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Some natural operations on vector fields

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RIASSUNTO – *Per ogni funtore H viene determinata una biiezione tra le fibre di H su $0 \in \mathbb{R}$ e l'insieme delle trasformazioni naturali che trasformano campi vettoriali in sezioni di H. Vengono descritte esplicitamente tutte le trasformazioni, a rilevamento naturale, della base, che trasformano campi vettoriali su varietà n-dimensionali in campi vettoriali sul funtore lineare fibrato tangente di ordine r sulle varietà n-dimensionali, purché risulti $n \geq 2$.*

ABSTRACT – *For every bundle functor H we determine a bijection between the fibre of H over $0 \in \mathbb{R}$ and the set of all natural transformations transforming vector fields into sections of H. We describe explicitely all natural base-extending transformations transforming vector fields on n-manifolds into vector fields on the linear tangent bundle functor of order r over n-manifolds, provided $n \geq 2$.*

KEY WORDS – *Natural bundle - Bundle functor - Natural transformation - natural base-extending transformation.*

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– Introduction

A natural bundle was introduced by A. NIJENHUIS [12], as a modern approach to the classical theory of geometrical objects. According to him geometrical objects on a manifold M are sections of some natural bundle FM over M.

Let G be a natural bundle. In differential geometry we studied many geometric (canonical) constructions transforming geometrical objects on

M into geometrical objects on GM. Geometrical objects on M are sections of a natural bundle $FM \to M$ and geometrical objects on GM are sections of a natural bundle $H(GM) \to GM$. If we denote by ΓFM the set of sections of $FM \to M$, then a construction (transformation)

$$\mathcal{D}_M : \Gamma FM \to \Gamma H(GM)$$

is called "natural" or "canonical" if for any embedding $\varphi : M \to N$ and sections $\sigma_1 \in \Gamma FM$, $\sigma_2 \in \Gamma FN$ which are φ-conjugate (i.e. $F\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$) the sections $\mathcal{D}_M(\sigma_1)$ and $\mathcal{D}_N(\sigma_2)$ are $G\varphi$-conjugate, c.f. [8]. The precise definitions will be recalled in Sections 2 and 3.

For example, the construction associating to any metric tensor on M its curvature tensor is natural. (Here FM is the natural bundle of metric tensors on M, $GM = M$ and HM is the natural bundle of tensor fields of type $(0,4)$.) Natural constructions with $GM = M$ have been studied by many authors, cf. [1],[3],[6],[14] etc.

In the case of a vector fibre bundle GM, J. GANCARZEWICZ [4], constructed for any $\sigma \in \Gamma GM$ a canonical vertical vector field σ^V on GM called the vertical lift of σ by the formula $\sigma^V(y) = \psi_y(\sigma(\pi(y)))$, where $\psi_y : G_{\pi(y)}M \to V_yGM = T_y(G_{\pi(y)}M)$ is the canonical isomorphism.

Many of "natural" constructions were studied in the case of a Weil functor G. G is called a Weil functor if $G(M_1 \times M_2) = G(M_1) \times G(M_2)$, cf. [6]. For example, I. KOLAR [7], gives a complete characterization of natural transformations of vector fields on M (i.e. $FM = TM$) into vector fields on GM (i.e. $H(GM) = T(GM)$), where G is a Weil functor.

In this paper we describe (completely) how for any vector field X on M one can construct canonically a vector field $\mathcal{D}_M(X)$ on the linear r-th order tangent bundle $T^rM = (J^r(M,\mathbb{R})_0)^*$ (i.e. we deal with canonical constructions, where $FM = TM$, $GM = T^rM$ and $H(GM) = T(T^rM)$). If G is a natural bundle then for any vector field X on M we can define a vector field GX on GM (called the complete lift of X to GM) via prolongations of local flows. If φ_t is a local flow of X, then $G\varphi_t$ is a local flow of GX. We prove that in the case of $GM = T^rM$ of vector tangent bundle of order r any natural (base-extending) transformation $\mathcal{D}$ of vector fields from M to vector fields on T^rM can be written in the form

$$(*) \qquad \mathcal{D}_M(X) = \lambda T^rX + \mu L_M + (D_M(X))^V,$$

if $\dim(M) \geq 2$, where $\lambda, \mu \in \mathbb{R}$, L_M is the Liouville vector field on T^rM defined by the fibre homotheties, $D_M(X) \in \Gamma T^rM$ is a canonical (with respect to X) section and $(D_M(X))^V$ denotes the vertical lift of $D_M(X)$. Moreover in Section 2 we define r canonical sections $\overset{(s)}{D}_M(X) \in \Gamma T^rM$, $s = 1, ..., r$, and we prove that $D_M(X)$ is a linear combination of the $\overset{(s)}{D}_M(X)$.

The main difficulty in proving the formula (*) is to show Lemmas 4.1 and 5.1. It seems that our methods also work to prove some form of the formula (*) for many other bundle functors G.

Section 2 can be read independently. In Section 2 we describe how for any vector field X on M (i.e. $FM = TM$) one can construct canonically a geometrical object $D_M(X) \in \Gamma HM$ on M (i.e. $GM = M$), where H is a bundle functor defined on the category of manifolds with smooth maps. We determine a bijection between the fibre of H over $0 \in \mathbb{R}$ and the set of all such constructions.

1 – Bundle functors

All manifolds in this paper are assumed to be paracompact, without boundary, second countable, finite dimensional and of class C^∞. Maps between manifolds will be assumed to be C^∞. Let $\mathcal{M}$ be the category of all manifolds and all maps, $\mathcal{FM}$ be the category of all fibered manifolds and their morphisms and $B : \mathcal{FM} \to \mathcal{M}$ be the base functor. Given a functor $H : \mathcal{M} \to \mathcal{FM}$ satisfying $B \circ H = \mathrm{id}_{\mathcal{M}}$, we denote by $p^H_M : HM \to M$ its value on a manifold M and by $H_xf : H_xM \to H_{f(x)}N$ the restriction of its value $Hf : HM \to HN$ in $f : M \to N$ to the fibres of HM over x and of HN over $f(x)$, $x \in M$.

DEFINITION 1.1. *([9]) A bundle functor is a functor $H : \mathcal{M} \to \mathcal{FM}$ satisfying $B \circ H = \mathrm{id}_{\mathcal{M}}$ and the localization condition: if $i : U \to M$ is the inclusion of an open subset, then $Hi : HU \to (p^H_M)^{-1}(U)$ is a diffeomorphism. We say that a bundle functor H is linear if $H : \mathcal{M} \to \mathcal{VB}$, where $\mathcal{VB} \subset \mathcal{FM}$ is the category of all vector bundles and their vector bundle morphisms.*

REMARK. If we replace (in Definition 1.1) $\mathcal{M}$ by the category $\mathcal{M}_n$ of all n-dimensional manifolds and their embeddings we obtain the concept of natural bundles over n-manifolds in the sense of A. NIJENHUIS [12]. Hence the restriction of a bundle functor to $\mathcal{M}_n$ is a natural bundle over n-manifolds.

EXAMPLE 1.1. Let $r \geq 1$ be an integer. Let $T^{r*}M = J^r(M, \mathbb{R})_0$ be the space of all r-jets of a manifold M into $\mathbb{R}$ with target 0. Since $\mathbb{R}$ is a vector space, $T^{r*}M$ has a canonical structure of a vector bundle over M. The dual vector bundle $T^rM := (T^{r*}M)^*$ is called the r-th order tangent bundle of M. Given a map $f : M \to N$, the jet composition $V \to V \circ j^r_x f$, $V \in T^{r*}_{f(x)}N$, determines a linear map $T^{r*}_{f(x)}N \to T^{r*}_x M$. The dual map $T^r_x M \to T^r_{f(x)}N$ is denoted by $T^r_x f$ and called the r-th order tangent map of f at x. This defines a functor $T^r : \mathcal{M} \to \mathcal{VB}$. Of course, T^r is a linear bundle functor. Functor T^1 is naturally isomorphic with the tangent bundle functor T. (The functor T associates to each manifold M the tangent bundle TM of M and to each map $f : M \to N$ the differential map $df : TM \to TN$ of f.) A natural isomorphism $I_M : TM \to T^1M$, $M \in \mathcal{M}$, is given by $I_M(v)(j^1_x\gamma) = v\gamma$, $v \in T_xM$, $j^1_x\gamma \in T^{1*}_x M$, $x \in M$.

The Weil functors of A-velocities, cf. [11], are also bundle functors.

Let M, N, P be manifolds. A parametrized family of maps $f_p : M \to N$, $p \in P$ is said to be smoothly parametrized if the resulting map $f : M \times P \to N$ is of class C^∞. We have the following proposition.

PROPOSITION 1.1. *([9]) Every bundle functor $H : \mathcal{M} \to \mathcal{FM}$ satisfies the regularity condition: if $f : M \times P \to N$ is a smoothly parametrized family, then the family $H(f_p) : HM \to HN$, $p \in P$, is also smoothly parametrized.*

Let us recall that a natural transformation between two natural bundles H_1, $H_2 : \mathcal{M}_n \to \mathcal{FM}$ is a family of maps $A_M : H_1M \to H_2M$, $M \in \mathcal{M}_n$, such that (a) for every embedding $\varphi : M \to N$ of two n-manifolds $H_2\varphi \circ A_M = A_N \circ H_1\varphi$, and (b) for every $M \in \mathcal{M}_n$ $p^{H_2}_M \circ A_M = p^{H_1}_M$. Since $T^r|\mathcal{M}_n$ is a functor in the category $\mathcal{VB}$, for every $k \in \mathbb{R}$ the homotheties

$$(1.1) \qquad (k)^r_M : T^rM \to T^rM, \qquad X \to kX$$

represent natural transformations of $T^r|\mathcal{M}_n$ into itself. In [10], I. KOLAR and G. VOSMANSKA proved the following proposition.

PROPOSITION 1.2. *Let $A_M : T^r M \to T^r M$, $M \in \mathcal{M}_n$, be a natural transformation of $T^r|\mathcal{M}_n$ into itself. Then there exists $k \in \mathbb{R}$ such that $A_M = (k)^r_M$ for all $M \in \mathcal{M}_n$.*

From Proposition 1.2 we obtain the following corollary.

COROLLARY 1.1. *If $r \geq 2$, then each natural transformation A_M : $T^r M \to TM$, $M \in \mathcal{M}_n$, is given by $A_M(w) = 0$.*

PROOF. We have a natural injection $i_M : T^1 M \to T^r M$, $M \in \mathcal{M}_n$ given by $i_M(w)(j^r_x\gamma) = w(j^1_x\gamma)$, $w \in T^1_x M$, $j^r_x\gamma \in T^{r*}_x M$, $x \in M$. Let $I_M : TM \to T^1 M$, $M \in \mathcal{M}_n$ be the natural isomorphism described in Example 1.1. Suppose that $A_M : T^r M \to TM$, $M \in \mathcal{M}_n$, is a natural transformation. Then $i_M \circ I_M \circ A_M$ is a natural transformation of $T^r|\mathcal{M}_n$ into itself. If $r \geq 2$, then $i_M \circ I_M \circ A_M$ is not surjective, and then (by Proposition 1.2) it is equal to $(0)^r_M$. Therefore $A_M(w) = 0$ for every $w \in T^r M$. □

2 – Natural transformations transforming vector fields into sections of some natural bundles

Let $H : \mathcal{M}_n \to \mathcal{FM}$ be a natural bundle. For every $M \in \mathcal{M}_n$ we denote by $\mathcal{X}(M)$ the vector space of all vector fields on M and by ΓHM the set of all sections of class C^∞ of the bundle $HM \to M$. We introduce the following definition.

DEFINITION 2.1. *A family $D = \{D_M\}$ of functions*

$$D_M : \mathcal{X}(M) \to \Gamma HM, \quad M \in \mathcal{M}_n,$$

is called a natural transformation transforming vector fields into sections of H iff the following naturality condition is satisfied: for every $M, N \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$ and every embedding $\varphi : M \to N$ the assumption $d\varphi \circ X = Y \circ \varphi$ implies $H\varphi \circ D_M(X) = D_N(Y) \circ \varphi$.

REMARK. (a)Every natural transformation D transforming vector fields into sections of H satisfies the following localization condition: if $X, Y \in \mathcal{X}(M)$, $M \in \mathcal{M}_n$, are two vector fields such that $X|U = Y|U$ on an open subset, then $D_M(X)|U = D_M(Y)|U$ (for, there exists $Z \in \mathcal{X}(U)$ such that $di \circ Z = X \circ i = Y \circ i$, where $i : U \to M$ is the inclusion, and then $D_M(X) \circ i = Hi \circ D_U(Z) = D_M(Y) \circ i$).

(b) The relationships between Definition 2.1 and the Category Theory are following. Given a natural bundle $H : \mathcal{M}_n \to \mathcal{FM}$ we define a functor $\Gamma_{loc}H : \mathcal{M}_n \to Sets$ as follows. For every $M \in \mathcal{M}_n$, $\Gamma_{loc}H(M)$ is the set of all locally defined C^∞-sections of the bundle $HM \to M$. For every embedding $\varphi : M \to N$ of two n-manifolds, $\Gamma_{loc}H(\varphi) := \varphi_* : \Gamma_{loc}H(M) \to \Gamma_{loc}H(N)$, $\varphi_*(\sigma) = H\varphi \circ \sigma \circ \varphi^{-1}$. If D is a natural transformation transforming vector fields into sections of H , then there exists one and only one natural transformation (of functors) $\tilde{D} : \Gamma_{loc}(T|\mathcal{M}_n) \to \Gamma_{loc}H$ such that $\tilde{D}|\mathcal{X}(M) = D_M$ for every $M \in \mathcal{M}_n$. (For every $X \in \Gamma_{loc}TM$ $\tilde{D}_M(X) : \mathrm{dom}(X) \to HM$ is defined by $\tilde{D}_M(X)(y) = D_M(\tilde{X})(y)$, where $\tilde{X} \in \mathcal{X}_0(M)$is such that $\mathrm{germ}_y(X) = \mathrm{germ}_y(\tilde{X})$.) On the other hand for every natural transformation (of functors) $\tilde{D} : \Gamma_{loc}(T|\mathcal{M}_n) \to \Gamma_{loc}H$ the family $D_M = \tilde{D}|\mathcal{X}(M)$, $M \in \mathcal{M}_n$, is a natural transformation transforming vector fields into sections of H.

We have the following lemma.

LEMMA 2.1. *Let D, D^* be two natural transformations transforming vector fields into sections of H such that $D_{\mathbb{R}^n}(\partial_1)(0) = D^*_{\mathbb{R}^n}(\partial_1)(0)$, where $\partial_1 = \frac{\partial}{\partial x^1}$ is the canonical vector field on $\mathbb{R}^n$. Then $D = D^*$.*

PROOF. Let $X \in \mathcal{X}(M)$, $M \in \mathcal{M}_n$ and $x_0 \in M$. It is sufficient to show that $D_M(X)(x_0) = D^*_M(X)(x_0)$.

Suppose that $X(x_0) \neq 0$. Then there exists a chart φ on M about x_0 such that $\varphi(x_0) = 0$, $\mathrm{im}\,\varphi = \mathbb{R}^n$ and $d\varphi^{-1} \circ \partial_1 = X \circ \varphi^{-1}$ on some open neighbourhood of 0. Using the naturality condition we deduce that $D_M(X)(x_0) = D^*_M(X)(x_0)$.

Now, we do not assume that $X(x_0) \neq 0$. There exist $Y \in \mathcal{X}(M)$ and two open subsets $U, V \subset M$ such that $Y(z) \neq 0$ for any $z \in V$, $X|U = Y|U$ and $x_0 \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$. Then $D^*_M(X)|U = D^*_M(Y)|U$, $D_M(X)|U = D_M(Y)|U$ and $D_M(Y)|V = D^*_M(Y)|V$. Therefore

$D_M(X)(x_0) = D^*_M(X)(x_0)$. □

We denote by Trans(H) the set of all natural transformations transforming vector fields into sections of H. (Since every natural transformation D transforming vector fields into sections of H is uniquely determined by $D_{\mathbb{R}^n}$, Trans(H) is a set.) If $H : \mathcal{M}_n \to \mathcal{VB}$, then Trans(H) has a vector space structure defined as follows. For any $D_1, D_2 \in$ Trans(H) and $\lambda \in \mathbb{R}$ we define $D_1 + D_2$, $\lambda D_1 \in$ Trans(H) to be the systems of functions

$$(D_1 + D_2)_M : \mathcal{X}(M) \to \Gamma HM,$$
$$(D_1 + D_2)_M(X) = (D_1)_M(X) + (D_2)_M(X)$$
$$(\lambda D_1)_M : \mathcal{X}(M) \to \Gamma HM,$$
$$(\lambda D_1)_M(X) = \lambda((D_1)_M(X)).$$

The purpose of this section is to determine the set Trans($H|\mathcal{M}_n$), where $H : \mathcal{M} \to \mathcal{FM}$ is a bundle functor.

EXAMPLE 2.1 We denote by $\mathcal{X}_0(M)$ the set of all vector fields on M with compact supports. Let $H : \mathcal{M} \to \mathcal{FM}$ be a bundle functor. For any $v \in H_0\mathbb{R}$ and $M \in \mathcal{M}_n$ we define $\tilde{D}^v_M : \mathcal{X}_0(M) \to \Gamma HM$ by

$$\tilde{D}^v_M(X)(y) = H_0(\Phi^X_y)(v),$$

$y \in M$, where $\Phi^X_y : \mathbb{R} \to M$ is defined by $\Phi^X_y(t) = \mathrm{Exp}(tX)(y)$. It follows from Proposition 1.1 that $\tilde{D}^v_M(X)$ is of class C^∞ for every $X \in \mathcal{X}_0(M)$. If $X \in \mathcal{X}_0(M)$, $Y \in \mathcal{X}_0(N)$ are two vector fields on n-manifolds and $\varphi : M \to N$ is an embedding such that $d\varphi \circ X = Y \circ \varphi$, then $\Phi^Y_{\varphi(y)} = \varphi \circ \Phi^X_y$ for any $y \in M$, and then $\tilde{D}^v_N(Y) \circ \varphi = H\varphi \circ \tilde{D}^v_M(X)$. By the localization condition of Definition 1.1 for any $f : \mathbb{R} \to M$, $H_0 f$ depends only on $\mathrm{germ}_0 f$. Then the family $\tilde{D}^v = \{\tilde{D}^v_M\}$ satisfies the following localization condition: if $X, Y \in \mathcal{X}_0(M)$, $M \in \mathcal{M}_n$ are two vector fields such that $X|U = Y|U$ on an open subset, then $\tilde{D}^v_M(X)|U = \tilde{D}^v_M(Y)|U$. We can therefore define a family $D^v_M : \mathcal{X}(M) \to \Gamma HM$, $M \in \mathcal{M}_n$ as follows. For any $X \in \mathcal{X}(M)$ and $y \in M$ we put

$$D^v_M(X)(y) = \tilde{D}^v_M(\tilde{X})(y),$$

where $\tilde{X} \in \mathcal{X}_0(M)$ is such that $\text{germ}_y(\tilde{X}) = \text{germ}_y(X)$. It is clear that the family $D^v = \{D^v_M\}$ is an element from $\text{Trans}(H|\mathcal{M}_n)$.

The main result in this section is the following theorem.

THEOREM 2.1. *The function*

$$P : H_0\mathbb{R} \to \text{Trans}(H|\mathcal{M}_n), \qquad P(v) = D^v,$$

is a bijection. The inverse bijection is given by $S(D) = Hq(D_{\mathbf{R}^n}(\partial_1)(0))$, *where* $q : \mathbb{R}^n \to \mathbb{R}$ *is the projection onto first factor. In particular, if* H *is linear, then* P *is a linear isomorphism.*

PROOF. If H is linear, then S is a linear map. We see that

$$S \circ P(v) = Hq(D^v_{\mathbf{R}^n}(\partial_1)(0)) = Hq \circ H_0(\mathbb{R} \ni t \to (t, 0) \in \mathbb{R}^n)(v) = v$$

for every $v \in H_0\mathbb{R}$. Therefore $S \circ P = \text{id}$ and $Hq(D^v_{\mathbf{R}^n}(\partial_1)(0)) = v$ for every $v \in H_0\mathbb{R}$. It remains to show that $P \circ S = \text{id}$. Consider $D \in \text{Trans}(H|\mathcal{M}_n)$. Let $v := S(D)$. We have to show that $D = P \circ S(D) = P(v) = D^v$. We see that

$$Hq(D^v_{\mathbf{R}^n}(\partial_1)(0)) = v = S(D) = Hq(D_{\mathbf{R}^n}(\partial_1)(0)).$$

It is obvious that for all $t \in \mathbb{R} - \{0\}$, $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_t(x^1, ..., x^n) = (x^1, tx^2, ..., tx^n)$ is a diffeomorphism preserving ∂_1. Therefore using the naturality condition and Proposition 1.1 we get

$$D_{\mathbf{R}^n}(\partial_1)(0) = H\varphi_t(D_{\mathbf{R}^n}(\partial_1)(0)) \to Hi \circ Hq(D_{\mathbf{R}^n}(\partial_1)(0))$$

as $t \to 0$, where $i : \mathbb{R} \to \mathbb{R}^n$ is given by $i(y) = (y, 0)$. Hence

$$\begin{aligned} D_{\mathbf{R}^n}(\partial_1)(0) &= Hi \circ Hq(D_{\mathbf{R}^n}(\partial_1)(0)) \\ &= Hi \circ Hq(D^v_{\mathbf{R}^n}(\partial_1)(0)) = D^v_{\mathbf{R}^n}(\partial_1)(0). \end{aligned}$$

Then $D = D^v$, because of Lemma 2.1. □

From Theorem 2.1 we have the following simple corollary.

COROLLARY 2.1. *Let* $D \in \text{Trans}(T|\mathcal{M}_n)$. *Then there exists* $\lambda \in \mathbb{R}$ *such that* $D_M(X) = \lambda X$ *for any* $M \in \mathcal{M}_n$ *and* $X \in \mathcal{X}(M)$.

PROOF. By Theorem 2.1, $\dim_{\mathbb{R}}(\mathrm{Trans}(T|\mathcal{M}_n)) = \dim_{\mathbb{R}} T_0\mathbb{R} = 1$. On the other hand, for every $\lambda \in \mathbb{R}$, the family $D_M = \lambda\, \mathrm{id}_{\mathcal{X}(M)}$, $M \in \mathcal{M}_n$, is an element of $\mathrm{Trans}(T|\mathcal{M}_n)$. □

EEXAMPLE 2.2 For every $s = 1, ..., r$, $M \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$ and $y \in M$ we have a linear map

$$\overset{(s)}{D}_M(X)(y) : T^{r*}_y M \to \mathbb{R}, \qquad j^r_y(\gamma) \to X^{(s)}\gamma(y),$$

where $X^{(s)} = X \circ X \circ ... \circ X$ $\quad$ ($s-times$). Hence for every $s = 1, ..., r$, $M \in \mathcal{M}_n$ and $X \in \mathcal{X}(M)$ we have a section $\overset{(s)}{D}_M(X) \in \Gamma T^r M$, $y \to \overset{(s)}{D}_M(X)(y)$. It is easy to verify that for every $s = 1, ..., r$ the family

$$\overset{(s)}{D}_M : \mathcal{X}(M) \to \Gamma T^r M, \qquad X \to \overset{(s)}{D}_M(X), \qquad M \in \mathcal{M}_n,$$

is an element of $\mathrm{Trans}(T^r|\mathcal{M}_n)$. It is easy to verify that the natural transformations $\overset{(1)}{D}, ..., \overset{(r)}{D}$ are linearly independent in $\mathrm{Trans}(T^r|\mathcal{M}_n)$. On the other hand $\dim(T^r_0\mathbb{R}) = r$. Therefore we have the following corollary of Theorem 2.1.

COROLLARY 2.2. *The natural transformations* $\overset{(1)}{D}, ..., \overset{(r)}{D}$ *described above form a basis (over* $\mathbb{R}$*) of* $\mathrm{Trans}(T^r|\mathcal{M}_n)$.

3 – Natural base-extending transformations transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$

We introduce the following definition.

DEFINITION 3.1. *A family* $\mathcal{D}$ *of functions*

$$\mathcal{D}_M : \mathcal{X}(M) \to \mathcal{X}(T^r M), \qquad M \in \mathcal{M}_n,$$

is called a natural base-extending transformation transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ *iff the following two conditions are satisfied:*

(a) (Naturality condition) for any $M, N \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$ and any embedding $\varphi : M \to N$ the assumption $d\varphi \circ X = Y \circ \varphi$ implies $dT^r\varphi \circ \mathcal{D}_M(X) = \mathcal{D}_N(Y) \circ T^r\varphi$, and
(b) (Regularity condition) if U is a manifold and $X : U \times M \to TM$ is a C^∞ map such that for every $t \in U$ $X_t : M \to TM$, $X_t(y) = X(t, y)$ is a vector field on M, then the mapping

$$U \times T^rM \ni (t, w) \to \mathcal{D}_M(X_t)(w) \in TT^rM$$

is of class C^∞.

Using similar arguments to these as in the proof of Lemma 2.1 we obtain the following lemmas.

LEMMA 3.1. *Let $\mathcal{D}, \mathcal{D}^1$ be two natural base-extending transformations transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ such that $\mathcal{D}_{\mathbb{R}^n}(\partial_1) = \mathcal{D}^1_{\mathbb{R}^n}(\partial_1)$ over $0 \in \mathbb{R}^n$. Then $\mathcal{D} = \mathcal{D}^1$.*

LEMMA 3.2. *Let $\mathcal{D}$ be a natural base-extending transformation transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ such that*

$$dp^{T^r}_{\mathbb{R}^n} \circ \mathcal{D}_{\mathbb{R}^n}(\partial_1)|T^r_0\mathbb{R}^n = 0.$$

Then for any $M \in \mathcal{M}_n$, $x \in M$ and $X \in \mathcal{X}(M)$ we have $dp^{T^r}_M \circ \mathcal{D}_M(X) = 0$ over x.

Denote by $\mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$ the set of all natural base-extending transformations transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$. For any $\mathcal{D}, \mathcal{D}^1 \in \mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$ and $\lambda \in \mathbb{R}$ define $\mathcal{D} + \mathcal{D}^1$, $\lambda\mathcal{D} \in \mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$ to be the systems of functions

$$(\mathcal{D}+\mathcal{D}^1)_M : \mathcal{X}(M) \to \mathcal{X}(T^rM), \qquad (\mathcal{D}+\mathcal{D}^1)_M(X) = \mathcal{D}_M(X) + \mathcal{D}^1_M(X),$$

$$(\lambda\mathcal{D})_M : \mathcal{X}(M) \to \mathcal{X}(T^rM), \qquad (\lambda\mathcal{D})_M(X) = \lambda(\mathcal{D}_M(X)).$$

Then $\mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$ is a vector space over $\mathbb{R}$.

We have the following examples of natural base-extending transformations transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$.

EXAMPLE 3.1 (Complete lifting) Let $X \in \mathcal{X}(M), M \in \mathcal{M}_n$. Let us recall that the complete lift $T^rX \in \mathcal{X}(T^rM)$ of X to T^rM is the vector field on T^rM satisfying the following condition: if $\{\varphi_t\}$ is a local flow of X defined near $y \in M$, then $\{T^r\varphi_t\}$ is a local flow defining T^rX over y. It is easy to see that the family T^r given by $\mathcal{X}(M) \ni X \to T^rX \in \mathcal{X}(T^rM)$, $M \in \mathcal{M}_n$, is an element from $\mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$. The family is called the complete lifting of vector fields to $T^r|\mathcal{M}_n$, cf. [5] .

Let us recall that the natural bundle $VT^r|\mathcal{M}_n$ is defined as follows. For every $M \in \mathcal{M}_n$ $VT^rM \to M$ is the vertical bundle of $p_M^{T^r} : T^rM \to M$ i.e. $VT^rM := \ker(dp_M^{T^r}) \subset TT^rM$. For every embedding $\varphi : M \to N$, $VT^r\varphi : VT^rM \to VT^rN$ is the restriction of $dT^r\varphi$. Since T^r is linear, we have the natural bundle isomorphism

$$(3.1) \qquad J_M : T^rM \oplus_M T^rM \to VT^rM, \qquad J_M(u,w) = \frac{d}{dt}(u+tw)_{|t=0},$$

where $M \in \mathcal{M}_n$.

EXAMPLE 3.2 (Liouville vector field) For any $M \in \mathcal{M}_n$ we have the Liouville vector field $L_M \in \mathcal{X}(T^rM)$ given by $L_M(w) = J_M(w,w)$, $w \in T^rM$. Of course, the family L given by

$$\mathcal{X}(M) \ni X \to L_M \in \mathcal{X}(T^rM), \qquad M \in \mathcal{M}_n,$$

is an element from $\mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$.

EXAMPLE 3.3 Let $D = \{D_M\} \in \mathrm{Trans}(T^r|\mathcal{M}_n)$ be a natural transformation transforming vector fields into sections of $T^r|\mathcal{M}_n$ (see Section 2). Then we define the family $D_M^V : \mathcal{X}(M) \to \mathcal{X}(T^rM)$, $M \in \mathcal{M}_n$, by

$$D_M^V(X)(w) = J_M(w, D_M(X)(p_M^{T^r}(w))), \qquad w \in T^rM.$$

It is easy to verify that $D^V = \{D_M^V\} \in \mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$. (The regularity of D^V is a consequence of Corollary 2.2.)

We have the following simple corollary.

COROLLARY 3.1. *The transformations* $\overset{(1)}{D}{}^V, \dots, \overset{(r)}{D}{}^V$, L, T^r *(*$\overset{(s)}{D}$ *is as in Example 2.2.) are linearly independent in* $\mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$.

Now, we formulate the following classification theorem.

THEROM 3.1. *Assume that $n \geq 2$. Then the natural base-extending transformations $T^r, L, \overset{(1)}{D}{}^V, \dots, \overset{(r)}{D}{}^V$ form a basis of* $\mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$.

REMARK. In [13], M. SEKIZAWA proved this theorem for $r = 1$. The proof of the theorem for $r = 2$ is given by M. DOUPOVEC, cf. [2].

The proof of Theorem 3.1 is given in Section 6.

4 – Decomposition lemma

The purpose of this section is to prove the following lemma.

LEMMA 4.1 (DECOMPOSITION LEMMA). *Let $\mathcal{D} \in \mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$, $r \geq 2$. Then there exist $\lambda, \mu \in \mathbb{R}$ and $D \in \mathrm{Trans}(T^r|\mathcal{M}_n)$ such that:*

$$dp^{T^r}_{\mathbb{R}^n} \circ \mathcal{D}^*_{\mathbb{R}^n}(\partial_1)|T^r_0\mathbb{R}^n = 0, \tag{4.1}$$

$$\mathcal{D}^*_{\mathbb{R}^n}(\partial_1)(0) = 0, \qquad (0 \in T^r_0\mathbb{R}^n) \tag{4.2}$$

$$\mathcal{D}^*_{\mathbb{R}^n}(0)|T^r_0\mathbb{R}^n = 0, \tag{4.3}$$

where $\mathcal{D}^* = \mathcal{D} - \lambda T^r - \mu L - D^V$ (T^r, L, D^V are described in Section 3).

PROOF. Consider the map

$$g : \mathbb{R} \times T^r_0\mathbb{R}^n \to T_0\mathbb{R}^n, \qquad g(t,w) = dp^{T^r}_{\mathbb{R}^n} \circ \mathcal{D}_{\mathbb{R}^n}(t\partial_1)(w),$$

$t \in \mathbb{R}$, $w \in T^r_0\mathbb{R}^n$. Using the regularity condition we see that g is of class C^∞. It follows from the naturality condition that for all $\tau \in \mathbb{R} - \{0\}$ we have

$$g(\tau t, B(\tau)(w)) = \tau g(t,w),$$

where $B(\tau) = T^r_0(\tau\,\mathrm{id}) \in \mathrm{End}(T^r_0\mathbb{R}^n)$. Therefore

$$g(t,w) = \frac{d}{d\tau} g(\tau t, B(\tau)(w))_{|\tau=0} = d_{(0,0)}g(t, B'(0)(w))$$

i.e. g is linear.

It is obvious that the family

$$C_M : \mathcal{X}(M) \to \mathcal{X}(M), \qquad C_M(X) = dp_M^{T^r} \circ \mathcal{D}_M(X) \circ 0_M, \qquad M \in \mathcal{M}_n,$$

($0_M : M \to T^r M$ is the 0-section) is an element of $\mathrm{Trans}(T|\mathcal{M}_n)$, and then it follows from Corollary 2.1 that there exists $\lambda \in \mathbb{R}$ such that $C_M = \lambda\, \mathrm{id}_{\mathcal{X}(M)}$ for any $M \in \mathcal{M}_n$. In particular, $g(1,0) = \lambda\partial_1(0)$.

On the other hand the family

$$A_M : T^r M \to TM, \qquad A_M(w) = dp_M^{T^r} \circ \mathcal{D}_M(0)(w), \qquad M \in \mathcal{M}_n,$$

is a natural transformation of $T^r|\mathcal{M}_n$ into $T|\mathcal{M}_n$ i.e. $A_M(w) = 0$ for any $M \in \mathcal{M}_n$ and $w \in T^r M$, because of Corollary 1.1. Then $g(0,.) = 0$.

Therefore $\mathcal{D} - \lambda T^r \in \mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$ satisfies the equalities

$$dp_{\mathbb{R}^n}^{T^r} \circ (\mathcal{D} - \lambda T^r)_{\mathbb{R}^n}(\partial_1)(w) = g(1,w) - \lambda\partial_1(0) = 0 \tag{4.4}$$

for any $w \in T_0^r\mathbb{R}^n$.

It follows from (4.4) and Lemma 3.2 that

$$\mathrm{im}(\mathcal{D} - \lambda T^r)_M(X) \subset VT^r M$$

for any $X \in \mathcal{X}(M)$ and $M \in \mathcal{M}_n$. We can therefore define a natural transformation of $T^r|\mathcal{M}_n$ into itself by

$$B_M : T^r M \to T^r M, \qquad B_M(w) = q_M \circ J_M^{-1} \circ (\mathcal{D} - \lambda T^r)_M(0)(w),$$

where $q_M : T^r M \oplus_M T^r M \to T^r M$ is the projection onto second factor and $J_M : T^r M \oplus_M T^r M \to VT^r M$ is defined in (3.1). We can also define $D \in \mathrm{Trans}(T^r|\mathcal{M}_n)$ by

$$D_M : \mathcal{X}(M) \to \Gamma T^r M, \qquad D_M(X) = q_M \circ J_M^{-1} \circ (\mathcal{D} - \lambda T^r)_M(X) \circ 0_M.$$

By Proposition 1.2 there exists $\mu \in \mathbb{R}$ such that

$$B_M(w) = \mu w$$

for every $w \in T^r M$ and $M \in \mathcal{M}_n$.

We prove that λ, μ, D satisfy the conditions of the lemma. It follows from the definitions of L and D^V (see Examples 3.2 and 3.3) and from (4.4) that

$$dp^{T^r}_{\mathbf{R}^n} \circ \mathcal{D}^*_{\mathbf{R}^n}(\partial_1)(w) = 0$$

for every $w \in T^r_0\mathbb{R}^n$, i.e. $\mathcal{D}^*$ satisfies the condition (4.1). From the definitions of D, D^V and L it follows that

$$D^V_{\mathbf{R}^n}(\partial_1)(0) = (\mathcal{D} - \lambda T^r)_{\mathbf{R}^n}(\partial_1)(0)$$

and $L_{\mathbf{R}^n}(\partial_1)(0) = 0$, and hence $\mathcal{D}^*_{\mathbf{R}^n}(\partial_1)(0) = 0$, i.e. $\mathcal{D}^*$ satisfies the condition (4.2). From the condition (4.1) and Lemma 3.2 we deduce that $\mathcal{D}^*_{\mathbf{R}^n}(0)(w) \in VT^r\mathbb{R}^n$ for any $w \in T^r_0\mathbb{R}^n$. From Corollary 2.2 we get that $D_{\mathbf{R}^n}(0)(0) = 0$. Hence

$$q_{\mathbf{R}^n} \circ J^{-1}_{\mathbf{R}^n} \circ \mathcal{D}^*_{\mathbf{R}^n}(0)(w) = B_{\mathbf{R}^n}(w) - \mu w = 0$$

for any $w \in T^r_0\mathbb{R}^n$. Therefore $\mathcal{D}^*$ satisfies Condition (4.3). □

5 – An algebraic lemma

From now on we use the following notation. We fix three positive integers n, r, m. Let

$$(5.1) \quad S = \{\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbb{N} \cup \{0\})^n : 1 \le |\alpha| = \alpha_1 + ... + \alpha_n \le r\}$$

and

$$(5.2) \qquad q = \mathrm{card}(S).$$

Let $x^i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, ..., n)$ be the projection onto i-th factor. Similarly, let $X^\alpha : \mathbb{R}^q \to \mathbb{R}$, $(\alpha \in S)$ be the projection onto $\alpha - th$ factor. For every $\alpha \in S$ we define the map

$$(5.3) \qquad x^\alpha : \mathbb{R}^n \to \mathbb{R}, \qquad x^\alpha = (x^1)^{\alpha_1} ... (x^n)^{\alpha_n}.$$

Similarly, for every $\Psi = (\Psi_\alpha; \alpha \in S) \in (\mathbb{N} \cup \{0\})^q$ we define the map

$$X^\Psi : \mathbb{R}^q \to \mathbb{R}, \qquad X^\Psi = \prod_{\alpha \in S} (X^\alpha)^{\Psi_\alpha}. \tag{5.4}$$

Let Ω be the linear isomorphism

$$\Omega : T_0^r \mathbb{R}^n \to \mathbb{R}^q, \qquad \Omega(w) = (w(j_0^r x^\alpha); \alpha \in S). \tag{5.5}$$

Given $l \in \mathbb{N}$ and $i \in \{1, ..., n\}$ let $\varphi_l^i : \mathbb{R}^n \to \mathbb{R}^n$ be the map defined by

$$\varphi_l^i(x) = x + (x^n)^l e_i, \tag{5.6}$$

where $x = (x^1, ..., x^n) \in \mathbb{R}^n$ and $e_j = (0, ..., 1, ..., 0) \in \mathbb{R}^n$, 1 in $j - th$ position.

The purpose of this section is to prove the following algebraic lemma.

LEMMA 5.1. *Let $h : \mathbb{R}^q \to \mathbb{R}^m$ be a polynomial map such that*

$$\frac{\partial}{\partial X^\beta} h = 0 \qquad \textit{and} \qquad \frac{\partial}{\partial X^\beta} (h \circ \Omega \circ T_0^r \varphi_l^i \circ \Omega^{-1}) = 0$$

for all $\beta \in S$ with $|\beta| = r$ and all integers $l \geq 2$, $i \in \{1, ..., n\}$. Then $h = const$.

PROOF. Suppose that $h \neq const$. Then there exists $i_0 \in \{1, ..., n\}$ such that

$$B = \{\alpha \in S : \frac{\partial}{\partial X^\alpha} h \neq 0 \qquad and \qquad \alpha_{i_0} \neq 0\} \neq \emptyset. \tag{5.7}$$

Let

$$l_0 = max\{r + 1 - |\alpha| : \alpha \in B\} \tag{5.8}$$

and

$$C = \{\alpha \in B : r + 1 - |\alpha| = l_0\}. \tag{5.9}$$

It is clear that $l_0 \geq 2$. Let

$$(5.10)\quad E = \{\Psi = (\Psi_\alpha; \alpha \in S) \in (\mathbb{N} \cup \{0\})^q : \Psi_\alpha = 0, \text{ if } \frac{\partial}{\partial X^\alpha} h = 0\}$$

and

$$(5.11)\qquad H = \{\Psi \in E : \Psi_\alpha = 0, \quad if \quad \alpha \in C\}.$$

It is obvious that for every $\Psi = (\Psi_\alpha : \alpha \in S) \in (\mathbb{N} \cup \{0\})^q$ there exists one and only one $\Psi^* \in (\mathbb{N} \cup \{0\})^q$ such that

$$(5.12)\qquad X^{\Psi^*} = \prod_{\alpha \in C} (X^{\alpha - e_{i_0} + l_0 e_n})^{\Psi_\alpha} \prod_{\alpha \in S - C} (X^\alpha)^{\Psi_\alpha}.$$

Now, we need the following two lemmas.

LEMMA 5.2. *The function*

$$E \ni \Psi \to \Psi^* \in (\mathbb{N} \cup \{0\})^q$$

is injective.

LEMMA 5.3. *Let $\alpha \in S$ be such that $\frac{\partial}{\partial X^\alpha} h \neq 0$. Then*

$$X^\alpha \circ \Omega \circ T_0^r \varphi_{l_0}^{i_0} \circ \Omega^{-1} = \begin{cases} X^\alpha, & \text{if } \alpha \in S - C \\ X^\alpha + \alpha_{i_0} X^{\alpha - e_{i_0} + l_0 e_n}, & \text{if } \alpha \in C \end{cases}$$

We continue the proof of Lemma 5.1. It follows from Lemma 5.3 that

$$(5.13)\qquad \begin{aligned} &X^\Psi \circ \Omega \circ T_0^r \varphi_{l_0}^{i_0} \circ \Omega^{-1} = \\ &\begin{cases} X^\Psi, & \text{if } \Psi \in H \\ X^\Psi + \prod_{\alpha \in C, \Psi_\alpha \neq 0} (\alpha_{i_0})^{\Psi_\alpha} X^{\Psi^*} + \ldots, & \text{if } \Psi \in E - H, \end{cases} \end{aligned}$$

where the dots denote a polynomial , each term of which contains at last one of the X^α with $\alpha \in C$.

Of course,

$$h = \sum_{\Psi \in E} X^{\Psi} a_{\Psi},$$

where $a_\Psi \in \mathbb{R}^m$ are vectors. It is clear that $a_\Psi \neq 0$ for some $\Psi \in E - H$.

It follows from the condition (5.13) that

$$h \circ \Omega \circ T_0^r \varphi_{l_0}^{i_0} \circ \Omega^{-1} = h + \sum_{\Psi \in E-H} \prod_{\alpha \in C, \Psi_\alpha \neq 0} (\alpha_{i_0})^{\Psi_\alpha} X^{\Psi^*} a_\Psi + \ldots,$$

where the dots denote a polynomial , each term of which contains at last one of the X^α with $\alpha \in C$. We see that for every $\Psi \in E - H$, X^{Ψ^*} is independent of X^α for all $\alpha \in C$ and it contains at last one of the X^γ with $\gamma \in S, |\gamma| = r$, for if $\alpha \in C$, then $|\alpha| \leq r - 1$ and $|\alpha - e_{i_0} + l_0 e_n| = r$. Then (owing to the assumptions of the lemma) we have

$$\sum_{\Psi \in E-H} \prod_{\alpha \in C, \Psi_\alpha \neq 0} (\alpha_{i_0})^{\Psi_\alpha} X^{\Psi^*} a_\Psi = 0.$$

Hence $a_\Psi = 0$ for all $\Psi \in E - H$, because of Lemma 5.2. This is a contradiction. Therefore $h = const.$

It remains to prove Lemmas 5.2 and 5.3.

PROOF OF LEMMA 5.2 Let $\Psi \in E$. We see that if $\alpha \in C$, then $|\alpha| \leq r - 1$ and $|\alpha - e_{i_0} + l_0 e_n| = r$, and then $\alpha - e_{i_0} + l_0 e_n \in S - C$. Therefore

$$\Psi^*_\beta = \begin{cases} \Psi_\beta, & \text{if } \beta \in S - C,\ |\beta| \leq r - 1 \\ \Psi_\alpha + \Psi_\beta, & \text{if } \beta = \alpha - e_{i_0} + l_0 e_n \text{ for some } \alpha \in C \end{cases}$$

because of (5.12). On the other hand, if $\beta \in S$, $|\beta| = r$, then $\Psi_\beta = 0$, because of (5.10). Hence $\Psi \in E$ is uniquely determined by Ψ^*. □

PROOF OF LEMMA 5.3 Let $(j_0^r x^\beta)^*$, $\beta \in S$ be the basis of $T_0^r \mathbb{R}^n$ dual to $j_0^r x^\beta$, $\beta \in S$. Then for every $(Y^\beta; \beta \in S) \in \mathbb{R}^q$ we have

$$
\begin{aligned}
X^\alpha \circ \Omega \circ T^r_0\varphi^{i_0}_{l_0} \circ \Omega^{-1}(Y^\beta; \beta \in S)
&= T^r_0\varphi^{i_0}_{l_0}(\Omega^{-1}(Y^\beta;\beta\in S))(j^r_0x^\alpha)\\
&= \Omega^{-1}(Y^\beta;\beta\in S)(j^r_0(x^\alpha\circ\varphi^{i_0}_{l_0}))\\
&= \sum_{\beta\in S} Y^\beta(j^r_0x^\beta)^*(j^r_0(x^\alpha\circ\ \varphi^{i_0}_{l_0}))\\
&= \begin{cases} Y^\alpha, & \text{if } \alpha\in S-C\\ Y^\alpha+\alpha_{i_0}Y^{\alpha-e_{i_0}+l_0e_n}, & \text{if } \alpha\in C\end{cases}
\end{aligned}
$$

as

$$
(5.14)\qquad j^r_0(x^\alpha\circ\varphi^{i_0}_{l_0}) = \begin{cases} j^r_0x^\alpha, & \text{if } \alpha\in S-C\\ j^r_0x^\alpha+\alpha_{i_0}j^r_0(x^{\alpha-e_{i_0}+l_0e_n}), & \text{if } \alpha\in C\end{cases}
$$

It remains to prove the formula (5.14). If $\alpha_{i_0}=0$, then $x^\alpha\circ\varphi^{i_0}_{l_0}=x^\alpha$ and $\alpha\in S-C$. So, we assume that $\alpha_{i_0}\neq 0$. Then

$$
x^\alpha\circ\varphi^{i_0}_{l_0} = x^\alpha+\alpha_{i_0}x^{\alpha-e_{i_0}+l_0e_n}+\sum_{k=2}^{\alpha_{i_0}} C^{\alpha_{i_0}}_k x^{\alpha-ke_{i_0}+kl_0e_n}
$$

because of the Newton formula. If $\alpha\in S-C$, then $\alpha\in B-C$ i.e. $l_0>r+1-|\alpha|$, and then

$$
|\alpha-ke_{i_0}+l_0ke_n| = |\alpha|-k+kl_0 > r+(l_0-1)(k-1)\geq r
$$

for $k=1,\ldots,\alpha_{i_0}$, because of $l_0\geq 2$. If $\alpha\in C$, then $l_0=r+1-|\alpha|$, and then $|\alpha-ke_{i_0}+kl_0e_n|>r$ for $k=2,\ldots,\alpha_{i_0}$ and $|\alpha-e_{i_0}+l_0e_n|=r$. These facts complete the proof of Lemma 5.3. □

6 – Proof of Theorem 3.1

From Corollary 3.1 and Lemma 4.1 it follows that Theorem 3.1 will be demonstrated after proving the following proposition.

PROPOSITION 6.1. *Let* $\mathcal{D}^*\in \mathrm{Trans}_{Ex}(T^r|\mathcal{M}_n)$ *be a natural base-extending transformation satisfying the conditions* (4.1), (4.2) *and* (4.3). *Then* $\mathcal{D}^*=0$.

PROOF. Let $S, q, x^i, X^\alpha, \Omega$ be as in Section 5. Using the condition (4.1) and Lemma 3.2 one can define

(6.1)
$$H : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q, \qquad H(t, X) := \Omega \circ q_{\mathbb{R}^n} \circ J_{\mathbb{R}^n}^{-1} \circ \mathcal{D}^*_{\mathbb{R}^n}(t\partial_1) \circ \Omega^{-1}(X),$$

where $J_M : T^rM \oplus_M T^rM \to VT^rM$ is the natural isomorphism given in (3.1) and $q_M : T^rM \oplus_M T^rM \to T^rM$ is the projection onto the second factor. It follows from the regularity condition that H is of class C^∞. Using the naturality condition with respect to the homotheties $\tau\,\mathrm{id}_{\mathbb{R}^n}$, $\tau \in \mathbb{R}^n - \{0\}$, we see that for every $\tau \in \mathbb{R}^n - \{0\}$, $\alpha \in S$, $t \in \mathbb{R}$ and $(Y^\beta; \beta \in S) \in \mathbb{R}^q$ we have

$$\tau^{|\alpha|} H^\alpha(t, Y^\beta; \beta \in S) = H^\alpha(\tau t, \tau^{|\beta|} Y^\beta; \beta \in S), \tag{6.2}$$

where H^α is defined by $H = (H^\alpha; \alpha \in S)$. On the other hand it follows from the condition (4.3) that

$$H^\alpha(0, .) = 0, \qquad \alpha \in S. \tag{6.3}$$

To discuss (6.2), we need the following simple property of globally defined smooth homogeneous functions, a proof of which can be found e.g. in [8].

LEMMA 6.1. *Let $g(x^i, y^p, ..., z^t)$ be a smooth function defined on $\mathbb{R}^m \times \mathbb{R}^n \times ... \times \mathbb{R}^p$, and let $a > 0, b > 0, ..., c > 0, d$ be real numbers such that*

$$k^d g(x^i, y^p, ..., z^t) = g(k^a x^i, k^b y^p, ..., k^c z^t) \tag{6.4}$$

for every real $k > 0$. Then g is a sum of polynomials of degrees ζ in x^i, η in $y^p, ..., \xi$ in z^t satisfying

$$a\zeta + b\eta + ... + c\xi = d. \tag{6.5}$$

If there are no non-negative integers $\zeta, \eta, ..., \xi$ with the property (6.5), *then g is the zero function.*

According to (6.3) and to this lemma we see that H^α is a polynomial in t and X^β, where $\beta \in S$ and $|\beta| \le |\alpha| - 1$. (H^α is independent of the X^β, $\beta \in S$, $|\beta| \ge |\alpha|$.) Therefore the map

$$h := H(1,.) : \mathbb{R}^q \to \mathbb{R}^q \tag{6.6}$$

is a polynomial such that $\frac{\partial}{\partial X^\alpha} h = 0$ for every $\alpha \in S$ with $|\alpha| = r$.

Let $\varphi_l^i : \mathbb{R}^n \to \mathbb{R}^n$ be given by (5.6). Then φ_l^i is a diffeomorphism of an open neighbourhood V_l^i at $0 \in \mathbb{R}^n$. Since $n \ge 2$, $\varphi_l^i | V_l^i$ preserves $\mathrm{germ}_0(\partial_1)$, and then

$$\Omega \circ T_0^r \varphi_l^i \circ \Omega^{-1} \circ h = h \circ \Omega \circ T_0^r \varphi_l^i \circ \Omega^{-1}$$

because of the naturality condition. Hence

$$\frac{\partial}{\partial X^\alpha}(h \circ \Omega \circ T_0^r \varphi_l^i \circ \Omega^{-1}) = 0$$

for all $\alpha \in S$ with $|\alpha| = r$ and all integers $l \ge 2$, $i \in \{1, ..., n\}$. It follows from Lemma 5.1 that $h = const.$

It follows from the condition (4.2) that $h(0) = 0$. Then $h = 0$, i.e. $\mathcal{D}^\bullet(\partial_1) | T_0^r \mathbb{R}^n = 0$. Therefore $\mathcal{D}^\bullet = 0$, because of Lemma 3.1. □

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