zeta) \in \mathbb{R}^{N_0}$ and $|\alpha| \leq m$,
implies that the latter part of (6.1),

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,t,u,Du,\dots,D^m u) \tag{6.2}$$

gives rise to a bounded continuous map S from the space $\mathcal{V} = L^p(0,T;V)$ with $V = W_0^{m,p}(\Omega)$ into its dual space $\mathcal{V}^* = L^{p'}(0,T;V^*)$ by the rule

$$\langle S(u), v\rangle = \sum_{|\alpha| \leq m} \int_Q A_\alpha(x,t,u,Du,\dots,D^m u) D^\alpha v, \quad u, v \in \mathcal{V}. \tag{6.3}$$

We shall assume in the sequel that $2 \leq p < \infty$. (The case $1 < p < 2$ can also be treated if we replace $\mathcal{V}$ by $\mathcal{W} = \mathcal{V} \cap L^2(Q)$ and modify the proofs accordingly cf. [9], [11]). Indeed, each $u \in \mathcal{V}$ with $u' \in \mathcal{V}^*$ also belongs to $C([0,T], L_2(\Omega))$ and the initial condition $u(x,0) = 0$ in Ω makes sense. Thus the operator $\frac{\partial}{\partial t}$ induces a linear map from the subset $D(L) = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*,\ v(0) = 0\}$ of $\mathcal{V}$ into $\mathcal{V}^*$ by

$$\langle Lu, v\rangle = \int_0^T \langle u'(t), v(t)\rangle dt, \quad u \in D(L),\ v \in \mathcal{V}. \tag{6.4}$$

Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\frac{\partial \varphi(t)}{\partial t}dt \quad \text{for all } \varphi \in C_0^\infty(0,T).$$

It can be shown (see [12] that L is a closed linear maximal monotone map. This is also true, if in $D(L)$ the initial condition $v(0) = 0$ is replaced by the periodicity condition $v(0) = v(T)$. A function $u \in \mathcal{V}$ is called a *weak solution* of the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) = h & \text{in } Q \\ D^\alpha u = 0 & \text{on } \partial\Omega \times [0,T] \text{ for all } |\alpha| \leq m-1 \\ u(x,0) = 0 & \text{in } \Omega \end{cases} \tag{6.5}$$

if and only if

$$(6.6) \qquad Lu + S(u) = h, \quad u \in D(L).$$

Thus we can apply the results of section 5 to the study of the existence of weak solutions for (6.5) as soon as the operator A satisfies relevant monotonicity and coercivity conditions.

Indeed, if we assume that the coefficients A_α satisfy the classical Leray-Lions condition

$$(A_2)_S \qquad \sum_{|\alpha|=m} \{A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*)\}(\zeta_\alpha - \zeta^*_\alpha) > 0$$

for all $(x,t) \in Q$, $\eta \in \mathbb{R}^{N_1}$ and $\zeta \neq \zeta^* \in \mathbb{R}^{N_2}$

or its weaker version

$$(A_2)_W \qquad \sum_{|\alpha|=m} \{A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*)\}(\zeta_\alpha - \zeta^*_\alpha) \geq 0$$

for all $(x,t) \in Q$, $\eta \in \mathbb{R}^{N_1}$ and $\zeta, \zeta^* \in \mathbb{R}^{N_2}$

and the strong coercivity condition

(A_3) There exist $c_0 > 0$ and $k_0 \in L_1(Q)$ such that

$$\sum_{|\alpha|\leq m} A_\alpha(x,t,\xi)\xi_\alpha \geq c_0|\xi|^p - k_0(x,t)$$

for all $(x,t) \in Q$ and $\xi \in \mathbb{R}^{N_0}$,

then the existence theorems of the present note are available.

A significant feature in the conditions $(A_2)_S$ and $(A_2)_W$ is that monotonicity is assigned only to the top order part

$$A^{(1)}(u) = \sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha A_\alpha(\cdot,\cdot,u,Du,\dots,D^m u).$$

The lower order part of A is denoted by

$$A^{(2)}(u) = \sum_{|\alpha|\leq m-1} (-1)^{|\alpha|} D^\alpha A_\alpha(\cdot,\cdot,u,Du,\dots,D^m u)$$

and the special case where $A^{(2)}$ is independent of $D^m u$ by $A^{(3)}$, i.e.,

$$A^{(3)}(u) = \sum_{|\alpha|\le m-1} (-1)^{|\alpha|} D^\alpha A_\alpha(\cdot,\cdot,u,Du,\dots,D^{m-1}u).$$

If we denote the corresponding mappings by S_1, S_2 and S_3, i.e.,

$$(6.7) \qquad \langle S_1(u),v\rangle = \sum_{|\alpha|=m} \int_Q A_\alpha(x,t,u,\dots,D^m u)D^\alpha v, \quad u,v\in\mathcal{V},$$

$$(6.8) \qquad \langle S_2(u),v\rangle = \sum_{|\alpha|\le m-1} \int_Q A_\alpha(x,t,u,\dots,D^m u)D^\alpha v, \quad u,v\in\mathcal{V},$$

$$(6.9) \qquad \langle S_3(u),v\rangle = \sum_{|\alpha|\le m-1} \int_Q A_\alpha(x,t,u,\dots,D^{m-1} u)D^\alpha v, \quad u,v\in\mathcal{V},$$

we have the following

PROPOSITION 1. *Let Ω be an open bounded subset in $\mathbb{R}^N$, Q the cylinder $\Omega\times[0,T]$, A the differential operator defined by (6.2) and L the linear maximal monotone operator defined by (6.4). If S, S_1, S_2 and S_3 are the mappings from $\mathcal{V}$ to $\mathcal{V}^*$ defined by (6.3), (6.7), (6.8) and (6.9), respectively, then the following assertions hold:*

(a) *If A satisfies (A_1) and $(A_2)_S$, then S is pseudomonotone with respect to $D(L)$.*
(b) *If A satisfies (A_1) and $(A_2)_S$ and (A_3), then S is of class (S_+) with respect to $D(L)$.*
(c) *If $A^{(1)}$ satisfies (A_1) and $(A_2)_W$, then S_1 is pseudomonotone with respect to $D(L)$.*
(d) *If $A^{(2)}$ satisfies (A_1), then S_2 is quasimonotone with respect to $D(L)$.*
(e) *If $A^{(3)}$ satisfies (A_1), then S_3 is completely continuous with respect to $D(L)$.*

PROOF. (a) This is the classical case (see [8]). It is shown in [10] that no coercivity condition is needed when Ω is a bounded domain in $\mathbb{R}^N$.

(b) We refer to [11].

(c) The proof is a straightforward modification of the elliptic case (cf. [7]). Indeed, let $\{u_n\} \subset D(L)$ with $u_n \rightharpoonup u$ in $\mathcal{V}$, $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$ and $\limsup\langle S_1(u_n), u_n - u\rangle \leq 0$. To prove assertion (c) we have to verify that $S_1(u_n) \rightharpoonup S_1(u)$ in $\mathcal{V}^*$ and $\langle S_1(u_n), u_n\rangle \to \langle S_1(u), u\rangle$. Obviously it suffices to show the above convergences for a subsequence.

Since $u_n \rightharpoonup u$ in $\mathcal{V}$, $D^\alpha u_n \rightharpoonup D^\alpha u$ in $L^p(Q)$ for all $\alpha \leq m$, $S_1(u_n) \rightharpoonup h$ in $\mathcal{V}^*$ and $A_\alpha(\cdot,\cdot,u_n,\ldots,D^m u_n) \rightharpoonup h_\alpha$ in $L^{p'}(Q)$ for all $|\alpha| = m$, for a subsequence. By Aubin's lemma $D^\alpha u_n \to D^\alpha u$ in $L^p(Q)$, $D^\alpha u_n(x,t) \to D^\alpha u(x,t)$ a.e. in Q for all $|\alpha| \leq m-1$ and there are functions $k_\alpha \in L^p(Q)$ such that $|D^\alpha u_n(x,t)| \leq k_\alpha(x,t)$ a.e. in Q, for a further subsequence. Here the action of $h \in \mathcal{V}^*$ is given by

$$\langle h, v\rangle = \int_Q \sum_{|\alpha|=m} h_\alpha D^\alpha v, \quad v \in \mathcal{V},$$

and

$$\limsup\langle S_1(u_n), u_n\rangle \leq \langle h, u\rangle.$$

By $(A_2)_W$ we have for all $v = (v_\alpha) \in \prod_{|\alpha|=m} L^p(Q)$

$$\begin{aligned}\langle S_1(u_n), u_n\rangle &= \int_Q \sum_{|\alpha|=m} A_\alpha(x,t,u_n,\ldots,D^m u_n) D^\alpha u_n \\ &\geq \int_Q \sum_{|\alpha|=m} A_\alpha(x,t,u_n,\ldots,D^m u_n) v_\alpha \\ &\quad + \int_Q \sum_{|\alpha|=m} A_\alpha(x,t,u_n,\ldots,D^{m-1}u_n, v) D^\alpha u_n \\ &\quad - \int_Q \sum_{|\alpha|=m} A_\alpha(x,t,u_n,\ldots,D^{m-1}u_n, v) v_\alpha.\end{aligned}$$

Thus

$$\begin{aligned}\langle h, u\rangle &\geq \limsup\langle S_1(u_n), u_n\rangle \\ &\geq \sum_{|\alpha|=m} h_\alpha v_\alpha + \limsup\{\int_Q \sum_{|\alpha|=m} A_\alpha(x,t,u_n,\dots,D^{m-1}u_n,v)D^\alpha u_n \\ &\qquad - \int_Q \sum_{|\alpha|=m} A_\alpha(x,t,u_n,\dots,D^{m-1}u_n,v)v_\alpha\}.\end{aligned}$$

Bearing in mind (A_1) and the Carathéodory condition we can use the dominated convergence theorem to obtain

$$A_\alpha(\cdot,\cdot,u_n,\dots,D^{m-1}u_n,v) \to A_\alpha(\cdot,\cdot,u,\dots,D^{m-1}u,v)$$

in $L^{p'}(Q)$. Therefore the above inequalites give

$$\sum_{|\alpha|=m}\int_Q\{A_\alpha(x,t,u,\dots,D^{m-1}u,v)-h_\alpha\}(v_\alpha-D^\alpha u) \geq 0 \qquad \forall\, v\in\prod L^p(Q).$$

Now we can employ Minty's trick. Indeed, setting $v_\alpha = D^\alpha u + sw_\alpha$ with $s>0$ and $w=(w_\alpha)\in\prod L^p(Q)$ and letting $s\to 0+$ we have

$$\sum_{|\alpha|=m}\int_Q\{A_\alpha(x,t,u,\dots,D^{m-1}u,D^m u) - h_\alpha(x,t)\}w_\alpha \geq 0$$
$$\text{for all } w=(w_\alpha)\in\prod L^p(Q).$$

Hence $A_\alpha(x,t,u,\dots,D^{m-1}u,D^m u) = h_\alpha(x,t)$ a.e. in Q for each $|\alpha|=m$ and thus $S_1(u)=h$, $S_1(u_n)\rightharpoonup S_1(u)$ in $\mathcal{V}^*$. It remains to show that $\langle S_1(u_n),u_n\rangle \to \langle S_1(u),u\rangle$. Since we already have that

$$\limsup\langle S_1(u_n),u_n\rangle \leq \sum_{|\alpha|=m}\int_Q h_\alpha D^\alpha u = \langle S_1(u),u\rangle,$$

it suffices to verify that

$$\liminf\langle S_1(u_n),u_n\rangle \geq \langle S_1(u),u\rangle.$$

As above with $v = u$ we get

$$\liminf \langle S_1(u_n), u_n\rangle = \liminf \sum_{|\alpha|=m} \int_Q A_\alpha(x,t,u_n,\dots,D^m u_n) D^\alpha u_n$$

$$\geq \sum_{|\alpha|=m} \int_Q A_\alpha(x,t,u,\dots,D^m u) D^\alpha u + \sum_{|\alpha|=m} \int_Q A_\alpha(x,t,u,\dots,D^m u) D^\alpha u$$

$$- \sum_{|\alpha|=m} \int_Q A_\alpha(x,t,u,\dots,D^m u) D^\alpha u = \langle S_1(u), u\rangle$$

completing the proof of assertion (c).

(d) Let $\{u_n\} \subset D(L)$ with $u_n \rightharpoonup u$ in $\mathcal{V}$ and $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$. As in the previous case, $D^\alpha u_n \to D^\alpha u$ in $L^p(Q)$ for all $|\alpha| \leq m-1$, for a subsequence. Since $\{A_\alpha(\cdot,\cdot,u_n,\dots,D^m u_n)\}$ remains bounded in $L^{p'}(\Omega)$ for all $|\alpha| \leq m-1$, we have

$$\lim \langle S_2(u_n), u_n - u\rangle = \lim \int_Q \sum_{|\alpha|\leq m-1} A_\alpha(x,t,u_n,\dots,D^m u_n)(D^\alpha u_n - D^\alpha u) = 0$$

and the assertion follows.

(e) Assume again that $\{u_n\} \subset D(L)$ with $u_n \rightharpoonup u$ in $\mathcal{V}$ and $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$. It suffices to show that $S_3(u_n) \to S_3(u)$ in $\mathcal{V}^*$ for a subsequence. By the same argument as in (c) $A_\alpha(\cdot,\cdot,u_n,\dots,D^{m-1}u_n) \to A_\alpha(\cdot,\cdot,u,\dots,D^{m-1}u)$ in $L^{p'}(Q)$. Hence

$$\|S_3(u_n) - S_3(u)\|$$

$$= \sup_{\|v\|=1} \Big| \int_Q \sum_{|\alpha|\leq m-1} \{A_\alpha(x,t,u_n,\dots,D^{m-1}u_n) - A_\alpha(x,t,u,\dots,D^{m-1}u)\} D^\alpha v \Big|$$

$$\leq \sum_{|\alpha|\leq m-1} \|A_\alpha(\cdot,\cdot,u_n,\dots,D^{m-1}u_n) - A_\alpha(\cdot,\cdot,u,\dots,D^{m-1}u)\|_{L^{p'}(Q)}$$

$$\to 0 \text{ as } n \to \infty.$$

□

We are now in the position to close our paper by results on the existence of weak solutions for the initial-boundary value problem (6.5). We can use the Proposition 1, Theorem 4 and the equivalence of (6.5) with the equation (6.6). Bearing in mind that the condition (A_3) on the operator A implies that the mapping $L+S$ satisfies the conditions (i) and (ii) of Theorem 4, we obtain

THEOREM 5. *Let Ω be a bounded open subset in $\mathbb{R}^N$ and Q the cylinder $\Omega \times [0,T]$. Then the following assertions hold*

(a) *If A satisfies (A_1), $(A_2)_S$ and (A_3), then the equation*

$$Lu + S(u) = h$$

admits a solution $u \in D(L)$ for any given $h \in \mathcal{V}^*$.

(b) *If $A^{(1)}$ satisfies (A_1) and $(A_2)_W$, $A^{(3)}$ satisfies (A_1) and $A^{(1)} + A^{(3)}$ satisfies (A_3), then the equation*

$$Lu + S_1(u) + S_3(u) = h$$

admits a solution $u \in D(L)$ for any given $h \in \mathcal{V}^*$.

(c) *If $A^{(1)}$ satisfies (A_1) and $(A_2)_W$, $A^{(2)}$ satisfies (A_1) and $A^{(1)} + A^{(2)}$ satisfies (A_3), then the equation*

$$Lu + S_1(u) + S_2(u) = h$$

is almost solvable in the sense that $(L + S_1 + S_2)(D(L))$ is dense in $\mathcal{V}^*$.

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