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Solutions of the Boussinesq equation generated by weak symmetries

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RIASSUNTO – *Si determinano due famiglie a tre parametri di soluzioni dell'equazione di Boussinesq. Queste soluzioni sono ottenute come soluzioni invarianti rispetto a gruppi di simmetria puntuale deboli che non appartengono alla classe di simmetria secondo Bluman e Cole. Una di queste famiglie non rientra tra le soluzioni ottenute da Clarkson e Kruskal a da Lou, con metodi diretti di riduzione.*

ABSTRACT – *We determine two families of* ∞^3 *solutions for the Boussinesq equation. They are invariant with respect to weak symmetry groups that do not belong to the class of Bluman and Cole's non-classical symmetries. One of these families is not obtainable using the methods of direct similarity reduction.*

KEY WORDS – *Boussinesq - Weak symmetries - Invariant solutions.*

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1 – Introduction

The Lie symmetry groups of a partial differential equation are continuous groups of point transformations, sending all solutions of the equation into other solutions of the same equation. Besides such classical groups (each one uniquely determined by a Lie algebra of vector fields, called "generators" of the group), other groups may be considered: they are called weak symmetry groups and are groups of point symmetries, under whose action a non-empty subclass of solutions is invariant.

The "non-classical" groups, introduced by BLUMAN and COLE [1], and some other groups, whose existence for the heat equation has been notified by OLVER and ROSENAU [2], are examples of this second kind of symmetry groups.

In [3], it is defined the procedure to characterize all weak symmetry groups that are admitted by a given partial differential equation. For any of those groups, it is possible to determine the corresponding family of invariant solutions: they are solutions of the invariance equation of the group and, at the same time, of the partial differential equation.

In this note, we determine two families of solutions S_w of the Boussinesq equation (BE), obtained as invariant solutions associated to weak symmetry groups, which are not "non-classical". A class of these solutions cannot be obtained using the methods of direct similarity reduction, proposed by CLARKSON and KRUSKAL [4] and by LOU [6].

In fact, in [5] we evidenced that, for any partial differential equation, the family of invariant solutions obtained by direct method (let us call it S_{CKL}) is included in the family of the invariant solutions, which correspond to non-classical symmetries (let us call it S_{NC}).

For the Boussinesq equation, all solutions of the family S_{CKL} have been characterized in [4] and [6]; in [7], LEVI and WINTERNITZ found the "non-classical" groups, to which are associated the solutions in S_{NC} that match the list of solutions in S_{CKL} given in [4]. The other solutions in S_{CKL}, given in [6], are also invariant solutions under the other non-classical groups, and here we give the explicit expression of all such group. In this way, we also show that, for the Boussinesq equation, the two families of solutions S_{NC} and S_{CKL} coincide.

A class of S_W is also enclosed in S_{NC}, and precisely is a solution in S_{CKL} given in [6]. The other class is not enclosed in S_{NC}, therefore it is not possible to find this class via the direct similarity reduction.

2 – Classical and weak symmetries for a partial differential equation

Let

$$\Delta(x, t, u, u_x, u_t, \dots) = 0 \tag{2.1}$$

be a differential equation of order n in $u(x,t)$, with $(x,t) \in X \subset \mathbb{R}^2$ and $u \in \mathsf{U} \subset \mathbb{R}^1$. Let us call $\mathsf{U}^{(n)}$ the space of all derivatives of u up to order n and assume that $\Delta \in C^\infty(X \times \mathsf{U} \times \mathsf{U}^{(n)})$. With any continuous group of point transformations, a vector field

$$\mathbf{v} \equiv (\xi(x,t,u), \tau(x,t,u), \eta(x,t,u))$$

is associated, together with an operator

$$pr^{(0)} \equiv \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}$$

and an invariance equation

$$I \equiv \eta - \xi u_x - \tau u_t = 0\,. \tag{2.2}$$

As it is well-known, the vector fields $\mathbf{v}$, associated to Lie symmetry groups, are defined by the linear and homogeneous system obtained by setting identically equal to zero,

$$pr^{(n)}\Delta\Big|_{\Delta=0} = 0\,. \tag{2.3}$$

In (2.3), $pr^{(n)}$ indicates the n-th prolonged operator of $\mathbf{v}$, which is proved to be dependent on ξ, η, τ and all their derivatives up to order n (for the explicit expression see [8]).

Instead, the vector fields $\mathbf{v}$, associated with non-classical symmetry groups (weak symmetry groups at the first step [3]) are defined by the system obtained by setting identically equal to zero

$$pr^{(n)}\Delta\Big|_{\Sigma=0} = 0 \tag{2.4}$$

on the submanifold $\Sigma = 0$, defined by (2.1), (2.2) and by the differential consequences of (2.2), up to order n.

As it has been proved in [3], (2.4) is equivalent to the condition that the system of equations (2.1) and (2.2) is complete, that is, such that any differential consequence is also an algebraic consequence.

There exists only one independent differential consequence, that is, the compatibility condition which implies the existence of solutions for both of the equations (2.1) and (2.2).

If we do not set (2.4) identically equal to zero (that is, if we do not take (2.4) as completeness condition), but, on the contrary, we consider it as another equation on $u(x,t)$ besides (2.1) and (2.2), we obtain a new system of equations, whose completeness condition is equivalent to set identically zero the following expression

$$pr_v^{(n)}\left[pr_v^{(n)}\Delta\Big|_{\Sigma=0}\right]_{\Sigma^*=0}=0 \tag{2.5}$$

on the manifold $\Sigma^*=0$, defined by $\Sigma=0$ and the equation (2.4).

This condition still defines a system of equations on ξ,τ,η, and its solutions are weak symmetry groups at the second step.

The procedure can be repeated n times, and gives other weak symmetries up to the n-th step, the last possible step [3].

Clearly, any Lie symmetry and any weak symmetry at the k-th step is also a weak symmetry at the $(k+h)$-th step, for any $1\le h\le n-k$.

After determining the differential systems, which define the vector fields of weak symmetries, it is possible to recover the invariant class of solutions for any single group we are able to find.

In the following, the partial derivatives will be denoted with the variable in subscript, and the ordinary derivatives with a supercscripted comma; any parameter, appearing during the exposition, is to be considered as arbitrary constant.

3 – Lie groups and non-classical groups for the Boussinesq Equation

Let us consider the Boussinesq equation

$$E\equiv u_{tt}+u_{xxxx}+uu_{xx}+(u_x)^2=0 \tag{3.1}$$

It is known [7] that the generators of the Lie groups are

$$\xi=ax+b,\quad \tau=2at+2c,\quad \eta=-2au. \tag{3.2}$$

The generators of the non-classical groups that have been determined in

[7], are the following:

$$\begin{aligned}\xi &= \alpha(t)x + \beta(t),\\ \tau &= 1,\\ \eta &= -\Big[2\alpha u + 2\alpha(\alpha' + 2\alpha)x^2 + 2(\alpha\beta' + \alpha'\beta + 4\alpha^2\beta)x +\\ &\quad + 2\beta(\beta' + 2\alpha\beta)\Big],\end{aligned} \tag{3.3}$$

where α and β are solutions of

$$\alpha'' + 2\alpha\alpha' - 4\alpha^3 = 0 \quad \text{and} \quad \beta'' + 2\alpha\beta - 4\alpha^2\beta = 0. \tag{3.4}$$

Besides (3.3), there exist other non-classical symmetry groups, which are defined by the generators

$$\xi = 1, \quad \tau = 0, \quad \eta = \gamma(x)u + \delta(x,t), \tag{3.5}$$

where $\gamma(x)$ and $\delta(x,t)$ are solutions of the system

$$\gamma'' + 5\gamma\gamma' + 2\gamma^3 = 0 \tag{3.6}$$

$$\delta_{xx} + 3\gamma\delta_x + (5\gamma' + 4\gamma^2)\delta + 30\gamma\gamma'^2 - 5\gamma^3\gamma' - 10\gamma^5 = 0, \tag{3.7}$$

$$\begin{aligned}&\delta_{xxxx} + 4\gamma'\delta_{xx} + (3\delta - 26\gamma\gamma' - 12\gamma^3)\delta_x + \delta_{tt} +\\ &+ 2\gamma\delta^2 - 12\gamma'^2 + 50\gamma^2\gamma' + 28\gamma^4 = 0.\end{aligned} \tag{3.8}$$

The (3.6) equation may be written using the differential operator $D = \left(\frac{d}{dx} + \gamma\right)$ as

$$D\{\gamma' + 2\gamma^2\} = 0.$$

In this form, it is not hard to find all the solutions:

s_1) $\gamma(x) = 0$;

s_2) $\gamma(x) = (2x + g_0)^{-1}$ (non trivial solution of $\gamma' + 2\gamma^2 = 0$)

s_3) $\gamma(f(x)) = f\left(\frac{f^6 g_1 + 2}{3}\right)^{1/2}$, where $f(x)$ is defined by

$$x + g_2 = 2\sqrt{3}\int f^{-2}(f^6 g_1 + 2)^{-1/2} df.$$

It reduces to $\gamma(x) = 2(x + g_2)^{-1}$ for $g_1 = 0$.

The (3.7) equation admits the following general integral:

$s_1^{\#}$) $\delta(x,t) = a_1(t)x + a_2(t)$ (corresponding to s_1);

$s_2^{\#}$) $\delta(x,t) = a_1(t)(2x + g_0) + a_2(t)(2x + g_0)^{-3/2} - 5(2x + g_0)^{-3}$ (corresponding to s_2);

$s_3^{\#}$) $\delta(x,t) = a_1(t)\gamma e^{-\int \gamma dx} + a_2(t)(1 - 2x\gamma)e^{-\int \gamma dx} + 3\gamma\gamma' + \gamma^3$ (corresponding to s_3).

It reduces to

$$\delta(x,t) = a_1(t)(x + g_2)^{-3} + a_2(t)(x + g_2)^{-2} \quad \text{when} \quad g_1 = 0.$$

The $a_1(t)$ and $a_2(t)$ are arbitrary functions of t.

The compatibility with (3.8) exists only in the following three cases, corresponding to which (3.5) defines all possible non-classical symmetries with $\tau = 0$:

c_1) $\gamma(x) = 0$ $\delta(x,t) = a_1(t)x + a_2(t)$ with $a_1'' + 3a_1^2 = 0$ and $a_2'' + 3a_1 a_2 = 0$;

c_2) $\gamma(x) = 2(x + g_2)^{-1}$, $\delta(x) = 48(x + g_2)^{-3}$;

c_3) $\gamma(x) = 2(x + g_2)^{-1}$, $\delta(x) = 0$.

The invariant solutions which correspond to Lie groups are defined by

$$u = \frac{w(z)}{at + c} \quad \text{with} \quad z = \frac{ax + b}{a\sqrt{2at + 2c}},$$

and $w(z)$ solution of

$$w'''' + a^2 z^2 w'' + 2ww'' + 7a^2 z w' + 2w'^2 + 8a^2 w = 0.$$

The invariant solutions, corresponding to the non-classical groups, having (3.3) as generators, are defined by [7]

$$u = w(z)k^2(t) - (\alpha x + \beta)^2 \quad z = xk(t) - \int \beta(t)k(t)dt,$$

where $k(t) = \exp[-\int \alpha(t)dt]$ and $w(z)$ is solution of

$$w'''' + ww'' + w'^2 + (AZ + B)w' + 2Aw - 2(Az + B)^2 = 0.$$

Here, A and B are constant parameters defined by:

$$A = \frac{\alpha^2 - \alpha'}{k^4}, \qquad B = \frac{\alpha\beta - \beta'}{k^3} + \frac{\alpha^2 - \alpha'}{k^4} \int \beta(t)k(t)dt.$$

These solutions coincide with those obtained using the direct reduction method by C.K. [4].

The invariant solutions, which correspond to the non-classical groups of generators (3.5), are defined by:

$$u = \beta_1(x)w(t) + \alpha_1(x,t)$$

with $\beta_1(x) = \exp(\int \gamma dx)$, $\alpha_1(x,t) = \beta_1 \int \delta\beta_1^{-1}dx$ and $w(t)$ solution of

$$w'' + A_1 w^2 + B_1 w + B_2 = 0$$

where A_1 is the constant parameter $A_1 = (\beta_1'^2 + \beta_1\beta_1'')/\beta_1$ and B_1 and B_2 are the following functions of t

$$B_1(t) = \left(\beta_1'''' + 2\frac{\partial \alpha_1}{\partial x}\beta_1'' + \alpha\beta_1'' + \frac{\partial^2 \alpha_1}{\partial x^2}\beta_1\right),$$

$$B_2(t) = \frac{1}{\beta_1}\left(\frac{\partial^2 \alpha_1}{\partial t^2} + \alpha_1\frac{\partial^2 \alpha_1}{\partial x^2} + \left(\frac{\partial \alpha_1}{\partial x^2}\right)^2 + \frac{\partial^4 \alpha_1}{\partial x^4}\right)$$

It is easy to recognize that such a class of solutions coincides with the class given by direct reduction in [6].

4 – Other symmetry groups and corresponding invariant solutions

The method described in section 2, applied to the Boussinesq equation, allows to characterize the particular system of partial differential equations, which defines the generators of weak symmetries.

Since the Boussinesq equation is an equation of the fourth order, besides the defining system of non-classical weak symmetries (whose solutions have been given in the previous section), there are three more systems which define generators of other weak symmetries. They correspond to the remaining possible steps to which one can impose the completeness of the system (3.1) and (2.2).

Those three systems may be obtained, fairly easily, by means of symbolic manipulation programs, by they are extremely long and involved be analyzed and, therefore, not so meaningful.

It is more useful to look for solutions of those systems, which are obtained using simplifying hypothesis on the form of the generators.

In the present section, we determine the generators for a few weak symmetry groups and compute the corresponding invariant solutions. The program we used in MACSYMA running on a "Symbolics 3620".

We assume, for simplicity,

$$\xi = \xi(x,t), \quad \tau = 1, \quad \eta = \eta(x,t). \tag{4.1}$$

The equation (2.4), for the Boussinesq equation, becomes of the form

$$\begin{aligned} E_1 \equiv & - 6\xi_{xx}u_{xxx} + u_{xx}(-4\xi_{xxx} + 4\xi^2\xi_x + 2u\xi_x + 2\xi\xi_t + \eta) + \\ & 2\xi_x u_x^2 + u_x(4\xi\xi_x^2 - u\xi_{xx} - 2\xi_t\xi_x - \xi_{tt} - \xi_{xxxx} + 2\eta_x) - \\ & - 4\xi\eta_x\xi_x + 4\eta_t\xi_x - 2\eta_x\xi_t + u\eta_{xx} + \eta_{xxxx} + \eta_{tt} = 0. \end{aligned} \tag{4.2}$$

We consider this one as new equation to be associated with (3.1) and (2.2).

The compatibility condition is now

$$pr^{(n)}E_1\Big|_{\substack{E=0\\ I=0\\ E_1=0}} \equiv pr^{(n)}\left[pr^{(n)}E\Big|_{\substack{E=0\\ I=0}}\right]\Bigg|_{\substack{E=0\\ I=0\\ E_1=0}} = 0. \tag{4.3}$$

Setting this relation identically zero, one obtains the defining system for weak symmetries at the second step, with generators of the kind (4.1).

Explicit computations show that, in order (4.3) to be true, it is necessary that

$$\xi = \alpha_2(t)x + \beta_2(t). \tag{4.4}$$

Using this form for ξ, (4.2) and (4.3) simplify a lot. The system of equations on $\alpha_2(t), \beta_2(t), \eta(x,t)$ that one obtains is still hard to be analyzed. In particular, it admits solutions if α_2, β_2 and η solve the system

$$\alpha_2' - 2\alpha_2^2 = 0 \tag{4.5}$$

$$\eta_t + (\alpha_2 x + \beta_2)\eta_2 + \gamma_0 x^2 + \gamma_1 x + \gamma_2 = 0 \tag{4.6}$$

$$\begin{aligned}(\text{where} \quad \gamma_0 &= 48\alpha_2^4;\\ \gamma_1 &= 2\alpha_2\beta_2'' + 14\alpha_2^2\beta_2' + 52\alpha_2^3\beta_2;\\ \gamma_2 &= 2\beta_2\beta_2'' + 2\beta_2'^2 + 6\alpha_2\beta_2\beta_2' + 12\alpha_2^2\beta_2^2)\end{aligned}$$

$$\left(\eta_t + (\alpha_2 x + \beta_2)\eta_x\right)_{xx} = 0 \tag{4.7}$$

$$\begin{aligned}&(2\alpha_2\eta_{xx} - 48\alpha_2^4)x + 2\beta_2\eta_{xx} + 2\alpha_2\eta_x + 2\eta_{xt} -\\ &-\beta_2'' - \alpha_2\beta_2'' + 2\alpha_2^2\beta_2' + 4\alpha_2^3\beta_2 = 0\end{aligned} \tag{4.8}$$

$$\begin{aligned}&(-48\alpha_2^3\eta_x + \alpha_2\eta_{xtt} - 4\alpha_2^2\eta_{xt})x - 4\alpha_2\eta_{xxxx} + 2\eta\eta_{xx} +\\ &+2\eta_x^2 - (3\beta_2'' + 6\alpha_2\beta_2' + 12\alpha_2^2\beta_2)\eta_x + \eta_{ttt} +\\ &+\beta_2\eta_{xtt} + 4\alpha_2\eta_{tt} - 2\beta_2'\eta_{xt} + 8\alpha_2^2\eta_t = 0\end{aligned} \tag{4.9}$$

which is compatible only if $\alpha_2 = 0$.

It can be checked that the follwing are solutions of the system (4.5), ..., (4.9):

$$\text{I)}\quad \alpha_2 = 0$$

$$\beta_2 = b_0 t^2 + b_1 t + b_2$$

$$\eta = -\frac{1}{3}(6b_0 x + 10b_0^2 t^3 + 15b_0 b_1 t^2 + $$

$$+ 6(b_0 b_2 + b_1^2)t + 6b_1 b_2);$$

and

$$\text{II)}\quad \alpha_2 = 0$$

$$\beta_2 = b_0 t^2 + b_1 t + b_2$$

$$\eta = 6b_0 x - 6b_0^2 t^3 - 9b_0 b_1 t^2 -$$

$$- 2(5b_0 b_2 + b_1^2)t - 2b_1 b_2.$$

By comparison with (3.3), it can be seen that, for $b_0 \neq 0$, the I) and II) give generators of weak symmetries which are not "non-classical".

Related to these generators we find the similarity variable

$$z = x - (2b_0 t^3 + 3b_1 t^2 + 6b_2 t)/6; \tag{4.10}$$

the invariant solutions are, respectively, of the form

$$u_I = -2b_0 tz - b_0^2 t^4 - 2b_0 b_1 t^3 - (2b_0 b_2 + b_1^2)t^2 - 2b_1 b_2 t + w_1(z) \tag{4.11}$$

and

$$u_{II} = 6b_0 tz - b_0^2 t^4 - 2b_0 b_1 t^3 - (2b_0 b_2 + b_1^2)t^2 - 2b_1 b_2 t + w_2(z) \tag{4.12}$$

Since we are not dealing with "non-classical" weak symmetries, by substituting (4.11) and (4.12) in the Boussinesq equation, we do not obtain ordinary differential equations in $w_1(z)$ or $w_2(z)$.

In fact, we obtain

(4.13)
$$t(-2b_0 w_1'' z - 6b_0 w_1' - 6b_0 b_1) + w''''_1 + (w_1 + b_2^2)w_1'' + w_1'^2 - b_1 w_1' - 2b_1^2 = 0$$

and

$$t(6b_0w_2''z+10b_0w_2'-30b_0b_1)+w'''_2+(w_2+b_2^2)w_2''+ \tag{4.14}$$
$$+w_2'^2-b_1w_2'-16b_0b_2-2b^2=0$$

Nevertheless, each of them is compatible at any t if the following conditions hold: for the (4.13), if

$$w_1(z)=-\frac{12}{z^2}-b_1z-b_2^2 \tag{4.15}$$

for any b_0, b_1, b_2; for the (4.14), if

$$w_2(z)=3b_1z+b_3 \tag{4.16}$$

for any b_3 and b_0, b_1, b_2 such that $4b_0b_2-b_1=0$.

If we substitute (4.15) in (4.11), by expressing z as function of x and t via (4.10), we obtain the class of solutions:

$$S_w^I: \qquad u(x,t)=-\frac{p_1(t)x^3+p_2(t)x^2+p_3(t)x+p_4(t)}{q_1x^3+q_2(t)x^2+q_3(t)x}$$

where

$$p_1(t)=+432b_0t+216b_1$$

$$p_2(t)=-216b_0^2t^4-432b_0b_1t^3+(-864b_0b_2-108b_1^2)t^2+ \\ -216b_1b_2t+216b_2^2$$

$$p_3(t)=+(144b_0^2b_2-36b_0b_1^2)t^5+(144b_0b_1b_2-54b_1^3)t^4+ \\ +(288b_0b_262-216b_1^2b_2)t^3-432b_1b_2^2t^2-432b_2^3t$$

$$p_4(t)=+8b_0^4t^{10}+40b_0^3b_1t^9+(48b_0^3b_2+78b_0^2b_1^2)7^8+ \\ +(192b_0^2b_1b_2+72b_0b_1^3)t^7+(96b_0^2b_2^2+288b_0b_1^2b_2+27b_1^4)t^6+ \\ +360b_0b_1b_2^2++162b_1^3b_2)t^5+(144b_0b_2^3+378b_1^2b_2^2)t^4+ \\ +432b_1b_2^3t^3+216b_2^4t^2+2592$$

and

$$q_1 \quad = + 216$$

$$q_2(t) = - 144 b_0 t^3 - 216 b_1 t^2 - 432 b_2 t$$

$$q_3(t) = = 24 b_0^2 t^6 + 72 b_0 b_1 t^5 + (144 b_0 b_2 + 54 b_1^2) t^4 + 216 b_1 b_2 t^3 + 216 b_2^2 t^2$$

Likewise, substituting (4.16) in (4.12), we obtain the class of solutions:

$$S_w^{II}: \qquad u(x,t) = \frac{r_1(t)x - r_2(t))}{4b_0}$$

where

$$r_1(t) = 24 b_0^2 t + 12 b_0 b_1$$

$$r_2(t) = 12 b_0^3 t^4 + 24 b_0^2 b_1 t^3 + 18 b_0 b_1^2 t^2 + 5 b_1^3 t - 4 b_0 b_3$$

Each one of these classes depends on three arbitrary parameters.

The S_w^I are solutions not invariant under any non-classical group, and therefore cannot be obtained using the direct reduction methods.

The S_w^{II} are solutions also invariant under the non-classical symmetry (3.5) in the c_1-case ($a_1 = 0, a_2 = 6b_0 t + 3b_1$); therefore it is possible to find, as particular case, this family in [6].

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