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Kantorovich majorants for nonlinear operators and applications to Uryson integral equations

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RIASSUNTO – *Si studiano maggiorazioni alla Kantorovich per operatori non lineari in spazi muniti di K-norme, ossia funzionali sublineari che assumono valori nel cono di elementi positivi di qualche spazio di Banach. La convergenza di successive approssimazioni per punti fissi del corrispondente maggiorante. I risultati astratti vengono illustrati tramite equazioni integrali non lineari di tipo Uryson, per le quali si possono calcolare i maggioranti di Kantorovich sotto forma esplicita.*

ABSTRACT – *Nonlinear Kantorovich majorants of nonlinear operators are studied in spaces equipped with a K-norm, i.e. a sublinear functional taking values in the cone of positive elements of some Banach space. Convergence of successive approximations for fixed points of such operators reduces then to convergence of successive approximations for fixed points of the corresponding majorant. The abstract results are illustrated by nonlinear integral equations of Uryson type, where one can represent the Kantorovich majorant quite explicitly.*

KEY WORDS. – *Kantorovich majorants - Nonlinear operators - Spaces with K-norms - Successive approximations - Fixed points - Uryson integral equations.*

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Already in the classical work of L.V. Kantorovich (see, e.g., [3]) on successive approximations of operator equations, the idea arose to use so-called K-*norms*, i.e. sublinear functionals taking values in the cone of positive elements of some ordered space.

This made it in fact possible to pass from the iterates of a given (usually, complicated) operator to those of another (usually, simple) operator in an ordered space, which is nowadays called *Kantorovich majorant.* In the first papers, Kantorovich majorants were throughout *linear*, and took values in a *finite-dimensional* space. Already in this simple case, many new interesting abstract theorems have been obtained; some results in this spirit may be found in the book [6]. In particular, the use of K-norms with values in some Euclidean space made it possible to strengthen various known existence theorems for integral equations or boundary value problems.

Essentially new results, however, may be obtained if one admits also K-norms in *infinite-dimensional* ordered Banach spaces. Here we mention the papers [1], [2], where the authors use the simple (though infinite-dimensional!) K-norm $]x[=|x(\cdot)|$ (see below).

This leads to completely new results, for instance, in the theory of nonlinear integral equations of Hammerstein type, and gives also a mathematically rigorous justification of A.M. Samojlenko's convergence method for periodic solutions to various types of non-autonomous differential equations.

In the present paper we apply the majorant method to nonlinear integral equations of Uryson type. In comparison to previous work, there are two essential new features. First, we consider K-norms with values in a large class of infinite-dimensional Banach spaces. Second, we admit nonlinear Kantorovich majorants as well. As a consequence, we obtain new existence and uniqueness theorems for Uryson equations in a rather elegant and straightforward way.

1 – K-normed spaces

Let B be a real Banach space which is ordered by some cone K. Given a real linear space X, a *K-norm* on X is a map $]\cdot[: X \to K$ which has the usual algebraic properties of a norm, i. e. $]x[=\theta$ iff $x=\theta$, $]\lambda x[=|\lambda|]x[$ for $\lambda \in \mathbb{R}$, and $]x+y[\leq]x[+]y[$ for x, $y \in X$. Of course, the case $B=\mathbb{R}$ and $K=[0,\infty)$ gives the usual definition of a norm. Given $x_0 \in X$ and $R \in K$, by $B(x_0,R)$ we shall denote throughout the *K-ball*

$$B(x_0,R)=\{x : x\in X,]x-x_0[\leq R\}. \tag{1.1}$$

Recall [3], [9] that a sequence $(x_n)_n$ in a K-normed space X is called *o-convergent* to $x_* \in X$ if there exists a monotonically decreasing sequence $(r_n)_n$ in K such that $\inf r_n = \theta$ and $]x_n - x_*[\leq r_n$; equivalently, this means that $\theta \leq z \leq]x_n - x_*[$ $(n = 1, 2, \dots)$ implies that $z = \theta$. Similarly, $(x_n)_n$ is called *o-fundamental* if $\theta \leq z \leq]x_m - x_n[$ $(m, n = 1, 2, \dots)$ implies that $z = \theta$. Denoting by $\vartheta(X)$ the set of all *o-fundamental* sequences in X, we call X *ϑ-sequentially complete* if every sequence $(x_n)_n \in \vartheta(X)$ is o-convergent. More generally, we define the *C-sequential completeness* of X for any subset C of $\vartheta(X)$. Examples will be given below. We say that an ordered Banach space B has the *W-property* (where W stands for Weierstrass) if every monotonically increasing sequence $(r_n)_n$ in $K \subset B$ which is bounded from above has a supremum.

2 – Kantorovich majorants

Let X be a K-normed space and A a (nonlinear) operator defined on some K-ball $B(x_0, R)$ and taking values in X. An operator $\Phi : K \to K$ is called *Kantorovich majorant* for A if

$$]A(x+h) - Ax[\leq \Phi(]x[+]h[) - \Phi(]x[) \tag{2.1}$$

for $]x - x_0[+]h[\leq R$. Of course, an operator A may have several Kantorovich majorants. It is an interesting and important problem to determine the minimal of all Kantorovich majorants, if there are any. Surprisingly, this is in fact possible in many cases. For the sake of simplicity, we take $x_0 = \theta$ in what follows.

The usefulness of Kantorovich majorants consists in the fact that, on the one hand, the operator Φ is often much easier to deal with than the operator A (for instance, Φ is always positive, by definition), and, on the other, many properties of Φ carry over to analogous properties of A (for instance, convergence of successive approximations, see Theorem 2 below).

Suppose that the Banach space B is a *K-space* in the sense of [3], [4], i.e. every bounded set in B has a supremum. For fixed $a, b \in K$ let

$$W(a,b) = \sup \sum_{j=1}^{m}]A(x_j + h_j) - Ax_j[, \tag{2.2}$$

where the sup is taken over all finite chains $\{r_0, r_1, \ldots, r_m\} \subset K$ such that $a = r_0 \leq r_1 \leq \ldots \leq r_{m-1} \leq r_m = b$, and $]h_j[\leq r_j - r_{j-1}$, $]x_j[\leq r_{j-1}$ $(j = 1, 2, \ldots, m)$. Obviously, the function (2.2) has the property that

$$W(a,b) + W(b,c) \leq W(a,c), \tag{2.3}$$

and equality holds in (2.3) only in the scalar case $B = \mathbb{R}$.

THEOREM 1. *Suppose that B is a K-space, and the function (2.2) is defined on the conic interval $< \theta, R >$. Then*

$$\Phi_A(z) =]A\theta[+ W(\theta, z) \tag{2.4}$$

is a Kantorovich majorant for the operator A. Moreover, Φ_A is minimal in the sense that

$$\Phi_A(z) \leq \Phi(z) \tag{2.5}$$

for any other Kantorovich majorant Φ of A.

PROOF. For $]x[\leq r$ and $]h[\leq \delta$ we have, by (2.2) and (2.3),

$$\begin{aligned}]A(x+h) - Ax[\leq W(r, r+\delta) \leq \\ \leq \quad W(\theta, r+\delta) - W(\theta, r) = \Phi_A(r+\delta) - \Phi_A(r). \end{aligned}$$

Moreover, if Φ is another Kantorovich majorant for A, and $\{r_0, r_1, \ldots, r_m\} \subset K$ satisfies $\theta = r_0 \leq r_1 \leq \ldots \leq r_{m-1} \leq r_m = z$, then

$$]A(x_j + h_j) - Ax_j[\leq \Phi(r_j) - \Phi(r_{j-1}) \qquad (j = 1, \ldots, m)$$

for $]x_j[\leq r_{j-1}$ and $]h_j[\leq r_j - r_{j-1}$, and hence

$$\sum_{j=1}^{m}]A(x_j + h_j) - Ax_j[\leq \sum_{j=1}^{m} [\Phi(r_j) - \Phi(r_{j-1})] \leq \Phi(z) - \Phi(\theta).$$

This implies that

$$\Phi_A(z) =]A\theta[+ W(\theta, z) \leq]A\theta[+ \Phi(z) - \Phi(\theta) = \Phi(z) \tag{2.6}$$

as claimed. □

3 – Some fixed point theorems

We say that an operator $\Phi : K \to K$ has the F^+*-property* (where F stands for Fatou) if, for any monotonically increasing sequence $(r_n)_n$ in K one has

$$\Phi(\sup\{r_n\}) \leq \sup\{\Phi(r_n)\}. \tag{3.1}$$

Likewise, Φ has the F^-*-property* if, for any monotonically decreasing sequence $(r_n)_n$ in K one has

$$\Phi(\inf\{r_n\}) \geq \inf\{\Phi(r_n)\}. \tag{3.2}$$

Observe that, if the operator Φ is, in addition, monotone with respect to the ordering induced by K, one has equality in both (3.1) and (3.2).

The following three lemmas give statements on fixed points of monotone F^+ or F^--operators which we shall need in the sequel; here $\mathrm{Fix}\Phi$ denotes the set of all fixed points of Φ.

LEMMA 1. *Suppose that B has the W-property, and $\Phi : K \to K$ is monotone and has the F^+-property. If* $\mathrm{Fix}\Phi \neq \emptyset$ *and* $\Phi(\theta) \geq \theta$, *the successive approximations*

$$r_{n+1} = \Phi(r_n) \qquad (r_0 = \theta) \tag{3.3}$$

are o-convergent to the minimal fixed point r_* of Φ.

PROOF. By the monotonicity of Φ, the sequence (3.3) is increasing and bounded from above by any element $r \in \mathrm{Fix}\Phi$. By the W-property of B, the element $r_* = \sup\{r_n\}$ exists, and by the F^+-property of Φ, equality holds in (3.1). □

It is not hard to see that, under the hypotheses of Lemma 1, the successive approximations (3.3) o-converge to r_* not only for $r_0 = \theta$, but for any initial value $r_0 \in < \theta, r_* >$. However, a more precise statement is possible:

LEMMA 2. *Suppose that B has the W-property, and $\Phi : K \to K$ is monotone and has the F^--property. If* $\mathrm{Fix}\Phi \neq \emptyset$, $\Phi(\bar{r}) \leq \bar{r}$ *(but* $\Phi(\bar{r}) \neq \bar{r}$*)*

for some $\tilde{r} \in K$, *and* Φ *has no fixed points in* $< r_*, \tilde{r} >$ *different from* r_*, *then the successive approximations*

$$r_{n+1} = \Phi(r_n) \qquad (\theta \leq r_0 \leq \tilde{r}) \tag{3.4}$$

are o-convergent to r_* *for any* $r_0 \in < \theta, \tilde{r} >$.

PROOF. Defining a sequence $(\tilde{r}_n)_n$ by $\tilde{r}_0 = \tilde{r}$ and $\tilde{r}_n = \Phi^n(\tilde{r})$ we have $\tilde{r}_{n+1} \leq \tilde{r}_n$, since Φ is monotone. By the F^--property of Φ, we see that $\tilde{r}_* = \inf \tilde{r}_n$ exists and satisfies $r_* \leq \tilde{r}_* \leq \tilde{r}$. Since, by assumption, $< r_*, \tilde{r} > \cap \mathrm{Fix}\Phi = \{r_*\}$, we have $\tilde{r}_* = r_*$, i.e. $(\tilde{r}_n)_n$ is o-convergent to r_*.

Choose an arbitrary initial value $\rho_0 \in < \theta, \tilde{r} >$, and consider the successive approximations

$$\rho_{n+1} = \Phi(\rho_n) \qquad (\theta \leq \rho_0 \leq \tilde{r}). \tag{3.5}$$

Again by the monotonicity of Φ, we have $r_n \leq \rho_n \leq \tilde{r}_n$. But both $(r_n)_n$ and $(\tilde{r}_n)_n$ o-converge to r_*, and so does $(\rho_n)_n$. □

In what follows, we write

$$\mathcal{A}(\Phi) = \{r : r \geq r_*, \Phi(r) \leq r, \quad \lim_{n\to\infty} \Phi^n(r) = r_*\}, \tag{3.6}$$

where $r_* = \min \mathrm{Fix}\Phi$ as before.

LEMMA 3. *Suppose that* B *has the* W*-property, and* $\Phi : K \to K$ *has the* F^-*-property and is convex, i.e.*

$$\Phi\left((1-\lambda)a + \lambda b\right) \leq (1-\lambda)\Phi(a) + \lambda\Phi(b) \tag{3.7}$$

for $\theta \leq a \leq b \leq R$. *If* $\mathrm{Fix}\Phi \neq \emptyset$, $r_* = \min \mathrm{Fix}\Phi$, $r_{**} \in \mathrm{Fix}\Phi$ *with* $r_* \leq r_{**}$, *and there are no fixed points of* Φ *in* $< r_*, r_{**} >$ *different from* r_* *and* r_{**}, *then*

$$r(\lambda) = (1-\lambda)r_* + \lambda r_{**} \in \mathcal{A}(\Phi) \qquad (0 < \lambda < 1). \tag{3.8}$$

PROOF. From the convexity of Φ it follows that

$$\Phi(r(\lambda)) \leq (1-\lambda)\Phi(r_*) + \lambda\Phi(r_{**}) = r(\lambda).$$

Moreover, since $< r_*, r(\lambda) > \cap \mathrm{Fix}\Phi = \{r_*\}$, by assumption, we conclude from Lemma 2 that $\Phi^n(r(\lambda))$ o-converges to r_* as $n \to \infty$. □

4 – Operators with majorants

Let X be a $\mathcal{C}$-sequentially complete K-normed space, $A : B(x_0, R) \to X$ a given operator, and $\Phi : K \to K$ some Kantorovich majorant of A on $B(x_0, R)$ having the F^+-property. We suppose that $r_* = \min \mathrm{Fix}\Phi$ is a $\mathcal{C}$-*fixed point* of Φ, i.e. the sequence (3.3) o-converges to r_* and belongs to $\mathcal{C} \subseteq \vartheta(X)$.

The following theorem is fundamental, since it makes it possible to pass from fixed points of the majorant operator Φ to fixed points of the majorized operator A:

THEOREM 2. *Under the above hypotheses, the operator A has a unique fixed point $x_* \in B(\theta, r_*)$, and the successive approximations*

$$x_{n+1} = Ax_n \qquad (x_0 = \theta) \tag{4.1}$$

o-converge to x_* and belong to $\mathcal{C}$. Moreover,

$$]x_* - x_n[\leq r_* - r_n \qquad (n = 0, 1, \ldots), \tag{4.2}$$

with $(r_n)_n$ defined by (3.3). *The fixed point x_* is also unique in the set*

$$\mathcal{U}(\Phi) = \bigcup\{B(\theta, r) : r \in \mathcal{A}(\Phi)\}. \tag{4.3}$$

PROOF. Since Φ is a Kantorovich majorant of A, we have

$$]x_m - x_n[\leq r_m - r_n \qquad (m \geq n),$$

and hence the sequence $(x_n)_n$ is $\mathcal{C}$-fundamental. Its $\mathcal{C}$-limit x_* is then a fixed point of A and satisfies (4.2). Suppose that x_{**} is another fixed point of A in $\mathcal{U}(\Phi)$. This means that $x_{**} = Ax_{**}$ and $]x_{**}[\leq \rho$ for some $\rho \geq r_*$ such that $\Phi(\rho) \leq \rho$ and $\lim_{n\to\infty} \Phi^n(\rho) = r_*$. Since Φ is a Kantorovich majorant of A, we may show, by induction, that

$$]x_{**} - x_n[\leq \rho_n - r_n \qquad (n = 0, 1, \dots).$$

We conclude that the sequence $(x_n)_n$ converges to x_{**} as well. □

Theorem 2 may be considered as a modification of the classical Kantorovich theorem (see [3, Th. 5.7]). Observe that we did not just obtain existence and uniqueness of $x_* \in \mathcal{U}(\Phi) \cap \mathrm{Fix}A$ in Theorem 2, but also some kind of "localization" of x_*, since $]x_*[\leq \min \mathrm{Fix}\Phi$. This important information may be employed for approximate computations.

5 – Kantorovich - Uryson majorants

In this section we shall apply the previous results to the nonlinear integral operator of Uryson type

$$Ax(t) = a(t) + \int_\Omega \kappa(t, s, x(s))ds \tag{5.1}$$

generated by some Carathéodory function $\kappa = \kappa(t, s, u)$. Here we may assume that $\kappa(t, s, 0) \equiv 0$, since we have already "separated" the function

$$a(t) = A\theta(t) = \int_\Omega \kappa(t, s, 0)ds \tag{5.2}$$

in the definiton (5.1). It is natural to expect that possible majorants of A have the form

$$\Phi(z)(t) = |a(t)| + \int_\Omega \ell(t, s, z(s))ds\,. \tag{5.3}$$

where $\ell = \ell(t, s, u)$ is some Carathéodory function which "majorizes" $\kappa(t, s, u)$ in a sense to be made precise. In fact, the following general result was proved in [10]:

LEMMA 4. *For a real function f on $[-R, R]$, its majorant ϕ is given by*

$$\phi(u) = |f(0)| + \int_0^{|u|} \sup_{|\xi| \leq v} |f'(\xi)| dv. \tag{5.4}$$

If we apply Lemma 4, for all $t \in \Omega$ and almost all $s \in \Omega$, to the scalar function

$$f(u) = \kappa(t, s, u) \tag{5.5}$$

(assuming that the kernel function κ admits a derivative κ_u' with respect to the last argument), we get

$$\phi(u) = |\kappa(t, s, 0)| + \int_0^{|u|} \sup_{|\xi| \leq v} |\kappa_u'(t, s, \xi)| \, dv.$$

Using now Fubini's theorem, we arrive at the following

THEOREM 3. *Let $\kappa = \kappa(t, s, u)$ be a Carathéodory function which has a derivative κ_u' with respect to the last variable. Then the Uryson operator* (5.1) *admits a Kantorovich majorant of the form* (5.3), *where*

$$\ell(t, s, u) = \int_0^{|u|} \sup_{|\xi| \leq v} |\kappa_u'(t, s, \xi)| \, dv. \tag{5.6}$$

In what follows, we shall call the operator (5.3), with ℓ given by (5.6), the *Kantorovich-Uryson majorant*. The question arises whether or not the Kantorovich-Uryson majorant (5.3) is minimal.

To answer this question in general function spaces X seems to be a difficult problem. However, in all important examples we know, the operator (5.3) is in fact the minimal Kantorovich majorant Φ_A (see (2.4)) of the operator (5.1).

For instance, a typical example for the applicability of our results is the following. Suppose that X is an *ideal space* of measurable functions [11] over a bounded domain Ω in Euclidean space; roughly speaking, this means that X carries a monotone norm with respect to the natural ordering (a.e.) of measurable functions. In this case we take $B = X$, K the cone of all (a.e.) nonnegative functions in B, and

$$]x[= |x(\cdot)| \tag{5.7}$$

as a natural K-norm. Assume now that the Carathéodory function $\kappa = \kappa(t, s, u)$ admits a derivative κ'_u with respect to the third variable, and that the corresponding Uryson operator (5.1) acts in the space X (many sufficient conditions, for general ideal spaces as well as for specific spaces arising in applications, may be found in [12]). Combining Theorem 2 and Theorem 3 yields now the following

THEOREM 4. *Suppose that the integral equation*

$$z(t) = |a(t)| + \int_\Omega \int_0^{z(s)} \sup_{|\xi| \le v} |\kappa'_u(t, s, \xi)|\, dv\, ds$$

has a minimal solution $z_* \in K$. Then the Uryson integral equation

$$x(t) = a(t) + \int_\Omega \kappa(t, s, x(s))\, ds$$

has a solution $x_* \in B(\theta, z_*)$ which is unique in the set (4.3).

Moreover, if the successive approximations

$$z_{n+1}(t) = |a(t)| + \int_\Omega \int_0^{|z_n(s)|} \sup_{|\xi| \le v} |\kappa'_u(t, s, \xi)|\, dv\, ds \tag{5.8}$$

o-converge to z_* and belong to $C \subseteq \vartheta(X)$, then the successive approximations

$$x_{n+1}(t) = a(t) + \int_\Omega \kappa(t, s, x_n(s))\, ds \tag{5.9}$$

converge to z_* and also belong to C.

6 – Linear Kantorovich majorants

Now we shall be interested in linear (or affine) Kantorovich majorants for the Uryson operator (5.1), i.e. majorants of the form

$$\Phi(z)(t) = |a(t)| + Qz(t)\,, \tag{6.1}$$

where Q is a linear operator in an ideal space X. More precisely, to each nonnegative function $R \in X$ we associate the kernel function

$$q_R(t,s) = \sup_{|u|,|v| \leq R} \frac{|\kappa(t,s,u) - \kappa(t,s,v)|}{|u-v|} \tag{6.2}$$

and the corresponding linear integral operator

$$Qz(t) = \int_\Omega q_R(t,s)z(s)ds\,. \tag{6.3}$$

Lemma 5. *The operator* (6.1), *where a and Q are given by* (5.2) *and* (6.3), *respectively, is a Kantorovich majorant for the Uryson operator* (5.1) *on the K-ball $B(\theta, R)$ if and only if the kernel function* (6.2) *is defined.*

Proof. If q_R is defined, we get for $|x| + |h| \leq R$

$$\begin{aligned}
&|A(x+h)(t) - Ax(t)| = \\
&\qquad = \left|\int_\Omega [\kappa(t,s,x(s)+h(s)) - \kappa(t,s,x(s))]ds\right| \leq \\
&\qquad \leq \int_\Omega q_R(t,s)|h(s)|ds = Q|h|(t) = \\
&\qquad = \Phi(|x|+|h|)(t) - \Phi(|x|)(t)\,,
\end{aligned}$$

i.e. Φ is a Kantorovich majorant for A, by the definition (5.7) of the natural K-norm in X. Conversely, suppose that (2.1) holds on $B(\theta, R)$, where Φ is given by (6.1). This implies, in particular, that

$$\left|\int_D [\kappa(t,s,x(s)+h(s)) - \kappa(t,s,x(s))]ds\right| \leq \int_D q_R(t,s)h(s)\,ds \tag{6.4}$$

for any measurable subset D of Ω. From (6.4) we conclude that

$$|\kappa(t,s,x(s)+h(s)) - \kappa(t,s,x(s))| \leq q_R(t,s)|h(s)|$$

for almost all t, $s \in \Omega$. But this means that the kernel function (6.2) is defined. □

It is a striking fact that even a *linear* integral operator may fail to have a Kantorovich majorant. In fact, since the Kantorovich majorant Φ acts in the cone K of nonnegative functions, any operator A with Kantorovich majorant is necessarily *regular* in the sense of [5], [12]. But even in $X = L_2$ there are linear integral operators which are not regular.

Lemma 5 gives a criterion for the affine operator (6.1) to be a Kantorovich majorant for the nonlinear Uryson operator (5.1). To apply Theorem 2, we have to guarantee the existence of a minimal fixed point z_* of the operator Φ. Here the following general result is useful which goes back essentially to [6], [7]:

LEMMA 6. *Let Q be a positive linear operator which acts in the cone K of nonnegative functions in some ideal space X. Suppose that the spectral radius $\rho(Q)$ of Q satisfies*

$$\rho(Q) < 1 \tag{6.5}$$

Then, for any function $a \in K$, the operator $\Phi(z) = a + Qz$ has the only fixed point $z_* = (I-Q)^{-1}\Phi(\theta)$, and the set $\mathcal{A}(\Phi)$ (see (3.6)) coincides with the whole cone K.

We remark that the condition (6.5) is not necessary, in general, for Φ to have a minimal fixed point, but only the condition $\rho(Q) \leq 1$. This may be illustrated, for example, by means of the multiplication operator $Qz(t) = tz(t)$. Observe, however, that (6.5) is in fact necessary in case Q is a compact operator.

Lemma 6 provides not only existence and uniquess of $z_* \in \mathrm{Fix}\Phi$ but also some precise information on the convergence of the successive approximations (4.1). From (6.1) it follows that

$$]Az_1 - Az_2[\leq Q]z_1 - z_2[\tag{6.6}$$

and hence

$$
\begin{aligned}
]x_n - x_*[&\leq \sum_{j=n}^{\infty}]x_j - x_{j+1}[\leq \\
&\leq \sum_{j=n}^{\infty} Q^j]A\theta[\leq (I-Q)^{-1} Q^n \Phi(\theta).
\end{aligned}
\tag{6.7}
$$

Suppose that there exists a positive normalized eigenfunction e_0 of Q which corresponds to the eigenvalue $\lambda = \rho(Q)$, and that

$$
]Q^k z[\leq d\|z\| e_0 \tag{6.8}
$$

for some $k \in \mathbb{N}$ and $d > 0$. From (6.7) we get then

$$
]x_n - x_*[\leq \frac{d\rho(Q)^{n-k}}{1-\rho(Q)} \|\Phi(\theta)\| e_0 ,
$$

and hence

$$
\|x_n - x_*\| \leq \frac{d\rho(Q)^{n-k}}{1-\rho(Q)} \|\Phi(\theta)\| .
$$

Condition (6.8) holds always (with $k = 0$) if e_0 is an interior point of the cone K. For $k = 1$, condition (6.8) means that Q acts from X into the space X_{e_0} of all measurable functions u for which the norm

$$
\|u\|_{e_0} = \inf\{\alpha : \alpha \geq 0, \quad -\alpha e_0 \leq u \leq \alpha e_0\}
$$

makes sense and is finite. A detailed discussion of this space, also from the viewpoint of integral equations, may be found in the recent monograph [7].

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