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On certain differential inequalities

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RIASSUNTO – *Nel presente lavoro si danno alcune proprietà di funzioni analitiche che soddisfano a certe disuguaglianze differenziali nel cerchio unitario.*

ABSTRACT – *The object of the present paper is to derive some properties of analytic functions satisfying certain differential inequalities in the unit disk.*

KEY WORDS – *Analytic - Differential inequality - Subordination - Multivalent function.*

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1 – Introduction

Let A be the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \tag{1.1}$$

which are analytic in the unit disk $U = \{z\colon |z| < 1\}$. Let P denote the class of functions $p(z) \in A$ satisfying

$$\mathrm{Re}\{p(z)\} > 0 \quad (z \in U). \tag{1.2}$$

MILLER [2] has proved that if $p(z) \in A$ satisfies

$$\mathrm{Re}\{p(z) + zp'(z)\} > 0 \quad (z \in U), \tag{1.3}$$

then $p(z) \in P$.

Recently, NUNOKAWA [3] has improved the result by MILLER [2] as follows:
If $p(z) \in A$ satisfies the condition (1.3), then $\mathrm{Re}\{p(z)\} > \log(4/e) > 0$.

In the present paper, we give some differential inequalities for the class P.

2 – Differential inequalities

We begin with the statement of the following result which can be found in [1, Vol.I, Theorem 7].

LEMMA. *If $p(z) \in P$, then*

$$p(z) \prec \frac{1+z}{1-z} \quad (z \in U), \tag{2.1}$$

where the symbol $\prec$ means the subordination.

Using the above lemma, we derive

THEOREM 1. *If $p(z) \in A$ satisfies*

$$|\alpha p(z) + \beta z p'(z)| < (\alpha - \beta) \log \frac{4}{e} \quad (z \in U), \tag{2.2}$$

then $p(z) \in P$, where

$$\beta < \begin{cases} \dfrac{2\alpha(\log 2 - 1)}{2\log 2 - 1} \fallingdotseq -1.588\alpha & (\alpha \geq 0) \\ \dfrac{2\alpha \log 2}{2\log 2 - 1} \fallingdotseq 3.588\alpha & (\alpha \leq 0). \end{cases}$$

PROOF. Since

$$\alpha p(z) + \beta z p'(z) = (\alpha - \beta)p(z) + \beta(zp(z))', \tag{2.3}$$

our condition (2.2) implies that

$$\begin{aligned} \left| \int_0^z (\alpha - \beta)p(t)dt + \beta z p(z) \right| &= \left| \int_0^z (\alpha p(t) + \beta t p'(t))dt \right| \\ &< (\alpha - \beta)|z| \log \frac{4}{e}. \end{aligned} \tag{2.4}$$

Suppose that there exists a point $z_0 \in U$ with $|z_0| = R < 1$ such that $\mathrm{Re}\{p(z_0)\} = 0$ and $\mathrm{Re}\{p(z)\} > 0 (|z| < |z_0|)$. Then Lemma gives that

$$\frac{R - |z|}{R + |z|} \leq \mathrm{Re}\{p(z)\} \leq \frac{R + |z|}{R - |z|} \quad (|z| < |z_0| = R). \tag{2.5}$$

Therefore, we have

$$\begin{aligned} &\mathrm{Re}\left\{ \frac{1}{R} \int_0^R (\alpha - \beta) p(\rho e^{i\theta}) d\rho + \beta p(z_0) \right\} \\ &\geq \frac{\alpha - \beta}{R} \int_0^R \left(\frac{R - \rho}{R + \rho} \right) d\rho = (\alpha - \beta) \log \frac{4}{e}, \end{aligned} \tag{2.6}$$

which contradicts our condition of the theorem. Thus we obtain that $\mathrm{Re}\{p(z)\} > 0$ for all $z \in U$, that is, that $p(z) \in P$.

Taking $\alpha = 0$ in Theorem 1, we have

COROLLARY 1. *If $p(z) \in A$ satisfies*

$$|zp'(z)| < \log \frac{4}{e} \quad (z \in U), \tag{2.7}$$

then $p(z) \in P$.

Making $\alpha > 0$ and $\beta = -2\alpha$, Theorem 1 leads to

COROLLARY 2. *If* $p(z) \in A$ *satisfies*

$$|p(z) - 2zp'(z)| < 3\log\frac{4}{e} = 1.1589\ldots \quad (z \in U), \tag{2.8}$$

then $p(z) \in P$.

3 – An application for multivalent functions

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N = \{1, 2, 3, \ldots\}) \tag{3.1}$$

which are analytic in the unit disk U. Then we have

THEOREM 2. *If* $f(z) \in A(p)$ *satisfies*

$$\left|(\alpha - \beta p)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{z^{p-1}}\right| < (\alpha - \beta)\log\frac{4}{e} \quad (z \in U), \tag{3.2}$$

then $f(z)/z^p \in P$, *where* β *is given as in Theorem 1.*

PROOF. Letting $p(z) = f(z)/z^p$ in Theorem 1, we see that our condition (3.2) implies $f(z)/z^p \in P$.

If we put $\alpha = 0$ in Theorem 2, we have

COROLLARY 3. *If* $f(z) \in A(p)$ *satisfies*

$$\left|\frac{f(z)}{z^p} - \frac{f'(z)}{pz^{p-1}}\right| < \frac{1}{p}\log\frac{4}{e} \quad (z \in U), \tag{3.3}$$

then $f(z)/z^p \in P$.

Finally, letting $\alpha > 0$ and $\beta = -2\alpha$, we see

COROLLARY 4. *If $f(z) \in A(p)$ satisfies*

$$\left|\frac{f(z)}{z^p} - \frac{2}{1+2p}\frac{f'(z)}{z^{p-1}}\right| < \frac{3}{1+2p}\log\frac{4}{e} \quad (z \in U), \tag{3.4}$$

then $f(z)/z^p \in P$.

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