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Extended Hermite interpolation with additional nodes and mean convergence of its derivatives

G. CRISCUOLO - B. DELLA VECCHIA -
G. MASTROIANNI(•)

RIASSUNTO – *Gli autori studiano la convergenza in media L^p delle derivate dell'interpolazione estesa di Hermite sugli zeri dei polinomi di Jacobi più nodi addizionali*

ABSTRACT – *The author study the weighted L^p convergence of derivatives of extended Hermite interpolation on the zeros of Jacobi polynomials plus additional points.*

KEY WORDS – *Hermite interpolation - Jacobi polynomials - Mean convergence.*

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1 – Introduction

The uniform and mean convergence properties of Hermite interpolating polynomials on the zeros of Jacobi polynomials were widely studied; interested reader should consult [14,15,20,22,23,25] and the references given within. In these papers the good behaviour of Hermite interpolating polynomials was proved only for particular matrices, substantially when the Jacobi parameters are less than 0.

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We ask if we can extend the previous convergence results to other interpolation matrices. Positive answers to this problem have been given in [2,4].

Very recently a collocation method for solving numerically singular integral equations was introduced in [12] and it was proved in [13] that the zeros of the polynomials $p_m^{(\alpha,-\alpha)}(x)p_m^{(-\alpha,\alpha)}(x)$, $-1 < \alpha < 1, \alpha \neq 0$, or $p_m^{(\alpha,1-\alpha)}(x)p_{m+1}^{(-\alpha,\alpha-1)}(x)$, $0 < \alpha < 1$ have an arcsin distribution. This property allowed to introduce the so called *extended matrices*

$$Y_1 = \{t_{k,2m}, k = 1, ..., 2m / t_{k,2m} \text{ zeros of } p_m^{(\alpha,-\alpha)}(x)p_m^{(-\alpha,\alpha)}(x)\}$$

$$Y_2 = \{t_{k,2m+1}, k = 1, ..., 2m+1 / t_{k,2m+1} \text{ zeros of } p_m^{(\alpha,1-\alpha)}(x)p_{m+1}^{(-\alpha,\alpha-1)}(x)\}$$

and to consider the corresponding *extended Hermite interpolating polynomial* interpolating the function f and f' at the zeros of Y_1 or Y_2. (See [6]).

Procedures of extended interpolation, of Lagrange, Hermite and Hermite-Fejér type, have been object of research in the last years [3, 5, 7-9, 21].

Extended interpolation turns out to be useful for the numerical evaluation of the interpolation error based on the zeros of orthogonal polynomials. Indeed, let $H_m(w^{(\alpha,-\alpha)}; f)$ be the polynomial interpolating f and f' at the zeros of $p_m(w^{(\alpha,-\alpha)})$ and $H_m(Y_1; f)$ be the Hermite polynomial interpolating f and f' at the nodes of the matrix Y_1 and suppose that $H_m(w^{(\alpha,-\alpha)}; f)$ and $H_m(Y_1; f)$ have the same order of convergence to f. In practice we assume the difference $|H_m(w^{(\alpha,-\alpha)}; f) - H_n(w^{(\alpha,-\alpha)}; f)|, n \geq m+1$, as the error of $H_m(w^{(\alpha,-\alpha)}; f)$. Therefore, if $n = m+1$, we need $4m+2$ evaluations of f and f', while, by using only $4m$ evaluations of f and f', we can compare $H_m(w^{(\alpha,-\alpha)}; f)$ by $H_m(Y_1; f)$, which is much more accurate.

It was proved [6] that by adding some additional nodes near the endpoints ± 1, the corresponding extended Hermite interpolating polynomial can approximate well in infinitely many ways a function and its derivatives simultaneously in uniform norm.

On the other hand, ERDÖS and TURAN in [10] proved that, for every nodes matrix, the Lebesgue constants of Hermite interpolating polynomial in uniform norm are greater or equal $O(log m)$ and then not bounded. Therefore it is more convenient to consider the convergence in weighted L^p norm, where the Lebesgue constants of Hermite interpolating polynomial introduced in [6] are bounded, as we prove in the present paper. In addition in Theorems 3.2-3.4 we give results about the weighted L^p convergence of the derivatives of the above Hermite interpolating polynomial. This interpolating process has the remarkable property that the convergence conditions are independent on the Jacobi parameter.

2 – Preliminaries and notations

Spaces of functions

We define C^q, L^p and $(L\log^+L)^p$ on the interval $[-1,1]$ in the usual way. Thus e.g. $f \in (L\log^+L)^p$, $0 < p \leq \infty$, if and only if

$$\|f\log^+|f|\|_p = \left\{\int_{-1}^{1} [|f(x)|\log^+|f(x)|]^p dx\right\}^{1/p} < \infty.$$

We recall that u is a *generalized Jacobi weight* ($u \in$GJ) if

$$u(x) = \phi(x)v^{(\alpha,\beta)}(x) = \phi(x)(1-x)^\alpha(1+x)^\beta, \quad -1 \leq x \leq 1, \quad \alpha, \beta > -1.$$

with ϕ nonnegative and $\phi^{\pm 1} \in L^\infty$. If in addition ϕ is continuous and its modulus of continuity ω satisfies $\int_0^1 \omega(\phi;t)t^{-1}dt < \infty$, then we say u is a generalized smooth Jacobi weight ($u \in GSJ$).

In the following we assume that

(2.1) $w^{(\alpha,-\alpha)}(x) = (1-x)^\alpha(1+x)^{-\alpha}, \quad -1 < \alpha < 1, \alpha \neq 0, -1 \leq x \leq 1,$

and we denote by $\{p_m^{(\alpha,-\alpha)}\}_{m=0}^\infty = \{p_m(w^{(\alpha,-\alpha)}\}_{m=0}^\infty$ the system of orthonormal polynomials corresponding to the Jacobi weight function $w^{(\alpha,-\alpha)}$, that is $p_m^{(\alpha,-\alpha)}$ is a polynomial of degree m with positive leading coefficient $\gamma_m(w^{(\alpha,-\alpha)})$ and $\int_{-1}^1 p_m^{(\alpha,-\alpha)}(x)p_n^{(\alpha,-\alpha)}(x)w(x)dx = \delta_{m,n}$. Then we denote by $\{x_{i,m}(w^{(\alpha,-\alpha)}\}_{i=1}^m = \{x_{i,m}\}_{i=1}^m$ the zeros of $p_m^{(\alpha,-\alpha)}$ labelled in increasing order.

Consider now the new weight $w^{-1}(x) = w^{(-\alpha,\alpha)}(x)$, and denote by $\{p_m^{(-\alpha,\alpha)}\}_{m=0}^\infty = \{p_m(w^{(-\alpha,\alpha)})\}_{m=0}^\infty$ the corresponding system of orthonormal polynomials. It is known that the zeros $x_{i,m}(w^{(-\alpha,\alpha)}) = x_{i,m}^*, i = 1, ..., m$, of $p_m(w^{(-\alpha,\alpha)})$ interlace with the zeros $x_{i,m}$ of $p_m^{(\alpha,-\alpha)}$, i.e.

$$x_{i,m} < x^*_{i,m}, i = 1, \dots, m, \text{ if } \alpha > 0$$

and

$$x^*_{i,m} < x_{i,m}, \quad i = 1, \dots, m, \text{ if } \alpha < 0$$

and they have an arcsin distribution. (See [13]).

3 – Main results

If f is a given differentiable function on $[-1,1]$, we denote by $H_m(Y_1; f)$ the corresponding extended Hermite interpolating polynomial on the zeros of $p_m^{(\alpha,-\alpha)} p_m^{(-\alpha,\alpha)}$, defined by

$$H_m^{(i)}(Y_1; f; x_{k,m}) = f^{(i)}(x_{k,m}), \quad i = 0, 1, \quad k = 1, \dots, m,$$

$$H_m^{(i)}(Y_1; f; x^*_{k,m}) = f^{(i)}(x^*_{k,m}), \quad i = 0, 1, \quad k = 1, \dots, m.$$

Together with the matrix Y_1, we can consider the following $2r$ additional points $y_{j,m} = -1 + \frac{j-1}{r}(1 + t_{1,2m})$, $j = 1, \dots, r$ and $z_{i,m} = t_{2m,2m} + \frac{i}{r}(1 - t_{2m,2m})$, $i = 1, \dots, r$, distributed on $[-1,1]$ as follows

$$-1 = y_{1,m} < \dots < y_{r,m} < t_{1,2m} < t_{2,2m} < \dots$$

$$\dots < t_{2m,2m} < z_{1,m} < \dots < z_{r,m} = 1.$$

Note that we can choose the additional nodes in many ways (see e.g. [2,3]); here we considered the equispaced case just for sake of simplification.

Then, we denote by $H_{m,r}(Y_1; f)$ the Hermite interpolating polynomial of degree $4m + 2r - 1$ interpolating f and f' on the nodes of the matrix Y_1 and on the additional points $y_{j,m}, j = 1, \dots, r$ and $z_{i,m}, i = 1, \dots, r$, defined by

$$H_{m,r}^{(i)}(Y_1; f; t_{k,2m}) = f^{(i)}(t_{k,2m}), \quad i = 0, 1, \quad k = 1, \dots, 2m,$$

$$\text{(3.1)} \qquad H_{m,r}(Y_1; f; y_{k,m}) = f(y_{k,m}), \quad k = 1, \dots, r,$$

$$H_{m,r}(Y_1; f; z_{k,m}) = f(z_{k,m}), \quad k = 1, \dots, r.$$

We complete the definition by putting $H_{m,0}(Y_1; f) = H_m(Y_1; f)$.

We will call $H_{m,r}(Y_1; f)$ the *extended Hermite interpolating polynomial.*

In [6] we proved that, by a suitable choise of the number of additional nodes, it is possible in infinitely many ways to approximate well a function f and its derivatives simultaneously by $H_{m,r}(Y_1; f)$. In addition error estimates optimal in some sense were given.

Now, denoting by $\mathcal{P}_n$ the set of algebraic polynomials of degree at most n, we let

$$E_n(f) = \min_{P \in \mathcal{P}_n} \|f - P\|, \quad f \in C,$$

where $\|.\|$ is the supremum norm on $[-1, 1]$.

For the Hermite polynomial $H_{m,r}(Y_1; f)$ defined by (3.1) we give some weighted L^p convergence theorems.

THEOREM 3.1. *Let w be the weight function defined by (2.1) and let $u \in (L\log^+ L)^p$, with $0 < p < \infty$.*
If

$$v^{(3/2-r,3/2-r)} \in L^1, \qquad uv^{(r-1,r-1)} \in L^p, \tag{3.2}$$

where r is a nonnegative integer, then for every function $f \in C^1$

$$\lim_{m\to\infty} \|[f - H_{m,r}(Y_1; f)]u\|_p = 0. \tag{3.3}$$

Furthermore

THEOREM 3.2. *Let $f \in C^q$, with $q \geq 1$. Let w be the weight function defined by (2.1). Assume $u \in GJ$ and $0 < p < \infty$.*
If

$$v^{(\frac{q}{2}-r+1,\frac{q}{2}-r+1)} \in L^1, \qquad u \in L^p, \qquad uv^{(r-\frac{\ell}{2}-1,r-\frac{\ell}{2}-1)} \in L^p, \tag{3.4}$$

where r and ℓ are nonnegative integers, with $\ell \leq q$, then, for $h = 0, ..., \ell$,

$$\left\|[f - H_{m,r}(Y_1; f)]^{(h)}u\right\|_p \leq \frac{\text{const}}{m^{q-h}} E_{m-q}(f^{(q)}), \tag{3.5}$$

with some constant independent of f and $m \geq 4q + 5$.

To complete the previous results, we remark that generally speaking the polynomial $H_m(Y_1; f)$ interpolating f and f' at the nodes of the matrix Y_1 does not converge to f. Theorem 3.1 assures that, when the hypotheses (3.2) are satisfied, then, by adding knots near the endpoints ± 1, we can obtain an interpolating polynomial realizing the L^p convergence to the given function f. In addition Theorem 3.2 garantees the simultaneous L^p approximation of the function and of its derivatives by extended Hermite interpolating polynomial.

From (3.2) and (3.4) it follows that, as in the uniform case [6], the number of additional nodes depends only on the order of differentiation ℓ and on the order of smoothness of f, q, but is independent on the Jacobi parameter α and there exist infinitely many good matrices for which (3.3) and (3.5) hold.

When the weight u is defined by

$$u(x) = v^{(\gamma,\gamma)}(x), \tag{3.6}$$

we can explicit the conditions (3.4). For instance, Theorem 3.2 implies

Corollary 3.3 . *Let $f \in C^q$, with $q \geq 1$. Let w and $u \in L^p$ be defined by* (2.1)*and* (3.6) *respectively. Let $0 \leq \ell \leq q$ and $0 < p < \infty$. Then, there exists an integer r, defined by*

$$\frac{\ell}{2} - \gamma - \frac{1}{p} + 1 < r < \frac{q}{2} + 2$$

such that, for $h = 0, ..., \ell$,

$$\left\{ \int_{-1}^{1} |f^{(h)}(x) - H_{m,r}^{(h)}(Y_1; f; x)|^p u^p(x) dx \right\}^{1/p} \leq \frac{\text{const}}{m^{q-h}} E_{m-q}(f^{(q)}),$$

with some constant independent of f and $m \geq 4q + 5$.

An useful consequence of Corollary 3.3 is the following

COROLLARY 3.4. *Let $f \in C^q$, with $q \geq 1$ and let $0 \leq \ell \leq q$. Let w and u be defined by* (2.1) *and* (3.6) *respectively. Then, there exists an integer r, defined by*

$$\frac{\ell}{2} - \gamma < r < \frac{q}{2} + 2$$

such that, for $h = 0, \ldots, \ell$,

$$\int_{-1}^{1} |f^{(h)}(x) - H_{m,r}^{(h)}(Y_1; f; x)| u(x) dx \le \frac{\text{const}}{m^{q-h}} E_{m-q}(f^{(q)}),$$

with some constant independent of f *and* $m \ge 4q + 5$.

The last corollary has interesting applications in quadrature processes, when we want to approximate integrals of the type $\int_{-1}^{1} f^{(q)}(x) u(x) dx$, $1 \le q < \infty$, $u \in GJ$, by an interpolatory product rule obtained by replacing f by an Hermite interpolating polynomial.

We remark that the same results as above can be obtained, if we consider the extended Hermite polynomial $H_{m,r}(Y_1; f)$ on the nodes of the matrix Y_2.

4 – Proofs of the main results

We assume in the following

$$\mu(x) = \phi(x) v^{(\gamma,\delta)}(x) \in \text{GJ}, \tag{4.1}$$

and denote by $x_{i,m}(\mu)$, $i = 1, 2, \ldots, m$ the zeros of the m–th orthonormal polynomial $p_m(\mu)$ corresponding to the weight μ and by $\lambda_{i,m}(\mu) = \lambda_m(\mu; x_{i,m}(\mu))$, $i = 1, 2, \ldots, m$, the Cotes numbers, where

$$\lambda_m(\mu; x) = \left[\sum_{i=0}^{m-1} p_i^2(\mu; x)\right]^{-1},$$

is the m–th Christoffel function.

For the convenience of the reader, we provide a collection of properties of generalized Jacobi polynomials $p_m(\mu)$ which will be applied in the sequel. Let $\mu \in$GJ and set $x_{i,m}(\mu) = \cos\theta_{i,m}$ for $0 \le i \le m + 1$ where $x_{0,m}(\mu) = -1$, $x_{m+1,m}(\mu) = 1$ and $0 \le \theta_{i,m} \le \pi$. Then

$$\theta_{i,m} - \theta_{i+1,m} \sim m^{-1}, \tag{4.2}$$

uniformly for $0 \le i \le m$, $m \in N$. (See [16, Theorem 9.22, p.166].)
Let $\mu \in$GJ and let μ be given by (4.1). Then

$$\lambda_{i,m}(\mu) \sim m^{-1}(1 - x_{i,m}(\mu))^{\gamma+\frac{1}{2}}(1 + x_{i,m}(\mu))^{\delta+\frac{1}{2}}, \tag{4.3}$$

uniformly for $1 \le i \le m$, $m \in N$. (See [16, Theorem 6.3.28, p.120].)

If in addition $\phi \in Lip_M 1$, then

$$\lambda'_m(u; x_{i,m}(u) \le \text{const}\ m^{-1}(1 - x_{i,m}(u))^{\gamma-1/2}(1 + x_{i,m}(u))^{\delta-1/2}, \tag{4.4}$$

uniformly for $0 \le i \le m$, $m \in N$ (see [19, Lemma 2, p. 36]). Moreover

$$|p_m(\mu; x)| \le \text{const}(\sqrt{1-x} + m^{-1})^{-\gamma-\frac{1}{2}}(\sqrt{1+x} + m^{-1})^{-\delta-\frac{1}{2}}, \tag{4.5}$$

uniformly for $-1 \le x \le 1$ and $m \in N$ (see [1, Theorem 1.1, p.226]), in particular,

$$|p_m(\mu; x)| \sim m^{\gamma+1/2} \sim p_m(\mu; 1), \quad 1 - m^{-2} \le x \le 1, \tag{4.6}$$

and

$$|p_m(\mu; x)| \sim m^{\delta+1/2} \sim |p_m(\mu; -1)|, \quad -1 \le x \le -1 + m^{-2}, \tag{4.7}$$

uniformly for $m \in N$. (See also [19]).

Let $\mu \in$GJ and $0 < p < \infty$. If c is a fixed positive number and v is an arbitrary, not necessarily integrable, Jacobi weight, then for every polynomial Q of degree at most cm

$$\sum_{i=1}^{m} |Q(x_{i,m}(\mu)|^p v(x_{i,m}(\mu))\lambda_{i,m}(\mu) \le \text{const} \int_{-1}^{1} |Q(t)|^p v(t)\mu(t)dt.$$

(See [16, Theorem 9.25, p.168].)

If μ is given by (4.1) then for any fixed $c > 0$ we define $\Delta_m(c)$ by

$$\Delta_m(c) = [-1 + cm^{-2}, 1 - cm^{-2}],$$

and let 1_m^c denote the characteristic function of $\Delta_m(c)$. Then there exists a $\bar{c} > 0$ such that for every polynomial Q of degree at most m

$$\||Q|^p\mu\|_1 \le \text{const}\, \||Q|^p\mu 1_m^{\bar{c}}\|_1\,, \tag{4.8}$$

(cf. Theorem 6.3.28 and Remark 6.3.29 in [16].)

Now we introduce the polynomials

$$A_0(x) = 1, \quad A_r(x) = \prod_{j=1}^{r}(x - y_{j,m}), \quad r > 0,$$

$$B_0(x) = 1, \quad B_r(x) = \prod_{j=1}^{r}(x - z_{j,m}), \quad r > 0.$$

Denoting by $L_n(V;h)$ the Lagrange polynomial of degree $n-1$ interpolating the bounded function h on the nodes of the matrix V, we can write

$$L_r(Z; \frac{f}{A_r Q_m}; x) = \frac{f(z_{1,m})}{A_r(z_{1,m})Q_m(z_{1,m})} + \tag{4.9}$$

$$+ \sum_{i=2}^{r}(x - z_{1,m})(x - z_{2,m})...(x - z_{i-1,m})\left[z_{1,m}, z_{2,m}, ..., z_{i,m}; \frac{f}{A_r Q_m}\right],$$

$$L_r(Y; \frac{f}{B_r Q_m}; x) = \frac{f(y_{1,m})}{B_r(y_{1,m})Q_m(y_{1,m})} + \tag{4.10}$$

$$+ \sum_{i=2}^{r}(x - y_{1,m})(x - y_{2,m})...(x - y_{i-1,m})\left[y_{1,m}, y_{2,m}, ..., y_{i,m}; \frac{f}{B_r Q_m}\right],$$

with

$$Q_m(x) = [p_m^{(\alpha,-\alpha)}(x)p_m^{(-\alpha,\alpha)}(x)]^2. \tag{4.11}$$

Here $[u_1, ..., u_p; h]$ is the divided difference of the function h at the points $u_1, ..., u_p$.

The following lemmas are needed to prove the results stated in the previous section.

LEMMA 4.1. (Telyakovskii-Gopengauz)[11,24] *Let $f \in C^q$. Then for $m \geq 4q+5$ there exists a sequence of polynomials $\{G_m\}$ such that for $|x| \leq 1$ and for $j = 0, 1, ..., q$*

$$|f^{(j)}(x) - G_m^{(j)}(x)| \leq \text{const}[\frac{\sqrt{1-x^2}}{m}]^{q-j}\omega(f^{(q)}; \frac{\sqrt{1-x^2}}{m}),$$

with some constant independent of f and m.

LEMMA 4.2. *For every $f \in C^q$, there exists a polynomial P_m of degree $m \geq 4q+5$, such that*

$$|f^{(i)}(x) - P_m^{(i)}(x)| \leq const\ [\frac{\sqrt{1-x^2}}{m}]^{q-i} E_{m-q}(f^{(q)}),$$

with $|x| \leq 1, i = 0, ..., q$ and for some constant independent of f and m.

PROOF. Let g_m be an algebraic polynomial of degree $m > q$, such that $\|f^{(q)} - g_m^{(q)}\| \leq E_{m-q}(f^{(q)})$. From Lemma 4.1 there exists a polynomial G_m, $m \geq 4q+5$, such that

$$|(f-g_m)^{(i)}(x) - G_m^{(i)}(x)| \leq \text{const}\ [\frac{\sqrt{1-x^2}}{m}]^{q-i} \omega((f-g_m)^{(q)}; \frac{1}{m}) \leq$$

$$\leq \text{const}\ [\frac{\sqrt{1-x^2}}{m}]^{q-i} \|(f-g_m)^{(q)}\| \leq \text{const}\ [\frac{\sqrt{1-x^2}}{m}]^{q-i} E_{m-q}(f^{(q)}),$$

from which the assertion follows, for $P_m = g_m + G_m$. □

Denoting by $r_m = f - P_m$ the remainder term, we have the following

LEMMA 4.3. *Let w be the weight function defined by* (2.1) *and let $L_r(Z), L_r(Y)$ and Q_m be the polynomials defined by* (4.9), (4.10) *and* (4.11) *respectively. For every function $f \in C^q, q \geq 1$, if $v^{(\frac{\alpha}{2}-r+1, \frac{\beta}{2}-r+1)} \in L^1$ then*

(4.12)
$$|L_r(Z; \frac{r_m}{A_r Q_m}; x)| \leq \frac{\text{const}}{m^q} E_{m-q}(f^{(q)}) \Big(\sqrt{1-x} + \frac{1}{m}\Big)^{q+2}, \quad |x| \leq 1,$$

(4.13)
$$|L_r(Z; \frac{r_m}{B_r Q_m}; x)| \leq \frac{\text{const}}{m^q} E_{m-q}(f^{(q)}) \Big(\sqrt{1+x} + \frac{1}{m}\Big)^{q+2}, \quad |x| \leq 1,$$

with some constant independent of m.

PROOF. To prove (4.12), firstly assume that the points $z_{i,m}, i = 1, \dots, r$ are all coincident. From (4.6) it follows

$$|Q_m(x)| \sim m^2, \quad z_{1,m} \le x \le 1.$$

Thus, by Markov-Bernstein inequality

$$|Q'_m(x)| \le \text{const}\, m^2 \parallel Q_m \parallel_{[z_{1,m},1]} \sim m^2 Q_m(1), \quad z_{1,m} \le x \le 1,$$

and

$$\Big| \Big[\frac{1}{Q_m(x)}\Big]' \Big| = \Big| \frac{Q'_m(x)}{Q^2_m(x)} \Big| \sim \frac{m^2}{Q_m(1)}, \quad z_{1,m} \le x \le 1.$$

In view of the last inequality and taking into account that

$$\Big[\frac{1}{Q_m(x)}\Big]^{(l)} = \frac{1}{Q_m(x)} \sum_{j=0}^{l-1} \binom{l}{j} \Big[\frac{1}{Q_m(x)}\Big]^{(j)} Q_m^{(l-j)}(x),$$

(see [18]), we deduce

$$\Big| \Big[\frac{1}{Q_m(x)}\Big]^{(l)} \Big| \le \text{const} \frac{m^{2l}}{Q_m(1)}, \quad z_{1,m} \le x \le 1. \tag{4.14}$$

Now, we recall that

$$[z_{1,m}, z_{2,m}, \dots, z_{i,m}; \frac{r_m}{A_r Q_m}] = \frac{1}{(i-1)!} \Big[\frac{r_m}{A_r(x) Q_m(x)}\Big]^{(i-1)}_{x=\xi_i},$$
$$z_{1,m} \le \xi_i \le z_{i,m}.$$

So, by Leibniz formula

$$\Big|\Big[z_{1,m}, z_{2,m}, \dots, z_{i,m}; \frac{r_m}{A_r Q_m}\Big]\Big| \le$$
$$\le \frac{1}{(i-1)!} \sum_{l=0}^{i-1} \binom{i-1}{l} \Big|\Big[\frac{r_m}{Q_m(x)}\Big]^{(l)}_{x=\xi_i}\Big| \, \Big|\Big[\frac{1}{A_r(x)}\Big]^{(i-1-l)}_{x=\xi_i}\Big| =$$
$$= \frac{1}{(i-1)!} \sum_{l=0}^{i-1} \binom{i-1}{l} \sum_{j=0}^{l} \binom{l}{j} \, | \, r_m^{(j)}(\xi_i) \, | \cdot$$
$$\cdot \Big|\Big[\frac{1}{Q_m(x)}\Big]^{(l-j)}_{x=\xi_i}\Big| \Big|\Big[\frac{1}{A_r(x)}\Big]^{(i-1-l)}_{x=\xi_i}\Big|.$$

On the other hand, by Lemma 4.2

$$|r_m^{(l)}(z_{k,m})| \leq \text{const}\ m^{-2(q-l)} E_{m-q}(f^{(q)}), \quad l = 0, ..., q, \quad k = 1, ..., r.$$

Since the function $1/A_r(x)$ and its derivatives are bounded for $x > 0$, by (4.14)

$$| \left[z_{1,m}, z_{2,m}, \dots, z_{i,m}; \frac{r_m}{A_r Q_m}\right] | \leq$$

$$\leq \text{const} \frac{E_{m-q}(f^{(q)})}{m^{2q} Q_m(1)} \sum_{l=0}^{i-1} \binom{i-1}{l} m^{2l} \sim \frac{E_{m-q}(f^{(q)}) m^{2i-2}}{Q_m(1) m^{2q}}.$$

Recalling (4.9) and since $(x - z_{1,m})(x - z_{2,m}) ... (x - z_{i-1,m}) \leq (\sqrt{1-x} + m^{-1})^{2i-2}$ for $|x| \leq 1$, we deduce

(4.15)
$$| L_r(Z; \frac{r_m}{A_r Q_m}; x) | \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{Q_m(1) m^{2q}} \sum_{i=0}^{r-1} [m\sqrt{1-x} + 1]^{2i}, \quad |x| \leq 1.$$

At first, assume that $|x| \leq 1 - m^{-2}$; then by (4.15)

$$| L_r(Z; \frac{r_m}{A_r Q_m}; x) | \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{Q_m(1) m^{2q-2r+2}} (1-x)^{r-1}.$$

Recalling that $Q_m(1) \sim m^2$, we obtain

$$\text{(4.16)} \quad | L_r(Z; \frac{r_m}{A_r Q_m}; x) | \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^{2(q-r+2)}} (1-x)^{r-1}, \quad |x| \leq 1 - m^{-2}.$$

The hypothesis $r \leq q/2 + 2$ assures that $q/2 - r + 2 \geq 0$. Then, since $m^{-2} \leq 1 - x^2$, from (4.16) we get

$$|L_r(Z; r_m/A_r Q_m; x)| \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q} (\sqrt{1-x})^{q+2}$$

that is (4.12). If $|x| \geq 1 - m^{-2}$, then the inequality (4.12) follows immediately by (4.15).

If the nodes are all simple, then, firstly assume that $|x| \leq 1 - dm^{-2}$, with d a positive constant. Then it results

$$|A_r(x)| \sim (1+x)^r$$

If we write the polynomial $L_r(Z; \frac{r_m}{A_r Q_m}; x)$ in the Lagrange form, we get

$$L_r(Z; \frac{r_m}{A_r Q_m}; x) = \sum_{k=1}^{r} \prod_{i \neq k} \frac{x - z_{i,m}}{z_{k,m} - z_{i,m}} \frac{r_m(z_{i,m})}{A_r(z_{i,m}) Q_m(z_{i,m})}.$$

By (3.2) and (3.3) we have

$$|A_r^{-1}(z_{i,m})| \leq \text{const} , \quad i = 1, ..., r.$$

On the other hand

$$\Big| \prod_{i \neq k} \frac{x - z_{i,m}}{z_{k,m} - z_{i,m}} \Big| \leq m^{2r-2} (\sqrt{1-x} + m^{-1})^{2r-2} \leq m^{2r-2} (1-x)^{r-1},$$
$$|x| \leq 1 - dm^{-2}.$$

and working similarly as above, the assertion follows. Analogously we can prove (4.13). □

LEMMA 4.4. [8] *Let $1 < p < \infty$, $0 < c \leq 1$, $\mu \in GSJ$ and $\phi \in GJ$. Let A be a polynomial of degree $\ell(m-1)$, with ℓ positive integer, such that $|A(x) p_m(\mu; x)| \leq \phi(x)$ for $x \in (-1,1)$ and $m = 1, 2...$ Given nonnegative integers r and s, and a function $u \in L^1$, if $v^{(r,s)}\mu \in L^1, \phi u \in L^p$ and $\phi u v^{(r,s)} \mu \in (L\log^+ L)^p$ then*

$$\sum_{i=1}^{m} \lambda_{i,m}(\mu) v^{(r,s)}(x_{i,m}(\mu)) \Big| \int_{-1}^{1} 1_m^c(x) F^{p-1}(x) u(x) \frac{A(x) p_m(\mu; x)}{x - x_{i,m}(\mu)} dx \Big| \leq$$

$$\leq \text{const} \, \|1_m^c F\|_p^{p-1} , \quad m = 1, 2...,$$

for every function $F \geq 0$ such that $F \in (L\log^+ L)^p$ with some constant independent of m and F.

Finally we recall that, if u is any weight function defined by (4.1), then, for every $x \in [-1, 1]$,

$$\sum_{\substack{k=1 \\ k\neq j^*}}^{m} \frac{(1 \pm x_{k,m}(u))^\rho}{m^2(x - x_{k,m}(u))^2} \leq (\sqrt{1 \pm x} + \frac{1}{m})^{2\rho-2} + \frac{1}{m}, \tag{4.17}$$

where j^* denotes the index corresponding to the closest knot(s) to x and $\rho \geq 0$ is a real number. The proof of this inequality follows directly from [16, Lemma 9, p. 109].

PROOF OF THE THEOREM 3.2. Let P_m be the m–th polynomial of the Lemma 4.2 corresponding to the function $f \in C^q, q \geq 1$ and let $r_m = f - P_m$ be the remainder term. For $0 < p < \infty$ and $h = 0, 1, \ldots, \ell$, we have

$$\left\|[f - H_{m,r}(Y_1; f)]^{(h)}u\right\|_p \leq \text{const}\left\{\left\|r_m^{(h)}u\right\|_p + \left\|H_{m,r}^{(h)}(Y_1; r_m)u\right\|_p\right\}.$$

Since $u \in$GJ, we can write u in the form

$$u(x) = \bar{\phi}(x)v^{(a,b)}(x), \qquad \bar{\phi}^{\pm} \in L^\infty.$$

By [17, Theorem 5, p. 242] and (4.8), there is a number $0 < c^* \leq 1$ such that

$$\begin{aligned}\left\|[f - H_{m,r}(Y_1; f)]^{(h)}u\right\|_p \leq \text{const}\Big\{ &\frac{E_{m-q}(f^{(q)})}{m^{q-h}}\, \|u\|_p \\ &+ \left\|H_{m,r}(Y_1; r_m)v^{(-\frac{h}{2},-\frac{h}{2})}u1_m^{c^*}\right\|_p m^h\Big\},\end{aligned}$$

where $1_m^{c^*}$ denotes the characteristic function of the set $\Delta_m(c^*)$. We can assume that c^* is sufficiently small. More precisely, we will assume that $|A_r(x)| \sim (1 + x)^r$ and $|B_r(x)| \sim (1 - x)^r$, for $|x| \leq 1 - c^* m^{-2}$.

From the definition one has that $H_{m,r}(Y_1; f)$ can be written as follows

(4.18)

$$\begin{aligned} H_{m,r}(Y_1; f; x) = {} & (A_r B_r)(x) p_m^2(w^{(\alpha,-\alpha)}; x) \cdot \\ & \cdot H_m(w^{(-\alpha,\alpha)}; \frac{r_m}{A_r B_r p_m^2(w^{(\alpha,-\alpha)})}; x) + \\ & + (A_r B_r)(x) p_m^2(w^{(-\alpha,\alpha)}; x) H_m(w^{(\alpha,-\alpha)}; \frac{r_m}{A_r B_r p_m^2(w^{(-\alpha,\alpha)})}; x) + \\ & + A_r(x) Q_m(x) L_r(Z; \frac{f}{A_r Q_m}; x) + B_r(x) Q_m(x) L_r(Y; \frac{f}{B_r Q_m}; x), \end{aligned}$$

where again $Q_m = [p_m^{(\alpha,-\alpha)} p_m^{(-\alpha,\alpha)}]^2$. Hence

$$\begin{aligned} \| \quad & H_{m,r}(Y_1; f) v^{(-\frac{h}{2},-\frac{h}{2})} u 1_m^{c^*} \|_p \le \\ & \le const \Big\{ \| p_m^2(w^{(\alpha,-\alpha)} H_m\Big(w^{(-\alpha,\alpha)}; \frac{r_m}{A_r B_r p_m^2(w^{(\alpha,-\alpha)})}\Big) v^{(r-\frac{h}{2},r-\frac{h}{2})} u 1_m^{c^*} \|_p + \\ & + \| p_m^2(w^{(-\alpha,\alpha)}) H_m\Big(w^{(\alpha,-\alpha)}; \frac{r_m}{A_r B_r p_m^2(w^{(-\alpha,\alpha)})}\Big) v^{(r-\frac{h}{2},r-\frac{h}{2})} u 1_m^{c^*} \|_p + \\ & + \Big\| Q_m v^{(-\frac{h}{2},r-\frac{h}{2})} L_r\Big(Z; \frac{r_m}{A_r Q_m}\Big) u 1_m^{c^*} \Big\|_p + \\ & + \Big\| Q_m v^{(r-\frac{h}{2},-\frac{h}{2})} L_r\Big(Y; \frac{r_m}{B_r Q_m}\Big) u 1_m^{c^*} \Big\|_p \Big\} := \\ & := \text{const } [I_1 + I_2 + I_3 + I_4]. \end{aligned}$$

To bound I_3, we recall that $v^{(\frac{q}{2}-r+1,\frac{q}{2}-r+1)} \in L^1$. Thus, from Lemma 4.3, by (4.5)

$$|1_m^{c^*}(x) Q_m(x)| \le \text{const } [v^{(1,1)}(x)]^{-1},$$

and

$$\begin{aligned} I_3 & \le \text{const} \frac{E_{m-q}(f^{(q)})}{m^q} \Big\| v^{(\frac{q}{2}-\frac{h}{2}+1,r-\frac{h}{2})} v^{(-1,-1)} u \Big\|_p \\ & \le \text{const} \frac{E_{m-q}(f^{(q)})}{m^q} \Big\| v^{(0,r-\frac{h}{2}-1)} u \Big\|_p \qquad (4.20) \\ & \le \text{const} \frac{E_{m-q}(f^{(q)})}{m^q}. \end{aligned}$$

Similarly,

$$I_4 \leq \text{const}\frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.21}$$

In order to find a bound for I_1, first we assume $1 < p < \infty$. Then, from the definition of Hermite interpolating polynomial, we can write [6]

$$H_m(w^{(-\alpha,\alpha)}; \frac{r_m}{A_r B_r p_m^2(w^{(\alpha,-\alpha)})}; x) =$$

$$= c_m p_m^2(w^{(-\alpha,\alpha)}; x)[\sum_{i=1}^{m} C_i r_m(x_{i,m}^*) + \sum_{i=1}^{m} C_i' r_m'(x_{i,m}^*)] :=$$

$$:= c_m p_m^2(w^{(-\alpha,\alpha)}; x)[\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5],$$

where

$$c_m = \frac{\gamma_{m-1}^4(w^{(-\alpha,\alpha)})\pi^2}{\gamma_m^4(w^{(-\alpha,\alpha)}) sin^2 \pi\alpha} < \infty,$$

$$C_i = \Big\{1 + (x - x_{i,m}^*)\Big[\frac{\lambda_{i,m}'(w^{(-\alpha,\alpha)}; x_{i,m}^*)}{\lambda_{i,m}(w^{(-\alpha,\alpha)})} + \\ - \frac{(A_r B_r)'(x_{i,m}^*)}{(A_r B_r)(x_{i,m}^*)} - 2\frac{p_m'(w^{(\alpha,-\alpha)}; x_{i,m}^*)}{p_m(w^{(\alpha,-\alpha)}; x_{i,m}^*)}\Big]\Big\} \times \\ \times \frac{\lambda_{i,m}^2(w^{(-\alpha,\alpha)}) p_{m-1}^4(w^{(-\alpha,\alpha)}; x_{i,m}^*)}{(x - x_{i,m}^*)^2 (A_r B_r)(x_{i,m}^*)},$$

$$C_i' = \frac{\lambda_{i,m}^2(w^{(-\alpha,\alpha)})}{(x - x_{i,m}^*)(A_r B_r)(x_{i,m}^*)}.$$

Letting $\bar{u} = v^{(r-\frac{\ell}{2}, r-\frac{\ell}{2})}u$, we get

$$S_5 := ||Q_m \Sigma_5 v^{(r-\ell/2, r-\ell/2)} u 1_m^{c^*}||_p \leq ||Q_m \Sigma_5 \bar{u} 1_m^{c^*}||_p := T_1.$$

Set $F_m = Q_m \Sigma_5$ and $\Psi = \text{sgn} F_m$. Recalling Lemma 4.2, by (4.3) we can write

$$T_1^p \leq \text{const}\frac{E_{m-q}(f^{(q)})}{m^q} \sum_{i=1}^{m} \lambda_{i,m}(w^{(-\alpha,\alpha)}) v^{(\frac{\ell}{2}-r+\alpha+1, \frac{\ell}{2}-r+\beta+1)}(x_{i,m}^*) \times \\ \times \left| \int_{-1}^{1} \Psi(x) |F_m(x)\bar{u}(x)|^{p-1} \bar{u}(x) 1_m^{c^*}(x) \frac{Q_m(x)}{x - x_{i,m}^*} dx \right|.$$

Thus, by Lemma 4.4

$$T_1 \leq \text{const}\frac{E_{m-q}(f^{(q)})}{m^q},$$

and therefore

$$S_5 \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.22}$$

Analogously, since $\dfrac{|p_m'(w^{(\alpha,-\alpha)};x^*_{i,m})|}{|p_m(w^{(\alpha,-\alpha)};x^*_{i,m})|} \leq m(1-x^{*2}_{i,m})^{-1/2}$, (cf. e.g. [3]), we get,

$$S_4 := ||Q_m\Sigma_4 v^{(r-h/2,r-h/2)}u1^{c^*}_m||_p \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.23}$$

Similarly, since

$$\frac{|(A_rB_r)'(x^*_k)|}{|(A_rB_r)(x^*_k)|} \leq \text{const}\ (1-x^{*2}_k)^{-1}$$

it results

$$S_3 := ||Q_m\Sigma_3 v^{(r-\ell/2,r-\ell/2)}u1^{c^*}_m||_p \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.24}$$

Then, by (4.4), we can prove

$$S_2 := ||Q_m\Sigma_2 v^{(r-h/2,r-h/2)}u1^{c^*}_m||_p \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.25}$$

To find a bound for $S_1 := ||Q_m\Sigma_1 v^{(r-\ell/2,r-\ell/2)}u1^{c^*}_m||_p$, by (4.3) and (4.17), we get

$$\Sigma_1 \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}[(\sqrt{1-x}+1/m)^{q+2-2r}(\sqrt{1+x}+1/m)^{q+2-2r}+$$

$$+\sum_{k\neq j^*}\frac{(1-x_k^2)^{q/2+2-r}}{m^2(x-x_k)^2}] \leq \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}\times$$

$$\times\{(\sqrt{1-x}+1/m)^{q+2-2r}[(\sqrt{1+x}+1/m)^{q+2-2r}+1/m]+$$
$$+(\sqrt{1+x}+1/m)^{q+2-2r}[(\sqrt{1-x}+1/m)^{q+2-2r}+1/m]\},$$

where j^* denotes the index corresponding to the closest knot to x. Hence

$$S_1 \le \text{const}\ \frac{E_{m-q}}{m^q}\|v^{(r-\ell/2-1,r-\ell/2-1)}[v^{(q/2+1-r,q/2+1-r)}+m^{-1}v^{(q/2+1-r,0)}+$$
$$+m^{-1}v^{(0,q/2+1-r)}]u\|_p$$

and by the assumptions, it follows

$$S_1 \le \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.26}$$

Working similarly we get

$$I_2 \le \text{const}\ \frac{E_{m-q}(f^{(q)})}{m^q}. \tag{4.27}$$

From (4.19)-(4.27) and the last considerations, we deduce (3.5) for $p > 1$. In the case $0 < p \le 1$, the inequality (3.5) can be proved following a procedure used in [18]. □

PROOF OF THE THEOREM 3.1 To prove Theorem 3.2 we needed a Markov-Bernstein type inequality. This is the reason for the assumption $u \in GJ$, which is necessary for such an inequality. When the derivatives of the function f do not need to be approximated, the condition on u can be relaxed and it is sufficient to assume $u \in (L\log^+ L)^p$. Then, following the proof of the Theorem 3.2 with $q = 1$, one can prove (3.3). □

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INDIRIZZO DEGLI AUTORI:

Giuliana Criscuolo - Biancamaria Della Vecchia - Istituto per Applicazioni della Matematica, C.N.R. - Via P.Castellino 111 - 80131 Napoli - Italy

Giuseppe Mastroianni - Dipartimento di Matematica - Università della Basilicata - Via N.Sauro 85 - 85100 Potenza - Italy; *and* Istituto per Applicazioni della Matematica, C.N.R. - Via P.Castellino 111 - 80131 Napoli - Italy