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Characterizations of Hammerstein operators

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RIASSUNTO – *Si estende agli operatori di Hammerstein il criterio, ottenuto in [14], per riconoscere operatori integrali non lineari. La doppia dimostrazione consiste nel fatto che la condizione successiva è cambiata prendendo una successione anziché due e non vengono fatte ipotesi sull'ordine della limitazione dell'operatore.*

ABSTRACT – *In this paper the criterion for recognizing non-linear integral operators obtained in [14] is improved for Hammerstein operators. The twofold improvement consists in that the sequential condition is changed taking one sequence instead of two ones and order boundedness of the operator is not assumed.*

KEY WORDS – *Hammerstein operators - Ideal spaces of measurable functions - Lattice calculus for order bounded operators - Equimeasurable sets.*

A.M.S. CLASSIFICATION: 47H30 - 47B65 - 45P05

– Introduction

This article is devoted to improving the characterization of order bounded Uryson operators obtained in [14] for a wide and important class of non-linear operators; namely, that of Hammerstein operators; so that this paper complements [14]. In that article it was shown a sequential criterion for integral representation of non-linear operators similar to those due to A.V. BUKHVALOV in [2] for linear operators and to L. DREWNOWSKI and W. ORLICZ in [3] for functionals. In [3], L. DREWNOWSKI and W. ORLICZ proved that an orthogonality additive fuctional $\Phi\colon E \longrightarrow \mathbb{R}$ has an integral representation if and only if the following condition holds.

Given $(f_n)_{n=1}^{\infty}$ *and* $(g_n)_{n=1}^{\infty}$ *two order bounded sequences in* E, *then* $f_n - g_n \longrightarrow 0(^*)$ *implies* $\Phi f_n - \Phi g_n \longrightarrow 0$.

Nevertheless, they were successful improving their condition for functionals in [4] taking just one sequence. On the other hand, it is obvious that for linear operators we need to consider only one sequence too. Thus, it is natural to ask whether Drewnowski and Orlicz' techniques also work for non linear operators. We point out that this is not the case for Uryson operators (the setting of [14]); however, in this paper we show that it is possible to modify Drewnowski and Orlicz' method for the class of Hammerstein operators. The main result (theorem 2.4) states that a necessary and sufficient condition for an orthogonally additive operator $T\colon E \longrightarrow M(X,\mu)$ to be a Hammerstein operator is the following:

The operator can be factorized as $T = L \circ N$, L *being a regular linear operator and* N *being a projection commuting operator, and if* $(f_n)_{n=1}^{\infty}$ *is an order bounded sequence in* E *such that* $f_n \longrightarrow f(^*)$, *then* $Tf_n(x) \longrightarrow Tf(x)$ *a.e.*

Incidentally, theorem 2.4 also includes another criterion which characterize Hammerstein operators. That condition is similar to the one proved by W. Schachermayer in [10] for linear operators. This fact can easily be proved for Hammerstein operators taking into account Schachermayer's theorem and the above condition, but it is not so obvious the generalization to Uryson operators since Schachermayer's proof cannot be applied. However, that criterion still holds for arbitrary order bounded Uryson operators and a forthcoming article will be devoted to prove it.

1 – Preliminaries

As in [14] we shall use methods from the theory of Riesz spaces (vector lattices) and from measure theory. We refer to the monographs [1, 7, 12, 15, 17] for terminology and basic results in the theory of Riesz spaces. Given E and F Riesz spaces, we denote by $L_b(E,F)$ the space of all order bounded linear operators from E into F and by $E_n^{\sim}$ the space of all order continuous functionals on E. We recall that an operator $T\colon E \longrightarrow F$ is called **orthogonally additive** if $T(f+g) = Tf + Tg$ whenever $f, g \in E$ are disjoint ($f \perp g$ in symbols), and T is called **order bounded** if it maps order bounded sets in E onto order bounded sets in F. In case that E is a vector sublattice of F, an operator is said to be **projection commuting** if $N \circ P = P \circ N$ holds for every order projection $P\colon E \longrightarrow F$ satisfying $P(E) \subset E$ (see [8]). Throughout this paper the characteristic function of a set B will be denoted by $\mathbf{1}_B$. Let (Y, ν) be a σ-finite and complete measure space. We shall denote by $M(Y, \nu)$ the set of all ν-measurable and ν-almost everywhere finite functions on Y with the usual identification of ν-almost equal functions. Recall that the sequence $(f_n)_{n=1}^{\infty}$ in $M(Y, \nu)$ is said that **(*)-converges** to f ($f_n \longrightarrow f(^*)$ in symbols) if every subsequence $(f_{n_k})_{k=1}^{\infty}$ contains a subsequence $(f_{n_{k_i}})_{i=1}^{\infty}$ such that $f_{n_{k_i}}(y) \longrightarrow f(y)$ ν-a.e. For every $B \subset Y$ with $\nu(B) < \infty$, (*)-convergence is the same as **convergence in measure** (that is, given $\varepsilon > 0$, we have that $\lim_{n\to\infty} \nu(B_n^{\varepsilon}) = 0$ holds, where $B_n^{\varepsilon} := \{y \in Y \mid |f_n(y) - f(y)| > \varepsilon\}$ for all $n \in \mathbb{N}$).

In what follows E will denote an order ideal in $M(Y, \nu)$. Without restrictions we shall assume that the carrier of E is Y (see [17, §86]). Thus, there exists an increasing sequence $(Y_n)_{n=1}^{\infty}$ of ν-measurable sets such that $Y = \bigcup_{n=1}^{\infty} Y_n$, $\nu(Y_n) < \infty$ and $\mathbf{1}_{Y_n} \in E$ for all $n \in \mathbb{N}$ [17, theorem 86.2].

Consider two σ-finite and complete measure spaces (X, μ) and (Y, ν), define the product measure space as usual and denote by $(X \times Y, \mu \times \nu)$ its completion. We recall that every $\mu \times \nu$-measurable set can be approximated by finite unions of generalised rectangles [see 5, p. 56]. To be precise, fixed a $\mu \times \nu$-measurable set Z and given $\varepsilon > 0$ one can find $\mu \times \nu$-measurable sets $A_k \times B_k$ for $k = 1, \dots, n$ such that

$$\mu \times \nu\left(Z \Delta \left(\bigcup_{k=1}^{n} A_k \times B_k\right)\right) < \varepsilon.$$

Let $K\colon X \times Y \longrightarrow \mathbb{R}$ be a $\mu \times \nu$-measurable function and let $N\colon Y \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function with $N(y, 0) = 0$ for ν-almost all $y \in Y$ and in such a way that the product function satisfies the Carathéodory conditions; that is,

(C_1) The function $K(\cdot,\cdot)N(\cdot,t)$ is $\mu\times\nu$-measurable for all $t\in\mathbb{R}$

(C_2) The function $K(x,y)N(y,\cdot)$ is continuous on $\mathbb{R}$ for $\mu\times\nu$-almost all $(x,y)\in X\times Y$.

An operator $T\colon E\longrightarrow M(X,\mu)$ will be called a Hammerstein operator with kernel $K(x,y)N(y,t)$ if for every $f\in E$

(1) The function $x\longrightarrow\int_Y |K(x,y)N(y,f(y))|d\nu$ is μ-a.e. finite.

(2) $(Tf)(x)=\int_Y K(x,y)N(y,f(y))d\nu\,\mu$-a.e.

That is, we consider that T is a Hammerstein operator if it is a Uryson operator (see [8]) with a special kernel. We remark that the above use of the concept of Hammerstein operator is slightly different of (and apparently wider than) the usual one (see [8]). The reason lies in the product by the function $K(x,y)$. In fact, if there exists a $\mu\times\nu$-measurable set Z such that $K(x,y)=0$ for all $(x,y)\in Z$, then the product may satisfies (C_1) and (C_2) even if the Carathéodory conditions on the function N (that is, the function $N(\cdot,t)$ is ν-measurable for all $t\in\mathbb{R}$, and $N(y,\cdot)$ is continuous on $\mathbb{R}$ for ν-almost all $y\in Y$) do not hold. In our case, it is also possible to factorize the operator as $T=L\circ N$ where $L\colon \mathrm{Dom}_Y(K)\longrightarrow M(X,\mu)$ is a linear integral operator (see [17, §93] for a detailed definition) and $N\colon E\longrightarrow \mathrm{Dom}_Y(K)$ is a Nemytskii operator (i.e., there is a function $N\colon Y\times\mathbb{R}\longrightarrow\mathbb{R}$ such that $(Nf)(y)=N(y,f(y))$ ν-a.e. for all $f\in E$). Obviously this factorization is not unique (which may be useful, see remark in [6, p. 377]). In what follows we consider that functions N generating Nemytskii operators satisfy $N(y,0)=0$ for ν-almost all $y\in Y$ since there is not loss of generality and this condition implies that the considered operator is projection commuting.

In our task of recognizing Hammerstein operators we shall apply some characterizations of linear integral operators which we next list. Firstly recall that a subset M of $M(X,\mu)$ is said to be **equimeasurable** if for all $\varepsilon>0$ and all $X_0\subset X$ with $\mu(X_0)<\infty$ there exists $X_1\subset X_0$ with $\mu(X_0\text{-}X_1)<\varepsilon$ such that $\{g\mathbb{1}_{X_1}|g\in M\}$ is a relatively norm-compact subset of $L^\infty(X_1,\mu|_{X_1})$. A.R. Schep observe in [13, lemma 3.1] that if the sequence $(g_n)_{n=1}^\infty$ forms an equimeasurable set in $M(X,\mu)$, then

$g_n \longrightarrow g(^*)$ implies $g_n(x) \longrightarrow g(x)$ μ-a.e. (see also [11] where it is proved a converse for convex sets).

THEOREM A. *For a regular linear operator* $L\colon E \longrightarrow M(X,\mu)$ *the following statements are equivalent.*

(1) *L is an integral operator.*

(2) *L lies in the band in $L_b(E, M(X,\mu))$ generated by $E_n^{\sim} \otimes M(X,\mu)$ (i.e., by operators of finite rank).*

(3) *If $(f_n)_{n=1}^{\infty}$ is a sequence in E such that $0 \le f_n \le g \in E$, then $f_n \longrightarrow 0(^*)$ implies $Lf_n(x) \longrightarrow 0$ μ-a.e.*

(4) *L is $(^*)$-continuous (i.e., it transforms order bounded $(^*)$-convergent sequences in E in $(^*)$-convergent sequences) and maps order bounded sets of E into equimeasurable sets.*

The equivalence (1) $\iff$ (2) was proved by A.V. BUKHVALOV in [2] (see also [17, theorem 95.1]) with additional hypothesis which were removed by B. DE PAGTER in [9]. (1) $\iff$ (3) is due to A.V. BUKHVALOV in [2] (see [17, theorems 96,5 and 96.8]). (1) $\iff$ (4) was stated by W. SCHACHERMAYER for L^p-spaces in [10] and later by A. R. Schep in its actual form. Finally we point out that the equivalence (3) $\iff$ (4) may be proved in a direct way (see [11] and [13]).

2 – Characterizing Hammerstein operators

As before, let E be an order ideal in $M(Y,\nu)$ and assume that Y is the carrier of E. We denote by E_S the set of all simple functions of E with rationals values. That is,

$$E_S := \left\{ p = \sum_{i=1}^{n} t_i \mathbf{1}_{B_i} \in E \,\middle|\, B_i \subset Y \ \nu\text{-measurable and } t_i \in \mathbb{Q} \text{ for } i = 1, \dots, n \right\}.$$

We shall assume that the sets $(B_i)_{i=1}^{n}$ are pairwise disjoint.

We shall arrive to the main result into two stages, in the same way as in [14]. We shall first see (theorem 2.1) that operators $T = L \circ N$ on E_S satisfying

For every order bounded sequence $(\mathbb{1}_{B_n})_{n=1}^{\infty}$ of characteristic functions of E, $\mathbb{1}_{B_n} \longrightarrow 0(^*)$ implies $T(t\mathbb{1}_{B_n})(x) \longrightarrow 0$ μ-a.e. for all $t \in \mathbb{Q}$.

are integrals operators with a Hammerstein type kernel $K(x,y)N(y,t)$. Observe that this kernel is only defined for rational values.

In a second step (theorem 2.3) our goal is to prove that if $T\colon E \longrightarrow M(X,\mu)$ is an operator such that its restriction to E_S is as above, then the function $K(x,y)N(y,\cdot)$ is uniformly continuous on the bounded sets of $\mathbb{Q}$ for $\mu\times\nu$-almost all $(x,y)\in X\times Y$ whenever the operator T is order continuous in the following sense.

If $(f_n)_{n=1}^{\infty}$ is an order bounded sequence in E such that $f_n(y) \longrightarrow f(y)$ ν-a.e. then $Tf_n(x) \longrightarrow Tf(x)$ μ-a.e.

Next one can follow the proof of [14, theorem 4.1] on account of theorem 2.3 instead of [14, theorem 3.1]. So that the kernel may be extended in such a way that it satisfies the Carathéodory conditions and represents the operator T on the whole space E.

THEOREM 2.1. *Let $T\colon E_S \longrightarrow M(X,\mu)$ be an order bounded and orthogonally additive operator such that $T = L\circ N$, where L is a regular operator defined on an ideal D in $M(Y,\nu)$ and the operator $N\colon E_S \longrightarrow D$ is projection commuting. Assume that given an order bounded sequence $(\mathbb{1}_{B_n})_{n=1}^{\infty}$ of characteristic functions of E, $\mathbb{1}_{B_n} \to 0(^*)$ implies $T(t\mathbb{1}_{B_n})(x) \longrightarrow 0$ μ-a.e. for all $t\in\mathbb{Q}$. Then there is a $\mu\times\nu$-measurable function $K\colon X\times Y \longrightarrow \mathbb{R}$ and there is a function $N\colon Y\times\mathbb{Q} \longrightarrow \mathbb{R}$ satisfying*

(a) *$N(y,0) = 0$ for ν-almost all $y\in Y$*

(b) *The function $N(\cdot,t)$ is ν-measurable for all $t\in\mathbb{Q}$*

and such that for every $p\in E_S$

$$Tp(x) = \int_Y K(x,y)N(y,p(y))d\nu\mu - a.e.$$

PROOF. Without loss of generality two assumptions can be done. On the one hand, there is an increasing sequence $(Y_m)_{m=1}^{\infty}$ of ν-measurable sets such that $Y = \bigcup_{m=1}^{\infty} Y_m$, $\nu(Y_m) < \infty$ and $\mathbb{1}_{Y_m} \in E$ for all $m \in \mathbb{N}$. So after obtaining an integral representation for functions $p \in E_S$ with supp(p) $\subset$ Y_m for some $m \in \mathbb{N}$, we can extend it to every $p \in E_S$. In fact, let $p = \sum_{i=1}^{n} t_i \mathbb{1}_{B_i} \in E_S$. Since, for all $i = 1, \ldots, n$, $\mathbb{1}_{B_i \cap (Y - Y_m)} \longrightarrow 0(^*)$, we have that

$$\lim_{m\to\infty} T\Big(t_i \mathbb{1}_{B_i \cap Y_m}\Big)(x) =$$
$$= T\Big(t_i \mathbb{1}_{b_i}\Big)(x) - \lim_{m\to\infty} T\Big(t_i \mathbb{1}_{B_i \cap (Y - Y_m)}\Big)(x) = T\Big(t_i \mathbb{1}_{B_i}\Big)(x)$$

holds for μ-almost all $x \in X$ and so $\lim_{n\to\infty} T(p\mathbb{1}_{Y_m})(x) = Tp(x)\mu$-a.e.

Now, it follows from $T(p\mathbb{1}_{Y_m})(x) = \int_Y K(x,y)N(y,p(y)\mathbb{1}_{Y_m}(y))d\nu$ μ-a.e. for all $m \in \mathbb{N}$ that $Tp(x) = \int_Y K(x,y)N(y,p(y))d\nu$ μ-a.e. Thus, we may assume that $\nu(Y) < \infty$ and $\mathbb{1}_Y \in E_S$. On the other hand, we may also assume that $Np \in L^\infty(Y,\nu)$ for every $p \in E_S$. Indeed, suppose that the result is proved for this class of functions. Let $p \in E_S$ and define $Y_m := \{y \in Y \mid |Np(y)| \leq m\}$, $m \in \mathbb{N}$. As above, from the representation for $T(p\mathbb{1}_{Y_m})$ for all $m \in \mathbb{N}$. we get a representation for the function Tp. So, we shall also assume that $D \subset L^\infty(Y,\nu)$. Since the space $L_b(D, M(X,\mu))$ is Dedekind complete, $L_b(D, M(X,\mu)) = (D_n^\sim \otimes M(X,\mu))^d \oplus (D_n^\sim \otimes M(X,\mu))^{dd}$ and consequently $L = L_1 + L_2$ with $L_1 \in (D_n^\sim \otimes M(X,\mu))^d$ and $L_2 \in (D_n^\sim \otimes M(X,\mu))^{dd}$. The operator L_2 is an integral operator by Theorem A. Then there is a $\mu\times\nu$-measurable function $K: X \times Y \longrightarrow \mathbb{R}$ such that for every $f \in D$

$$L_2 f(x) = \int_Y K(x,y) f(y) d\nu\, \mu - \text{a.e.}$$

Hence

$$(L_2 \circ N)(p)(x) = \int_Y K(x,y) Np(y) d\nu\, \mu - \text{a.e.}$$

On the other hand, define $N(y,t) := N(t\mathbb{1}_Y)(y)\nu$-a.e. for all $t \in \mathbb{Q}$. It is easy to see that the function N satisfies conditions (a) and (b). Moreover, if $p = \sum_{i=1}^{n} t_i \mathbb{1}_{B_i} \in E_S$, the sets $(B_i)_{i=1}^{n}$ being mutually disjoint, then

$$Np(y) = \sum_{i=1}^{n} N\Big(t_i \mathbb{1}_{B_i}\Big)(y) = \sum_{i=1}^{n} \mathbb{1}_{B_i}(y) N\Big(t_i \mathbb{1}_Y\Big)(y) = N(y, p(y))\nu - \text{a.e.}$$

Thus, the result follows from the following fact: $(L_1 \circ N)(p)(x) = 0$ μ-a.e. for every $p \in E_S$. More conveniently, it follows from $(L_1 \circ N)(t\mathbb{1}_B)(x) = 0$ μ-a.e. for all $t \in \mathbb{Q}$ and for all ν-measurable set B.

Let B be a ν-measurable set and let $t \in \mathbb{Q}$. Firstly suppose that $N(t\mathbb{1}_B) \geq 0$. Since $L_1 = L - L_2$ and L_2 is integral, if $(\mathbb{1}_{B_n})_{n=1}^{\infty}$ is a sequence of characteristic functions of E such that $\mathbb{1}_{B_n} \longrightarrow 0(^*)$, then $(L_1 \circ N)(t\mathbb{1}_{B_n})(x) \longrightarrow 0$ μ-a.e. Let A be a μ-measurable set such that $\mathbb{1}_A \in F$ and define $L'f(x) := \mathbb{1}_A(x) \int_Y f(y) d\nu \mu$-a.e. for all $f \in D$. On account of $D \subset L^\infty(Y, \nu)$, it is well defined and it follows that $L' \in D_n^\sim \otimes M(X, \mu)$. Hence, $|L_1| \wedge L' = 0$. Keeping in mind the Abramovich theorem [1, theorem 3.16] we have that

$$(|L_1| \wedge L')(t\mathbb{1}_B) = \inf\Big\{(|L_1| \circ N)(t\mathbb{1}_{B_1}) + (L' \circ N)(t\mathbb{1}_{B\text{-}B_1}) | B \subset B_1\Big\} = 0\,.$$

Now, using the same procedure as in the linear case (see [2] or [17, theorem 96,5]), one obtains that $(L_1 \circ N)(t\mathbb{1}_B)(x) = 0$ μ-a.e.
Finally consider an arbitrary $t\mathbb{1}_B \in E_S$. Put $B^+ := \{y \in B | N(y, t) > 0\}$ and $B^- = B\text{-}B^+$. It is straightforward that $N^+(y, t\mathbb{1}_B(y)) = N(y, t\mathbb{1}_{B^+}(y))$ and $N^-(y, t\mathbb{1}_B(y)) = N(y, t\mathbb{1}_{B^-}(y))$. Hence

$$(L_1 \circ N)(t\mathbb{1}_B)(x) = (L_1 \circ N)(t\mathbb{1}_{B^+})(x) - (L_1 \circ N)(t\mathbb{1}_{B^-}(x) = 0\mu - \text{a.e.}$$

□

LEMMA 2.2. *Let $(f_{ni})_{i=1}^{m(n)}{}_{n=1}^{\infty}$ be a family of functions of $M(X, \mu)$ which form an equimeasurable set. Assume that for each $\sigma = (\sigma(n))_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \{1, \dots, m(n)\}$, $f_{n\sigma(n)}(x) \longrightarrow f(x)$ holds μ-a.e. Then there is a decreasing sequence $(h_n)_{n=1}^{\infty}$ in $M(X, \mu)$ satisfying*

(1) *$|f_{ni}(x) - f(x)| \leq h_n(x)\mu$-a.e. for all $i \in \{1, \dots, m(n)\}$ and for all $n \in \mathbb{N}$.*

(2) *$\inf_{n\in\mathbb{N}} h_n(x) = 0\mu$-a.e.*

Proof. Consider the finite sequences $(f_{ni})_{i=1}^{m(n)}$ for $n \in N$ and concatenate all of them into one single sequence. Then our hypothesis imply that this sequence $(^*)$-converges to f. Since the sequence lies in an equimeasurable set, it actually converges to f μ-a.e. from where the required result follows easily. □

Theorem 2.3. *Let $T: E \longrightarrow M(X,\mu)$ be an orthogonally additive operator such that there is a $\mu \times \nu$-measurable function $K: X \times Y \longrightarrow \mathbb{R}$ and there is a function $N: Y \times \mathbb{Q} \longrightarrow \mathbb{R}$ satisfying*

(1) *$N(y,0) = 0$ for ν-almost all $y \in Y$*

(2) *The function $N(\cdot,t)$ is ν-measurable for all $t \in \mathbb{Q}$ and such that for every $p \in E_S$*

$$Tp(x) = \int_Y K(x,y)N(y,p(y))d\nu \ \mu - a.e.$$

Moreover, assume that for every order bounded sequence $(f_n)_{n=1}^{\infty}$ in E, $f_n(y) \longrightarrow f(y)$ ν-a.e. implies $Tf_n(x) \longrightarrow Tf(x)$ μ-a.e. Then for $\mu \times \nu$-almost all $(x,y) \in X \times Y$, the function $K(x,y)N(y,\cdot)$ is uniformly continuous on the bounded sets of $\mathbb{Q}$.

Proof. First we shall reduce our general hypothesis to a simpler case. Since the measures μ and ν are σ-finite and Y is the carrier of E we may suppose that $\mu(X) < \infty$, $\nu(Y) < \infty$ and $\mathbb{I}_Y \in E$. Furthermore , we may also assume that the operator $f \longrightarrow N(\cdot, f(\cdot))$ is order bounded. Indeed, suppose that the result is true in that particular case and let u be a weak unit in $\mathrm{Dom}_Y(K)$ (note that such weak unit always exists and we may take $0 < u(y) < \infty$ for every $y \in Y$). Now, given the function N, define

$$N_n(y,t) := N(y,t)^+ \wedge nu(y) - N(y,t)^- \wedge nu(y)$$

for all $n \in N$. It is obvious that $N_n(y,t) \longrightarrow N(y,t)$ for all $y \in Y$ and all $t \in \mathbb{Q}$. Each of these functions generates an order bounded operator and so each function $K(x,y)N_n(y,\cdot)$ is uniformly continuous on the bounded sets for $\mu \times \nu$-almost all $(x, y) \in X \times Y$. Consider the sets

$$Z_n := \Big\{(x, y) \in X \times Y \,\big|\, \text{The function } K(x, y)N_n(y, \cdot) \text{ is uniformly continuous on the bounded sets}\Big\}$$

and fix $(x, y) \notin \bigcup_{n=1}^{\infty} Z_n$. If $K(x, y) = 0$, then it is obvious that the function $K(x, y)N(y, \cdot)$ is uniformly continuous on the bounded sets. Otherwise, note that if $s \in \mathbb{Q}$ satisfies $|N(y, s)| < nu(y)$ for some $n \in \mathbb{N}$, then $N_m(y, s) = N_n(y, s)$ for every $m \geq n$ and so $N_n(y, s) = N(y, s)$. Fixing $t \in \mathbb{Q}$ it is possible to find $n \in \mathbb{N}$ such that $|N(y, t)| < nu(y)$ and consequently $N_n(y, t) = N(y, t)$. It follows from $|N_n(y, t)| < nu(y)$ and the uniform continuity of the function $K(x, y)N_n(y, \cdot)$ that there exists an open bounded neighborhood V of t such that $|K(x, y)N_n(y, s)| < n|K(x, y)|u(y)$ for every $s \in V$. So $K(x, y)N_n(y, s) = K(x, y)N(y, s)$ for every $s \in V \cap \mathbb{Q}$ and hence the function $K(x, y)N(y, \cdot)$ is uniformly continuous on $V \cap \mathbb{Q}$. From this local property we next pass to the general case by relative compactness. Let $a, b \in \mathbb{Q}$ with $a < b$ and consider $t \in [a, b] \cap \mathbb{Q}$. Then there is an opern neighborhood V_t of t such that the function $K(x, y)N(y, \cdot)$ is uniformly continuous on $V_t \cap \mathbb{Q}$. Applying the classical Borel's theorem we may get $t_1, t_2, \dots, t_n \in [a, b] \cap \mathbb{Q}$ such that $[a, b] \subset \bigcup_{i=1}^{n} V_{t_i}$; thus, $K(x, y)N(y, \cdot)$ is uniformly continuous on $[a, b] \cap \mathbb{Q}$.

Therefore, the function $K(x, y)N(y, \cdot)$ is uniformly continuous on the bounded sets of $\mathbb{Q}$ and, consequently, we can suppose that the operator $f \longrightarrow N(\cdot, f(\cdot))$ is order bounded.

Keep $s \in \mathbb{Q}$ $s \geq 0$, fixed. Since the operator T is order bounded and by [14, proposition 2.7], given $s\mathbb{I}_Y \in E_s$ it is possible to find a positive and $\mu \times \nu$-measurable function M_s satisfying

(i) If $p \in \left[-s\mathbb{I}_Y, s\mathbb{I}_Y\right] \cap E_s$, then $\left|K(x, y)N(y, p(y))\right| \leq M_s(x, y)$ $\mu \times \nu$-a.e.

(ii) The function $M_s(x, \cdot)$ is ν-integrable for μ-almost all $x \in X$.

(iii) The function $h(x) := \int_Y M_s(x, y) d\nu$ is μ-a.e. finite.

In the same way as in [14, theorem 3.1] there is not loss of generality in supposing that the function h is μ-integrable and so the function M_s is $\mu \times \nu$-integrable by Tonelli-Hobson's theorem.

We remark that the proof of [14, theorem 3.1] is based in its first stage, that is, on the following fact:

The map $\Phi\colon \Big([-s\mathbb{1}_Y, s\mathbb{1}_Y]\cap E_s, \|\cdot\|_\infty\Big) \longrightarrow \mathbb{R}$ *defined by*

$$\Phi(p) := \int\limits_{X\times Y} K(x,y)N(y,p(y))d\mu\times\nu$$

is uniformly continuous.

So, to obtain the result it is enough to see that for every sequences $(p_n)_{n=1}^\infty$ and $(q_n)_{n=1}^\infty$ in E_s with $|p_n|$, $|q_n| \le s\mathbb{1}_Y$ for all $n\in\mathbb{N}$, it follows from $\lim\limits_{n\to\infty}\|p_n - q_n\|_\infty = 0$ that

$$\lim_{n\to\infty}\int\limits_{X\times Y} K(x,y)N(y,p_n(y)) - K(x,y)N(y,q_n(y))d\mu\times\nu = 0\,.$$

Let $(p_n)_{n=1}^\infty$ and $(q_n)_{n=1}^\infty$ be sequences in $[-s\mathbb{1}_Y, s\mathbb{1}_Y]$ such that $\lim\limits_{n\to\infty}\|p_n - q_n\|_\infty = 0$. For every $n\in N$, put $\delta_n := \sup_{k\ge n}\|p_n - q_n\|_\infty$. Then the sequence $(\delta_n)_{n=1}^\infty$ is decreasing and converges to 0. If $K(x,y)N(y,p_n(y)) - K(x,y)N(y,q_n(y)) \longrightarrow 0$ in $\mu\times\nu$-measure, then the conclusion follows from the dominated convergence theorem. Assume that the sequence $\Big(K(x,y)N(y,p_n(y)) - K(x,y)N(y,q_n(y))\Big)_{n=1}^\infty$ does not converge to 0 in $\mu\times\nu$-measure to get a contradiction. In order to use the same arguments as L. Drewnowski and W. Orlicz do in [4, theorem 2.3] the following facts are needed.

Let $(p_n)_{n=1}^\infty$ *and* $(q_n)_{n=1}^\infty$ *be two sequences in* $[-s\mathbb{1}_Y, s\mathbb{1}_Y]\cap E_s$ *and assume that there is a function* $u\in E$ *such that* $p_n(y)\longrightarrow u(y)$ *and* $q_n(y)\longrightarrow u(y)$ ν*-a.e.*

Claim 1. *If for each* $n\in\mathbb{N}$, $W_n\subset X\times Y$ *denotes a finite union of generalised rectangles and* $Z_n := X\times Y \text{-} W_n$, *then we have that*

$$\lim_{n\to\infty}\int\limits_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n}(x,y) + N(y,q_n(y))\mathbb{1}_{Z_n}(x,y)\Big)d\mu\times\nu = \\ = \int\limits_X (Tu)(x)d\mu\,.$$

CLAIM 2. *We have that*

$$\lim_{n\to\infty} \int_{X\times Y} |K(x,y)|\, |N(y,p_n(y)) - N(y,q_n(y))| d\mu\times\nu = 0$$

and consequently the sequence $\Big(K(x,y)(N(y,p_n(y)) - N(y,q_n(y))\Big)_{n=1}^{\infty}$ *converges to* 0 *in* $\mu\times\nu$*-measure.*

PROOF OF CLAIM 1. Suppose that for every $n\in\mathbb{N}$, $W_n = \bigcup_{i=1}^{m(n)} A_{ni}\times B_{ni}$ where $A_{ni}\subset X$ and $B_{ni}\subset Y$. We may and will assume that for each $n\in\mathbb{N}$, the sets $(A_{ni})_{i=1}^{m(n)}$ and $(B_{ni})_{i=1}^{m(n)}$ are mutually disjoint. So,

$$Z_n = \left(\bigcup_{i=1}^{m(n)} A_{ni}\times(Y\text{-}B_{ni})\right)\cup\left(\left(X\text{-}\bigcup_{i=1}^{m(n)} A_{ni}\right)\times Y\right).$$

To apply Lemma 2.2 we need to put our limit in a suitable form, so that we make the following computation.

$$\int_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n}(x,y) + N(y,q_n(y))\mathbb{1}_{Z_n}(x,y)\Big)d\mu\times\nu =$$

$$= \int_{X\times Y} K(x,y)\Big(\sum_{i=1}^{m(n)}\Big[N(y,p_n(y))\mathbb{1}_{A_{ni}}(x)\mathbb{1}_{B_{ni}}(y) +$$

$$+ N(y,q_n(y))\mathbb{1}_{A_{ni}}(x)\mathbb{1}_{Y\text{-}B_{ni}}(y)\Big] + N(y,q_n(y))\mathbb{1}_{[X\text{-}\bigcup_{i=1}^{m(n)} A_{ni}]}(x)\Big)d\mu\times\nu =$$

$$= \int_X \Big(\sum_{i=1}^{m(n)} \mathbb{1}_{A_{ni}}(x)\int_Y K(x,y)N(y,p_n(y)\mathbb{1}_{B_{ni}}(y) + q_n(y)\mathbb{1}_{Y\text{-}B_{ni}}(y))d\nu +$$

$$+ \mathbb{1}_{[X\text{-}\bigcup_{i=1}^{m(n)} A_{ni}]}(x)\int_Y K(x,y)N(y,q_n(y))d\nu\Big)d\mu =$$

$$= \int_X \Big(\sum_{i=1}^{m(n)} \mathbb{1}_{A_{ni}}(x)T(p_n\mathbb{1}_{B_{ni}} + q_n\mathbb{1}_{Y\text{-}B_{ni}})(x) +$$

$$+ \mathbb{1}_{[X\text{-}\bigcup_{i=1}^{m(n)} A_{ni}]}(x)T(q_n)(x)\Big)d\mu.$$

It follows from $p_n(y) \longrightarrow u(y)$ and $q_n(y) \longrightarrow u(y)$ ν-a.e. that for every $\sigma = (\sigma(n))_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \{1, \dots, m(n)\}$, we have

$$p_n(y)\mathbb{I}_{B_{n\sigma(n)}}(y) + q_n(y)\mathbb{I}_{Y\text{-}B_{n\sigma(n)}}(y) \longrightarrow u(y)\, \nu - \text{a.e.}$$

Since this sequence is order bounded in E, we obtain that $T(p_n\mathbb{I}_{B_{n\sigma(n)}} + q_n\mathbb{I}_{Y\text{-}B_{n\sigma(n)}})(x) \longrightarrow Tu(x)$ μ-a.e. On the other hand, since the family $\{p_n\mathbb{I}_{B_{ni}} + q_n\mathbb{I}_{Y\text{-}B_{ni}} | i = 1, \dots, m(n); n \in \mathbb{N}\}$ is order bounded in E and the operator $f \longrightarrow N(\cdot, f(\cdot))$ is order bounded, it follows from Theorem A that the set

$$\left\{T(p_n\mathbb{I}_{B_{ni}} + q_n\mathbb{I}_{Y\text{-}B_{ni}}) | i = 1, \dots, m(n); n \in \mathbb{N}\right\}$$

is equimeasurable. Now, by Lemma 2.2, there is a decreasing sequence $(h_n)_{n=1}^{\infty}$ in $M(X, \mu)$ satisfying

(1) $|T(p_n\mathbb{I}_{B_{ni}} + q_n\mathbb{I}_{Y\text{-}B_{ni}})(x) - Tu(x)| \leq h_n(x)$ μ-a.e. for all i with $1 \leq i \leq m(n)$ and for all $n \in \mathbb{N}$.
(2) $\inf_{n\in\mathbb{N}} h_n(x) = 0$ μ-a.e.

We may suppose without loss of generality that $|Tq_n(x) - Tu(x)| \leq h_n(x)$ also hold μ-a.e. for all $n \in \mathbb{N}$ since $Tq_n(x) \longrightarrow Tu(x)$ μ-a.e. On the other hand, we may also assume that $h_n \leq 2h$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
&\Big| \int_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{I}_{W_n}(x,y) + N(y,q_n(y))\mathbb{I}_{Z_n}(x,y)\Big)d\mu\times\nu + \\
&- \int_X (Tu)(x)d\mu\Big| \leq \\
&\leq \int_X \Big(\sum_{i=1}^{m(n)} \mathbb{I}_{A_{ni}}(x)\Big|T(p_n\mathbb{I}_{B_{ni}} + q_n\mathbb{I}_{Y\text{-}B_{ni}})(x) - Tu(x)\Big| + \\
&+ \mathbb{I}_{[X\text{-}\bigcup_{i=1}^{m(n)} A_{ni}]}(x)\Big|T(q_n)(x) - Tu(x)\Big|\Big)d\mu \leq \\
&\leq \int_X \sum_{i=1}^{m(n)} \mathbb{I}_{A_{ni}}(x)h_n(x) + \mathbb{I}_{[X\text{-}\bigcup_{i=1}^{m(n)} A_{ni}]}(x)h_n(x)d\mu = \int_X h_n(x)d\mu.
\end{aligned}$$

Since $h_n \leq 2h$ for all $n \in \mathbb{N}$, h being a μ-integrable function, and $h_n(x) \longrightarrow 0$ μ-a.e., it follows from the dominated convergence theorem that $\lim_{n\to\infty} \int_X h_n(x)d\mu = 0$. Thus,

$$\lim_{n\to\infty} \int_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n}(x,y)+N(y,q_n(y))\mathbb{1}_{Z_n}(x,y)\Big)d\mu\times\nu=$$
$$=\int_X (Tu)(x)d\mu$$

as desired. □

PROOF OF CLAIM 2. Define $Z_n^+ := \Big\{(x,y)\in X\times Y \Big| K(x,y)\Big(N(y,p_n(y))-N(y,q_n(y))\Big) \geq 0\Big\}$ and $Z_n^- := X \times Y \text{-} Z_n^+$ for all $n \in \mathbb{N}$. Since the function M_s is $\mu\times\nu$-integrable, for every $\eta > 0$ there is $\zeta > 0$ such that $\mu\times\nu(W) < \zeta$ implies $\int_W M_s(x,y)d\mu\times\nu < \eta/4$. On the other hand, given the set Z_n^+ one may find $W_n^+ \subset X\times Y$, finite union of generalised rectangles, such that $\mu\times\nu(Z_n^+\Delta W_n^+) < \zeta$. Now define the sets $W_n^- := X\times Y\text{-}W_n^+$. It is immediate that $\mu\times\nu(Z_n^-\Delta W_n^-) < \zeta$.

To apply claim 1 compute

$$\int_{X\times Y} |K(x,y)|\,|N(y,p_n(y)) - N(y,q_n(y))|d\mu\times\nu =$$
$$= \int_{Z_n^+} K(x,y)\Big(N(y,p_n(y)) - N(y,q_n(y))\Big)d\mu\times\nu-$$
$$- \int_{Z_n^-} K(x,y)\Big(N(y,p_n(y)) - N(y,q_n(y))\Big)d\mu\times\nu \leq$$
$$\leq \Big| \int_{W_n^+} K(x,y)\Big(N(y,p_n(y)) - N(y,q_n(y))\Big)d\mu\times\nu-$$
$$- \int_{W_n^-} K(x,y)\Big(N(y,p_n(y)) - N(y,q_n(y))\Big)d\mu\times\nu\Big|+$$
$$+2\int_{W_n^-\Delta Z_n^+} M_s(x,y)d\mu\times\nu + 2\int_{W_n^-\Delta Z_n^-} M_s(x,y)d\mu\times\nu \leq$$

$$\leq \Big| \int\limits_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n^+}(x,y) + N(y,q_n(y))\mathbb{1}_{W_n^-}(x,y)\Big)d\mu\times\nu -$$
$$- \int\limits_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n^-}(x,y) + N(y,q_n(y))\mathbb{1}_{W_n^+}(x,y)\Big)d\mu\times\nu\Big| + \eta .$$

Now claim 1 yields

$$\lim_{n\to\infty}\int\limits_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n^+}(x,y)+N(y,q_n(y))\mathbb{1}_{W_n^-}(x,y)\Big)d\mu\times\nu = \\ = \int\limits_X (Tu)(x)d\mu$$

and

$$\lim_{n\to\infty}\int\limits_{X\times Y} K(x,y)\Big(N(y,p_n(y))\mathbb{1}_{W_n^-}(x,y)+N(y,q_n(y))\mathbb{1}_{W_n^+}(x,y)\Big)d\mu\times\nu = \\ = \int\limits_X (Tu)(x)d\mu .$$

Hence,

$$\lim_{n\to\infty}\int\limits_{X\times Y} \Big|K(x,y)\Big|\,\Big|N(y,p_n(y)) - N(y,q_n(y))\Big|d\mu\times\nu = 0$$

and so claim 2 is proved. □

Next, we finish the proof of theorem 2.3. Note that given $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ sequences in $[-s\mathbb{1}_Y, s\mathbb{1}_Y]\cap E_s$ such that $\lim\limits_{n\to\infty}\|p_n-q_n\|_\infty = 0$, to obtain the desired contradiction, claim 2 cannot be applied since we cannot assume that there is a function u as above. Nevertheless, we may proceed as in the proof of [4, theorem 2.3]. Using the regularity of the space $M(X\times Y,\mu\times\nu)$ (see [4], or [7,§71] and [15, chapter VI], we get a function $u\in E\cap[-s\mathbb{1}_Y, s\mathbb{1}_Y]$, which is not necessarily a member of E_s, and two sequences $(p_n')_{n=1}^{\infty}$, $(q_n')_{n=1}^{\infty}$ in $[-s\mathbb{1}_Y, s\mathbb{1}_Y]\cap E_s$ such that $u(y) = \lim\limits_{n\to\infty} p_n'(y)$ ν-a.e. and $u(y) = \lim\limits_{n\to\infty} q_n'(y)$ ν-a.e. Furthermore, we also have that the sequence $\Big(K(x,y)(N(y,p_n'(y)) - N(y,q_n'(y)))\Big)_{n=1}^{\infty}$ does not converge in $\mu \times \nu$-measure to 0. Claim 2 finally gives the required contradiction. □

By [8, corollary 6.4], every Hammerstein operator with range the space $M(X,\mu)$ is order bounded and, moreover, by [8, remark (R_3)] inevery order interval $[-f,f]$ in E, there exists $v \in [-f,f]$ such that $|K(x,y)|\ |N(y,g(y))| \leq |K(x,y)|\ |N(y,v(y)))|$ for all $g \in [-f,f]$. The above facts will be used in the following result. Let us observe that although the definition of Hammerstein operator in [8] is not exactly equal to that used in this paper, the same reasoning it was made in [8] also works in our case.

THEOREM 2.4. *For an orthogonally additive operator* $T\colon E \longrightarrow M(X,\mu)$ *the following assertions are equivalent.*

(1) *T is a Hammerstein operator.*

(2) (a) *T can be factorized as $T = L \circ N$, L being a regular linear operator and N being a projection commuting operator.*

(b) *If $(f_n)_{n=1}^{\infty}$ is an order bounded sequence in E such that $f_n \longrightarrow f(^*)$, then $Tf_n(x) \longrightarrow Tf(x)$ μ-a.e.*

(3) (a) *T can be factorized as $T = L \circ N$, L being a regular linear operator and N being a projection commuting operator.*

(b) *T maps order bounded sets in E onto equimeasurable sets.*

(c) *If $(f_n)_{n=1}^{\infty}$ is an order bounded sequence in E such that $f_n(y) \longrightarrow f(y)$ ν-a.e. then $Tf_n(x) \longrightarrow Tf(x)$ μ-a.e.*

PROOF. (1) $\iff$ (2). To prove this equivalence we essentially have to proceed as in the proof of [14, theorem 4.1] applying theorem 2.3 instead of [14, theorem 3.1] and keeping in mind the previous remark.
(1) $\Longrightarrow$ (3) (a). This is obvious.
(1) $\Longrightarrow$ (3) (b). Suppose that T is a Hammerstein operator with kernel $K(x,y)N(y,t)$ and decompose it as usual defining $L\colon \mathrm{Dom}_Y(K) \longrightarrow M(X,\mu)$ by $Lf(x) := \int_Y K(x,y)f(y)d\nu$ μ-a.e. and $N\colon E \longrightarrow \mathrm{Dom}_Y(K)$ by $Nf(y) := N(y,f(y))$ ν-a.e. Note that there is not loss of generality in assuming that T is positive (that is, for every $f \in E$ we have $Tf(x) \geq 0$ μ-a.e.; see also [8, proposition 5.3]) since we always can write a Hammerstein operator as a difference of two positive ones.

As a consequence of the previous remark we obtain that given $f \in E$, $f \geq 0$, it is possible to find $v \in [-f, f]$ such that $0 \leq K(x,y)N(y,g(y)) \leq K(x,y)N(y,v(y))$ for all $g \in [-f, f]$. Define $u(y) := N(y,v(y))$, let $g \in [-f, f]$ and consider the set $B := \left\{y \in Y | N(y,g(y)) \leq u(y)\right\}$. Then $K(x,y) = 0$ for $\mu \times \nu$-almost all $(x,y) \in X \times (Y \sim B)$ and consequently $T(g) = T(g\mathbb{1}_B) = L \circ N(g\mathbb{1}_B) \in L([0,u])$. Thus, $T([-f,f]) \subset L([0,u])$ and it follows from Theorem A that $T([-f,f])$ is an equimeasurable set.
(2) $\Longrightarrow$ (3) (c). This is immediate.
(3) $\Longrightarrow$ (2). Let $(f_n)_{n=1}^{\infty}$ be an order bounded sequence in E such that $f_n \longrightarrow f(^*)$. The condition (3) (c) implies that $Tf_n \longrightarrow Tf(^*)$ while it follows from the condition (3) (b) that the set $\{Tf_n | n \in \mathbb{N}\}$ is equimeasurable. Hence, $Tf_n(x) \longrightarrow Tf(x)$ μ-a.e. and so condition (2) (b) is proved. □

It is clear that the Drewnowski and Orlicz theorem must be a particular case of the above result. To see it we only have to decompose orthogonality additive functionals. Details are included in the following result. Condition (2) is the one proved by L. DREWNOWSKI and W. ORLICZ in [3], while condition (3) is similar to that generality used for linear functionals (see [17, theorem 86.3]).

PROPOSITION 2.5. *Let $\Phi: E \longrightarrow \mathbb{R}$ be an orthogonality additive functional. The following statements are equivalent.*

(1) *There exists a function $N: Y \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the Carathéodory conditions such that*

$$\Phi f = \int_Y N(y, f(y))d\nu \quad \textit{for every} \quad f \in E.$$

(2) *Given an order bounded sequence $(f_n)_{n=1}^{\infty}$ in E, $f_n \longrightarrow f(^*)$ implies $\Phi f_n \longrightarrow \Phi f$.*

(3) *Given an order bounded sequence $(f_n)_{n=1}^{\infty}$ in E, $f_n(y) \longrightarrow f(y)$ ν-a.e. implies $\Phi f_n \longrightarrow \Phi f$.*

PROOF. (1) $\Longrightarrow$ (3). This is a straightforward application of the dominated convergence theorem keeping in mind that every integral functional is order bounded [3, Lemma 2.2].

(3) $\Longrightarrow$ (2). This is easy to see by assuming the opposite and passing to subsequences to get a contradiction.

(2) $\Longrightarrow$ (1). As in the proof of Theorem 2.1, we may suppose that $\nu(Y) < \infty$ and $\mathbb{I}_Y \in E$ without loss of generality. Consider that X is a singleton (i.e., $X = \{x\}$ and so $M(X, \mu) = \mathbb{R}$). To apply our characterization of Hammerstein operators we only have to see that the functional Φ can be factorized. Fix $f \in E$. Since $\nu(B) = 0$ implies $\Phi(f\mathbb{I}_B) = 0$, by Radon-Nikodym theorem there is $Nf \in L^1(Y, \nu)$ such that $\Phi(f\mathbb{I}_B) = \int_B Nf\, d\nu$ for every $B \subset Y$. Thus, we have defined an operator $N\colon E \longrightarrow L^1(Y, \nu)$ which is projection commuting by [3, Lemma 3.1 (a)]. Now it is enough to define $L\colon L^1(Y, \nu) \longrightarrow \mathbb{R}$ by $Lg := \int_Y g d\nu$ for all $g \in L^1(Y, \nu)$ to factorize the functional as $\Phi = L \circ N$. Note that L is a positive linear operator, so that it is regular. □

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