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A geometrical stability for T-minimum solutions of Hamiltonian systems

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RIASSUNTO – *In questo lavoro si provano alcuni risultati di stabilità per traiettorie associate ad una classe di soluzioni periodiche di certi sistemi Hamiltoniani autonomi, rispetto ad alcune perturbazioni delle superfici di energia fissata che le contengono. Si ottengono alcune opportune estensioni anche al caso nonautonomo.*

ABSTRACT – *In this paper one proves some stability results for trajectories associated with a class of periodic solutions of autonomous Hamiltonian systems, with respect to some perturbations of fixed energy surfaces containing them. One obtains some suitable extensions to the nonautonomous case too.*

KEY WORDS – *Minimum solutions - Hamiltonian systems - Stability.*

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– Introduzione

In [3] some stability results concerning a class of periodic solutions of autonomous Hamiltonian systems with a fixed minimal period T, the so-called T-minimum solutions, were stated, by taking into account the minimality property of the critical points associated with the solutions via the well known duality result by CLARKE and EKELAND [1]. Also, some definitions and results related to the stability with respect to some families of "fixed energy" surfaces were given. In [4] some extensions to the nonautonomous case were obtained.

Indeed it seems that the case concerning the "energy surfaces" was not completely investigated, since the results are based on a connection between "solution" and "trajectory lying on a fixed surface", which is not very meaningful in the sense that it doesn't explain enough the meaning of the "geometrical stability" of the Hamiltonian trajectories related to the T-minimum solutions.

The aim of this paper is twofold. From one side, we wish to put in a more geometrical formulation the stability results concerning the "fixed energy surfaces" in the sense that we prove them for some closed Hamiltonian trajectories which ly just on "converging" energy surfaces. The definitions and statements are based on two general facts: firstly, the Hamiltonian trajectories lying on a fixed level $\{H(z) = c > 0\}$ (where H is the Hamiltonian function) are homothetic to the trajectories of any other level $\{H(z) = c' > 0\}$, in case that H is positively homogeneous; secondly the Hamiltonian trajectories on a fixed suitably "smooth" surface S don't depend on the Hamiltonian function representing S, in the sense that they are the same unless of a suitable time - reparametrization.

The other type of results refers to the nonautonomous case and concerns the possibility of considering in this case too some stability properties related to the Hamiltonian surfaces, which are connected to the T-minimum solutions, but depend on the time (in the nonautonomous case, one cannot say that the energy of a solution is a time-constant).

The main result yields a suggestive geometrical meaning, in terms of one - parameter families of surfaces, of the fact, stated in [4], that, for a suitable class of nonautonomous Hamiltonian systems, the T-minimum solutions "converge" to a T-minimum solution of an autonomous system.

1 – The autonomous case

Let us consider the following Hamiltonian system

$$(H) \qquad J\dot{z} = H'(z)$$

where $J(x, y) = (y, -x)$ $\forall(x, y) \in \mathbb{R}^{2N}$ and H belongs to the class

$$\mathfrak{H}_1 = \{H \in C^1(\mathbb{R}^{2N}, \mathbb{R}) | H \quad \text{is a convex function such that}$$

$$(H_1) \quad a_1|z|^\alpha \leq H(z) \quad \forall z \in \mathbb{R}^{2N}$$
$$(H_2) \quad H(z) \leq a_2|z|^\alpha \quad \forall z \in \mathbb{R}^{2N}$$
$$(H_3) \quad H'(z) \cdot z \leq \alpha H(z) \quad \forall z \in \mathbb{R}^{2N}$$

for some $a_1, a_2 > 0$ and $\alpha \in (1,2)\}$.

Let F^* be the functional defined on the space

$$L_0^\beta = \left\{ v \in L^\beta(0,T;\mathbb{R}^{2N}) | \int_0^T v(t)dt = 0 \right\}, \quad \text{with } \beta = \frac{\alpha}{\alpha - 1}, \ T > 0$$

as

$$F^*(v) = \int_0^T G(v) - \frac{1}{2}\int_0^T L^{-1}v \cdot v$$

where G is the Legendre transform of H, that is

$$G(v) = \sup \left\{ v \cdot z - H(z) | z \in \mathbb{R}^{2N} \right\}$$

and $L = J\dfrac{d}{dt}$ is defined from the space

$$H_{\neq}^{1,\beta} = \left\{ z \in H^{1,\beta}(0,T;\mathbb{R}^{2N}) \middle| \int_0^T z(t)dt = 0, \quad z(0) = z(T) \right\}$$

into L_0^β.

In [1] CLARKE and EKELAND proved, for any $T > 0$, the existence of a solution z of (H) with minimal period T, where $z = G'(u)$ and u is a minimizing point of F^* on L_0^β.

We call *T-minimum solution of* (H) any solution z constructed in such a way (see [3]).

Let us consider now the following class $\mathfrak{S}_1 = \{S | S$ is the boundary of a bounded, closed, convex subset B of $\mathbb{R}^{2N}$ with $0 \in B\}$.

Let be $S \in \mathfrak{S}_1$ and let us define

$$H_S(z) = \begin{cases} (\lambda(z))^\alpha & z \in \mathbb{R}^{2N} - \{0\} \\ 0 & z = 0 \end{cases}$$

where $\lambda(z)$ is the gauge function of S that is the unique positive number such that $z = \lambda(z)\bar{z}$ with $\bar{z} \in S$.

Obviously H_S is a positively homogeneous function of degree α and one has $S = \{v \in \mathbb{R} | H_S(v) = 1\}$.

Let us consider the following Hamiltonian system

$$(H_S) \qquad J\dot{z} = H'_S(z)$$

In [3] the following definition was given

DEFINITION 1. *We say that z is a T-minimum solution with respect to S if it is a T-minimum solution of (H_S).*

Let us recall the definition of Γ-convergence (see e.g. [2]).

DEFINITION 2. *Let $\{S_n\}$ be a sequence of subset in $\mathbb{R}^{2N}$. We say that $\{S_n\}$ Γ-converges to S_0 ($S_0 = \Gamma - \lim_{n\to\infty} S_n$) if*

$$1) \quad \forall v_0 \in S_0 \quad \exists v_n \in S_n \mid v_0 = \lim_{n\to\infty} v_n$$
$$2) \quad \forall v_{n_j} \in S_{n_j} \mid \lim_{n\to\infty} v_{n_j} = v_0 \implies v_0 \in S_0$$

The following proposition was proved in [3]

PROPOSITION 1. *Let $S_0, S_n \in \mathfrak{S}_1$ $\forall n \in \mathbb{N}$, $S_0 = \Gamma - \lim_{n\to\infty} S_n$. Then any sequence $\{z_n\}$, with z_n T-minimum solution with respect to S_n admits a subsequence converging in $H^{1,\beta}_{\neq}$ to a T-minimum solution with respect to S_0.*

REMARK 1. One obtains, as an obvious corollary of Proposition 1, that, if $\{z_n\}$ is a sequence of T-minimum solutions with rspect to S_n converging in $H^{1,\beta}_{\neq}$, then its limit is a T-minimum solution with respect to S_0. Of course, this type of remark can be done also for the following convergence results, which will have always a similar "structure" as the statement of Proposition 1.

Let us observe that the trajectory associated with a T-minimum solution with respect to S might not belong to S. Anyway it is obvious that, any solution - trajectory $\overline{z}$ can be carried upon S by a (unique) homothety, that is there exists a unique $r > 0$ such that

$$(1) \qquad z = r\overline{z}$$

belongs to S. From (1) and from the homogeneity property of H_S one has

$$1 = H_S(r\overline{z}) = r^{\alpha} H_S(\overline{z})$$

then

$$r = \left(\frac{1}{H_S(\overline{z})}\right)^{1/\alpha}$$

Therefore it is natural to give the following.

DEFINITION 3. *We say that a trajectory z in S is a T- minimum trajectory with respect to S if there exists a T-minimum solution $\overline{z}$ of (H_S) such that z is homothetic to $\overline{z}$, that is such that*

$$z = \left(\frac{1}{H_S(\overline{z})}\right)^{1/\alpha} \cdot \overline{z}$$

REMARK 2. We point out that the T-minimum trajectories with respect to S are solutions of the system

$$(H_S) \qquad J\dot{z} = r^{2-\alpha} H_S'(z) = \widetilde{H}_S'(z)$$

where $\widetilde{H}_S = r^{2-\alpha} H_S$.

Moreover the surface $S = \{v \in \mathbb{R}^{2N} | H_S(v) = 1\}$ can be represented also as $S = \{v \in \mathbb{R}^{2N} | \widetilde{H}_S(v) = r^{2-\alpha}\}$.

Also for the T-minimum trajectories w.r. to S it is possible to state a stability result given by the following

THEOREM 1. *Let $S_0, S_n \in \mathfrak{S}_1$ $\forall n \in \mathbb{N}$, $S_0 = \Gamma - \lim_{n \to \infty} S_n$.*

Then any sequence $\{z_n\}$ with z_n T-minimum trajectory w.r. to S_n admits a subsequence converging in $H^{1,\beta}_{\neq}$ to a T-minimum trajectory w.r. to S_0.

PROOF. From proposition 1, it follows that there exists a subsequence $\overline{z}_{n_j}$ converging in $H^{1,\beta}_{\neq}$ to $\overline{z}_0$. From Lemma 1 of [3] one has that $z_{n_j} = \left(1/H_{S_{n_j}}\left(\overline{z}_{n_j}\right)\right)^{1/\alpha} \overline{z}_{n_j}$ converges in $H^{1,\beta}_{\neq}$ to $z_0 = (1/H(\overline{z}))^{1/\alpha}\, \overline{z}_0$ which is a T-minimum trajectory w.r. to H_{S_0}, since $\overline{z}_0$ is a T-minimum solution of (H_{S_0}) as follows from Proposition 1. $\square$

Let us consider now the following class

$$\mathfrak{H}_2 = \{H \in C^2(\mathbb{R}^{2N};\mathbb{R}) | H \quad \text{is a convex function}, \\ H(z) \geq 0 \quad \forall z \in \mathbb{R}^{2N}, H(z) = 0 \leftrightarrow z = 0\}$$

and, for any $b > 0$, let us consider the surface

$$S_b = \{v \in \mathbb{R}^{2N} | H(v) = b\} \quad \text{where} \quad H \in \mathfrak{H}_2.$$

In [3] the following definition was introduced

DEFINITION 4. *We say that z is a T-minimum solution w.r. to (H, b) if it is a T-minimum solution w.r. to S_b, that is if it is a T-minimum solution of*

$$(H_{S_b}) \qquad J\dot{z} = H'_{S_b}(z)$$

and the following proposition was proved

PROPOSITION 2. *Let $H \in \mathfrak{H}_2, b_0, b_n > 0$ and $\lim_{n\to\infty} b_n = b_0$. Then any sequence $\{z_n\}$ with z_n T-minimum solution w.r. to (H, b_n) admits a subsequence converging in $H^{1,\beta}_{\neq}$ to a T-minimum solution w.r. to (H, b_0).*

In this case too, a solution - trajectory of (H_{S_b}) might not belong to S_b but still, in a similar way as for the previous case, one can carry it upon S_b by a suitable homothety.

Let us observe that, if there exist two different functions H_1 and H_2 in $\mathfrak{H}_2$ such that

$$S = \{v \in \mathbb{R}^{2N} | H_1(v) = c_1\} = \{v \in \mathbb{R}^{2N} | H_2(v) = c_2\}$$

then the closed Hamiltonian trajectories on S related to the system (H_1) coincide, in a geometrical sense, with those related to (H_2). Indeed there exists a continous function λ on $\mathbb{R}^{2N}$ such that, for any $v \in S$, one has $H'_1(v) = \lambda(v)H'_2(v)$. From this relation it easily follows that, if z_1 is a solution of (H_1) and its trajectory belongs to S, then $z_2(t) = z_1(\gamma(t))$

is a solution of (H_2), where $\gamma(t)$ is the maximal solution of the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = \lambda(z_1(\gamma(t))) \\ \gamma(0) = 0 \end{cases}$$

Obviously it is defined one the whole real line, thanks to the boundedness of $(\lambda \cdot z_1)$.

Now, the surface

$$S_b = \{v \in \mathbb{R}^{2N} | H(v) = b\}$$

can also be represented as

$$S_b = \{v \in \mathbb{R}^{2N} | H_{S_b}(v) = 1\}.$$

Let z be a solution of (H) with $H(z) = b$. The previous argument implies that $z(\gamma_b(t))$ is a solution of (H_{S_b}) where $\gamma_b(t)$ is the solution of the problem

$$\begin{cases} \dot{\gamma}_b(t) = \lambda_b(z(\gamma_b(t))) \\ \gamma_b(0) = 0 \end{cases} \tag{2}$$

and λ_b is the continuous function defined on S_b such that, for any $v \in S_b$,

$$H'(v)\lambda_b(v) = H'_{S_b}(v) \tag{3}$$

that is

$$\lambda_b(v) = \frac{H'_{S_b}(v) \cdot v}{H'(v) \cdot v} = \frac{\alpha H_{S_b}(v)}{H'(v) \cdot v} = \frac{\alpha}{H'(v) \cdot v} \tag{4}$$

This argument suggests the following

Definition 5. *Let z be a solution of (H) with $H(z) = b$. We say that z is a T-minimum trajectory w.r. to (H, b) if $z(\gamma_b(t))$ is a T-minimum solution of (H_{S_b}), where γ_b is the solution of (2) and λ_b is given by (4).*

Also for the T-minimum trajectories w.r. to the energy levels one can state a stability result given by the following

THEOREM 2. *Let* $b_0, b_n > 0$ $\forall n \in \mathbb{N}$ *and* $\lim_{n\to\infty} b_n = b_0$. *Then any sequence* $\{z_n\}$ *with* z_n *T-minimum trajectory w.r. to* (H, b_n) *admits a subsequence converging in* $H^{1,\beta}_{\neq}$ *to a T-minimum trajectory w.r. to* (H, b_0).

PROOF. First of all let us prove that $\{\gamma_{b_n}\}$ uniformly converges to $\{\gamma_{b_0}\}$ where, for any $n \in \mathbb{N}$, $\gamma_{b_n}(t)$ is the maximal solution of the problem

$$\begin{cases} \dot\gamma_{b_n}(t) = \lambda_n(z_n(\gamma_{b_n}(t))) = F_n(\gamma_{b_n}(t)) \\ \gamma_{b_n}(0) = 0 \end{cases} \tag{5}$$

λ_n is the continuous function defined on S_{b_n} such that $\forall v \in S_{b_n}$

$$H'(v)\lambda_n(v) = H'_{S_b}(v) \tag{6}$$

and $F_n(\cdot) = (\lambda_n \cdot z_n)(\cdot)$. Let us observe that $\forall n \in \mathbb{N}$

$$\lambda_n(v) = \frac{\alpha}{H'(v)v} \tag{7}$$

By exploiting a well known result about the continuous dependence of solutions of Cauchy problems on the datas, it is enough to prove the following statements:

$$F_n \quad \text{uniformly converges to} \quad F_0 \tag{8}$$

(9) For any compact set $I \subset \mathbb{R}$ there exist some $c = c(I)$

such that $|F_n(t_1) - F_n(t_2)| \leq c|t_1 - t_2|$ $\forall t_1, t_2 \in I$ $\forall n \in \mathbb{N}$

$$F_n \quad \text{are equibounded} \tag{10}$$

First of all, (8) follows from (7) and from the assumption that $\{z_n\}$ converges to z_0 in $H^{1,\beta}_{\neq}$.

Let us prove (9). From (8) it follows that

$$|F_n(t_1) - F_n(t_2)| = |(\lambda_n \cdot z_n)(t_1) - (\lambda_n \cdot z_n)(t_2)| = \\ = \alpha \left| \frac{1}{H'(z_n(t_1)) \cdot z_n(t_1)} - \frac{1}{H'(z_n(t_2)) \cdot z_n(t_2)} \right|$$

For any $n \in \mathbb{N}$ let us consider the function

$$g_n(t) = \frac{1}{H'(z_n(t)) \cdot z_n(t)} \quad t \in [0, T]$$

Taking into account that z_n is a solution of (H) and that $v \cdot Jv = 0$ $\forall v \in \mathbb{R}^{2N}$, one has that

$$g_n'(t) = \frac{H''(z_n(t))JH'(z_n(t)) \cdot z_n(t)}{(H'(z_n(t)) \cdot z_n(t))^2} \tag{11}$$

Since $H(z_n) = b_n$ and $\{b_n\}$ converges, then (H_1) implies

$$\|z_n\|_\infty \leq \text{const} \tag{12}$$

On the other side, $b_0 = \lim_{n\to\infty} b_n > 0$ so (H_2) implies

$$\|z_n\|_\infty \geq \text{const} > 0 \tag{13}$$

Then (11), (12), (13) and the continuity of H'', H' yield

$$|g_n'(t)| \leq \text{const} \quad \forall n \in \mathbb{N}, t \in [0, T] \tag{14}$$

so (9) follows from Lagrange theorem.

Finally (10) derives from (8), (13) and the continuity of H'. Therefore (8), (9), (10) hold and $\{\gamma_{b_n}\}$ uniformly converges to γ_{b_0}.

At this point one observes that $\{z_n\}$ admits a subsequence $\{z_{n_j}\}$ converging in $H^{1,\beta}_{\neq}$, since (12) holds and also

$$\|\dot{z}\|_\infty \leq \text{const} \tag{15}$$

and z_n is a solution of (H_n).

Let $z_0 = \lim_{n\to\infty} z_{n_j}$ in $H^{1,\beta}_{\neq}$. Obviously z_0 is a solution of (H), let us prove that z_0 is in fact a T-minimum trajectory that is $(z_0 \cdot \gamma_{b_0})$ is a T-minimum solution of $(H_{S_{b_0}})$.

Indeed, unless of another subsequence, from one side the subsequence $\overline{z}_{n_j}(t) = z_{n_j}(\gamma_{b_n}(t))$ converges in $H^{1,\beta}_{\neq}$ to a T-minimum solution $\overline{z}_0$ of (H_{S_b}), thanks to Proposition 1, on the other side the uniform convergence of $\{\gamma_{b_n}\}$ to γ_{b_0} yields

$$\overline{z}_0 = z_0(\gamma_{b_0}(t))$$

and the thesis follows. □

2 – The nonautonomous case

Let H be a function defined from $\mathbb{R}_+ * \mathbb{R}^{2N}$ into $\mathbb{R}$ such that

$$H(\cdot, z) \in C^0(\mathbb{R}_+) \qquad \forall z \in \mathbb{R}^{2N} \tag{16}$$

$$H(t, \cdot) \in C^1(\mathbb{R}^{2N}) \qquad \forall t \in \mathbb{R}_+ \tag{17}$$

$$H(t, \cdot) \quad \text{is strictly convex on} \quad \mathbb{R}^{2N} \qquad \forall t \in \mathbb{R}_+ \tag{18}$$

$$\begin{array}{l} H(\cdot, z) \quad \text{is periodic on} \quad \mathbb{R}_+, \quad \text{with minimal period} \\ T \quad \text{for any} \quad z \in \mathbb{R}^{2N} \end{array} \tag{19}$$

$$\begin{array}{l} a_1|z|^\alpha \leq H(t,z) \leq a_2|z|^\alpha \quad \forall t \in \mathbb{R}_+, \quad \forall z \in \mathbb{R}^{2N} \\ \text{with} \quad \alpha \in (1,2), \quad a_1, a_2 > 0 \end{array} \tag{20}$$

Let us consider, for any $n \in \mathbb{N}$, the following Hamiltonian system

$$(H_n) \qquad \begin{cases} J\dot{z}(t) = H'(nt, z(t)) \\ z \quad \text{is } T\text{-periodic with minimal period} \quad T > 0 \end{cases}$$

It is easy to check that, as in the autonomous case, (H_n) admits a solution z_n which can be obtained through the duality principle by Clarke and

Ekeland. More precisely it is possible to show that the functional $F_n^*(v)$ defined on L_0^β as

$$F_n^*(v) \int_0^T G(nt, v(t)) - \frac{1}{2}\int_0^T L^{-1}v(t)\cdot v(t)$$

(where G is the Legendre transform of H in the z-variable that is

$$G(t,v) = \sup\{v\cdot z - H(t,z) | z \in \mathbb{R}^{2N}\} \quad \forall (t,v) \in \mathbb{R}_+ \times \mathbb{R}^{2N})$$

admits a negative minimum on L_0^β and, if u_n is a minimum point then $z_n = G'(nt, u_n(t))$ is a solution of (H_n). We call *T-minimum solution of* (H_n) any solution obtained in such a way (see [4]).

Let us recall the following result for T-minimum solution of (H_n), proved in [4]

PROPOSITION 3. *Let* $H(t,z) = \overline{H}(z)\varphi(t)$ *where* $\overline{H} \in C^1(\mathbb{R}^{2N})$, $\varphi \in C^0(\mathbb{R}_+)$ *and*

(21) $$\overline{H} \quad \textit{is strictly convex on} \quad \mathbb{R}^{2N}$$

(22) $$a_1|z|^\alpha \le \overline{H}(z) \le a_2|z|^\alpha \quad \forall z \in \mathbb{R}^{2N}, a_1, a_2 > 0, \alpha \in (1,2)$$

(23) $$\varphi \quad \textit{is periodic on} \quad \mathbb{R}_+, \quad \textit{with minimal period } T$$

(24) $$\textit{there exists} \quad c > 0 \quad \textit{such that} \quad \varphi(t) \ge c \quad \forall t \in \mathbb{R}_+$$

Let us consider the following Hamiltonian system

$$(\overline{H}) \qquad J\dot{z} = \overline{H}'(z)\varphi_0$$

where $\varphi_0 = \frac{1}{T}\int_0^T \varphi(t)dt$. *Then, putting* $\{H_n(t,z)\} = \{H(nt,z)\} = \{\overline{H}(z)\varphi(nt)\}$, *one has that any sequence* $\{z_n\}$ *with* z_n *T-minimum solution of* (H_n) *admits a subsequence converging in* $H_{\neq}^{1,\beta}$ *to a T-minimum solution of* $(\overline{H})$.

Let us consider now, for a fixed $T > 0$, the following class $\mathfrak{S}_2 = \{S\colon \mathbb{R}_+ \longrightarrow \mathbb{R}^{2N} | \quad \forall t \geq 0, S(t)$ is the boundary of a bounded, closed, convex subset B of $\mathbb{R}^{2N}$ with $0 \in B$ and T is the minimum positive number such that $S(t+T) = S(t) \quad \forall t \geq 0\}$

Let us show how it is possible to define a concept of T-minimum solution related to some suitable elements S in the class $\mathfrak{S}_2$ and to give a related convergence result, by exploiting Proposition 3. Indeed, let $S \in \mathfrak{S}_2$ and define the gauge function $\lambda(z,t)$ of $S(t)$, for any $t \in \mathbb{R}_+$ and $z \in \mathbb{R}^{2N}$ as the unique number such that $z = \lambda(z,t)z_t$ with $z_t \in S(t)$ if $z \neq 0, \lambda(0,t) = 0$. Then, put for an arbitrarily fixed number $\alpha \in (1,2)$,

$$H_S(t,z) = H_{S(t)}(z) = (\lambda(z,t))^\alpha \qquad \forall t \in \mathbb{R}_+, z \in \mathbb{R}^{2N} \tag{25}$$

one can state the following

LEMMA 1. *Let $S \in \mathfrak{S}_2$. Then the corresponding H_S defined as in* (25) *can be written as*

$$H_S(t,z) = \overline{H}(z)\varphi(t) \tag{26}$$

with $\overline{H} \in C^1(\mathbb{R}^{2N})$ homogeneous of degree α and satisfiyng (21) *and $\varphi \in C^0(\mathbb{R}_+)$ satisfiying* (23), (24), *if and only if there exists a continuous function ψ on $\mathbb{R}_+$, satisfiyng the assumptions*

$$\psi \quad \text{is periodic on } \mathbb{R}_+ \text{ with minimal period} \quad T \tag{27}$$

$$\exists c' > 0 \quad s.t. \quad \psi(t) \geq c' \quad \forall t \in \mathbb{R}_+ \tag{28}$$

such that

$$S(t) = \{v \in \mathbb{R}^{2N} | \overline{H}(v) = \psi(t)\}. \tag{29}$$

Moreover $\psi(t) = (\varphi(t))^{-1}$.

PROOF. Let H_S be represented by (26). Then obviously $S(t)$ can be written as in (29), with $\psi(t) = (\varphi(t))^{-1}$ as, by definition, $S(t) = \{v \in \mathbb{R}^{2N} | H_S(t,v) = 1\}$.

On the contrary, let there exist a pair of functions $\overline{H}$, φ satisfiyng (21), (23), (24) and (22) with the further α-homogeneity assumption, in such a way that (29) holds. It is enough to prove that the function

$$\lambda(t,z) = \left(\frac{\overline{H}(z)}{\psi(t)}\right)^{1/\alpha} \qquad z \in \mathbb{R}^{2N}, \quad t \geq 0$$

is the gauge function of $S(t)$, or, equivalently, the relation

$$z(t)\left(\frac{\psi(t)}{\overline{H}(z)}\right)^{1/\alpha} \qquad \in S(t). \tag{30}$$

Indeed (30) is an obvious consequence of the α-homogeneity of $\overline{H}$ and (29). So the thesis follows. □

By virtue of Lemma 1, it makes sense the following

DEFINITION 6. *Let $\psi \in C^0(\mathbb{R}_+)$ satisfiyng* (27), (28) *and let $\overline{H}$ satisfy* (21) *and* (22) *with the further α-homogeneity assumption. Let us take $S \in \mathfrak{S}_2$ of the form given by* (29). *From now on we will say that S is of the type $(\overline{H}, \psi)$. Then one says that z is a T-minimum solution w.r. to S if z is a T-minimum solution of the system*

$$(H_S) \qquad J\dot{z} = H'_S(t,z)\,.$$

Obviously one can also consider,for any $n \in \mathbb{N}$, the element $S_n \in \mathfrak{S}_2$ defined by

$$S_n(t) = S(nt) = \{v \in \mathbb{R}^{2N} | \overline{H}(v) = \psi(nt)\} \tag{31}$$

and define a T-minimum solution z w.r. to S_n as a T-minimum solution of the system

$$(H_{S_n}) \qquad J\dot{z} = H'_{S_n}(t,z)$$

REMARK 3. Let us recall the previous Definition 4. In case that the function H appearing in the definition of the surface S_b is homogeneous of degree α, the Definition 6 is an obvious generalization of Definition 4, where the function $\psi(t)$ is simply the constant number b.

By taking into account Definition 6, at this point the following result can be stated as an easy consequence of Proposition 3 and Lemma 1.

THEOREM 3. *Let $S \in \mathfrak{S}_2$ be of the type $(\overline{H}, \psi)$ and let $S_n(t)$ be as in* (31). *Let $\psi_0 = (\varphi_0)^{-1} = \left(\frac{1}{T}\int_0^T \varphi(t)dt\right)^{-1}$ and let $S_0 = \left\{v \in \mathbb{R}^{2N} | \overline{H}(v) = \psi_0\right\}$. Then any sequence $\{z_n\}$ with z_n T-minimum solution w.r. to $S_n(t)$ admits a subsequence converging in $H^{1,\beta}_{\neq}$ to a T-minimum solution w.r. to S_0.*

As in the autonomous case, also in the non-autonomous one, it is possible to give a suitable definition of T-minimum trajectory and a related convergence result. Indeed, let $S \in \mathfrak{S}_2$ be of the type $(\overline{H}, \psi)$, then it follows that, for any $\overline{z}$: $\mathbb{R}_+ \longrightarrow \mathbb{R}$, with $\overline{z}(t) \neq 0 \quad \forall t \geq 0$, the trajectory

$$z(t) = \left(\frac{\psi(t)}{\overline{H}(\overline{z}(t))}\right)^{1/\alpha} \overline{z}(t) \qquad t \geq 0$$

verifies the relation

$$z(t) \in S(t) \qquad \forall t \geq 0$$

One can, in particular, choose $\overline{z}$ as a T-minimum solution w.r. to S: indeed $\overline{z}(t) \neq 0 \quad \forall t \geq 0$, as a consequence of the more general result

LEMMA 2. *Let $H(t,z)$ satisfy assumptions* (16), ..., (20) *and let us consider the Hamiltonian system (H_n) for any $n \in \mathbb{N}$. Then there exists a constant number $d \geq 0$ such that, for any T-minimum solution of (H_n), one has*

$$\|z\|_{L^\alpha} \geq d, \qquad \|\dot{z}\|_{L^\alpha} \geq d \tag{32}$$

PROOF. Let z be a T-minimum solution of (H_n) and let $v = Lz$ the corresponding minimum point of the associated functional F_n^*. By the properties of the Legendre transform, one has

$$\int_0^T H(nt, z) = -F_n^*(v) + \frac{1}{2}\int_0^T L^{-1}v \cdot v \tag{33}$$

On the other side

$$L^{-1}v \cdot v > 0 \tag{34}$$

as $L^{-1}v = H'(nt, z)$ and, by convexity of H with $H(0) = 0$,

$$H'(nt, z) \cdot z \geq H(nt, z) > 0 \tag{35}$$

Therefore (33), (34), (35) imply

$$\int_0^T H(nt, z) > -F_n^*(v) \tag{36}$$

Nowe let $\overline{v}$ be an eigenvector of L^{-1} of the type

$$\overline{v}(t) = \left(a\sin\left(\frac{2\pi}{T}t\right) + b\cos\left(\frac{2\pi}{T}t\right); a\cos\left(\frac{2\pi}{T}t\right) - b\sin\left(\frac{2\pi}{T}t\right)\right)$$

$a, b \in \mathbb{R}^N$. By using the first relation in (20) and the continuity of L^{-1}, one has

$$F_n^*(\overline{v}) \leq c_1\left(|a|^2 + |b|^2\right)^{\beta/2} - c_2\left(|a|^2 + |b|^2\right) \tag{37}$$

for suitable constant number $c_1, c_2 > 0$, independent of n. As $\beta > 2$, it is easy to see that the function $g(a, b) = c_1(|a|^2 + |b|^2)^{\beta/2} - c_2(|a|^2 + |b|^2)$ has a negative minimum, say M, which is independent of n so (37) yields

$$F_n^*(\overline{v}) \leq m < 0 \qquad \forall n \in \mathbb{N} \tag{38}$$

Then the minimality property of v, (36) and (38) imply

$$\int_0^T H(nt, z) > -m > 0 \qquad \forall n \in \mathbb{N} \tag{39}$$

At this point the first relation in (20) and (39) yield

$$\|z\|^{\alpha}_{L^\alpha} > -m/_{a_1} > 0$$

and the first relation in (32) follows.

On the other hand, since z is a solution of (H_n), by the Hölder inequality, (35) and the first relation of (20), one obtains

$$\int_0^T |\dot{z}|^\beta = \int_0^T |H'(z)|^\beta \geq \int_0^T \left(\frac{H(z)}{|z|}\right)^\beta \geq a_1 \int_0^T (|z|)^{\beta(\alpha-1)} = a_1 \int_0^T |z|^\alpha$$

so the second relation in (32) is a consequence of the first one.

REMARK 4. One has to remark that, if $\overline{z}$ is a solution of (H_S) with S of the type $(\overline{H}, \psi)$, (not necessarily a T-minimum solution), then $\overline{H}(\overline{z}(t))$ is a constant number for any $t \geq 0$, so one shall put $\overline{H}(\overline{z}(t) = h_{\overline{z}}$ $\forall t \geq 0$. Indeed, by observing that $\nabla\overline{H}(\overline{z}(t)) \cdot \dot{\overline{z}}(t) = 0$ one has, putting $\varphi(t) = (\psi(t))^{-1}$:

$$\frac{d}{dt}H_S(t, \overline{z}(t)) = \frac{d}{dt}\left(\overline{H}(\overline{z}(t))\varphi(t)\right) = \varphi'(t)\overline{H}(\overline{z}(t))$$

then

$$\frac{\frac{d}{dt}\left(\overline{H}(\overline{z}(t))\varphi(t)\right)}{\overline{H}(\overline{z}(t))\varphi(t)} = \frac{\varphi'(t)}{\varphi(t)}$$

therefore

$$\frac{d}{dt}\lg\left(\overline{H}(\overline{z}(t))\varphi(t)\right) = \frac{d}{dt}\lg(\varphi(t)),$$

so

$$\lg\left(\overline{H}(\overline{z}(t))\varphi(t)\right) = \lg(\varphi(t)) + \text{constant}$$

which implies $\overline{H}\ (\overline{z}(t)) = h_{\overline{z}}$, where $h_{\overline{z}}$ only depends on $\overline{z}$ (and not on t).

Really this observation will be not essential for the next definition and theorem, but we have preferred to pointwise it for completeness.

At this point one can give the following

DEFINITION 7. *Let $S \in \mathfrak{S}_2$ of the type $(\overline{H}, \psi)$ and let $S_n(t) = S(nt)$ $\forall n \in \mathbb{N}$, $\forall t \geq 0$. Then one says that z is a T-minimum trajectory w.r.*

to S_n *if there exists a* T*-minimum solution* $\overline{z}$ *w.r.* S_n *such that*

$$z(t) = \left(\frac{\psi(nt)}{h_{\overline{z}}}\right)^{1/\alpha} \overline{z}(t) \qquad \forall t \geq 0$$

Obviously, in particular, $z(t)$ *belongs* $S_n(t)$ $\forall t \geq 0$.

REMARK 5. Definition 7 generalizes Definition 3 in case that $S = \{v \in \mathbb{R}^{2N} | \overline{H}(v) = b\}$ where b is a constant positive number. Obviously it is not a generalization of Definition 5. Indeed it doesn't make sense, in the nonautonomous case, develop an analogous argument as the one related to the autonomous case on which is based Definition 5.

Now let us give the convergence result related to T-minimum trajectories. In this nonautonomous case one can state that the "limit trajectory" does not belong to the "limit surface" $S_0 = \{v \in \mathbb{R}^{2N} | \overline{H}(v) = \psi_0\}$ where $\psi_0 = \left(\frac{1}{T}\int_0^T \varphi(t)dt\right)^{-1}$, but it is true unless of a suitable homothety. Precisely one can state the following result

THEOREM 4. *Let* $S \in \mathfrak{S}_2$ *be of the type* $(\overline{H}, \psi)$ *and let* $S_n(t) = S(nt)$ $\forall n \in \mathbb{N}$, $\forall t \geq 0$. *Let* $S_0 = \{v \in \mathbb{R}^{2N} | \overline{H}(v) = \psi_0\}$ *with* $\psi_0 = \left(\frac{1}{T}\int_0^T \varphi(t)dt\right)^{-1}$.

Then, any sequence $\{z_n\}$ *with* z_n T*-minimum trajectory w.r. to* $S_n(t)$ *admits a subsequence converging, in the weak* $*$*-topology of* $L^\infty(0, T; \mathbb{R}^{2N})$*, to an element* $\tilde{z}_0$ *which is homothetic to a* T*-minimum trajectory* z_0 *w.r. to* S_0*. More precisely*

$$\tilde{z}_0(t) = \frac{\psi_\alpha}{(\psi_0)^{1/\alpha}} z_0(t) \qquad \forall t \geq 0$$

where

$$\psi_\alpha = \frac{1}{T}\int_0^T (\psi(t))^{1/\alpha} dt$$

REMARK 6. In case that $\psi(t)$ is a constant number $b > 0$ for any $t \geq 0$, so $S_n(t)$ reduces to the unique surface $S = \{v \in \mathbb{R}^{2N} | \overline{H}(v) = b\}$ then $\psi_\alpha = (\psi_0)^{1/\alpha}$ so $z_0(t) = z(t)$ and Theorem 4 yields the compactness of the set of T-minimum trajectories w.r. to S. Indeed it can also be deduced from Theorem 2.

PROOF OF THEOREM 4. From Theorem 3 it follows that there exists a subsequence $\overline{z}_{n_j}$ converging in $H^{1,\beta}_{\neq}$ to $\overline{z}_0$ which is a T-minimum solution w.r. to S_0. Then $\overline{H}(\overline{z}_{nj}) = h_{\overline{z}_{n_j}}$ converges to $\overline{H}(\overline{z}_0) = h_{\overline{z}_0}$ so $\overline{z}_{n_j}/\left(h_{\overline{z}_{n_j}}\right)^{1/\alpha}$ converges to $\overline{z}_0/(h_{\overline{z}_0})^{1/\alpha}$ in $H^{1,\beta}_{\neq}$.

Since $\psi(n_j t)$ converges to ψ_α in the weak *-topology of $L^\infty(0,T;\mathbb{R}^{2N})$, then

$$z_n = \frac{\psi(n_j t)}{\left(h_{\overline{z}_{n_j}}\right)^{1/\alpha}} \overline{z}_{n_j} \quad \text{converges to}$$

$$\tilde{z}_0 = \frac{\psi_\alpha}{(h_{\overline{z}_0})^{1/\alpha}}$$

in the weak *-topology of $L^\infty(0,T;\mathbb{R}^{2N})$. Then $\tilde{z}_0$ is homothetic to the T-minimum trajectory

$$z_0 = \left(\frac{\psi_0}{h_{\overline{z}_0}}\right)^{1/\alpha} \overline{z}_0 \quad \text{since one obtains}$$

$$\tilde{z}_0 = \frac{\psi_\alpha}{(\psi_\alpha)^{1/\alpha}} z_0.$$

□

REMARK 7. By looking at the proof of Theorem 4, one realizes that the main reason for which the "limit-trajectory" of T-minimum trajectories is not of this type is essential the fact that the "mean operation" does not obviously commute with the "exponential operation".

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