Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 1013-1029

A Riquier-like problem for a hyperbolic partial differential equation

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RIASSUNTO – *Si studia, per un'equazione differenziale iperbolica di ordine 2p, un problema con condizioni al contorno analoghe a quelle che, per le equazioni ellittiche, originano il problema di Riquier. Estendendo il metodo introdotto da G. Fichera [4] per le equazioni del secondo ordine, si stabiliscono le condizioni necessarie e sufficienti per l'esistenza della soluzione.*

ABSTRACT – *The paper concerns a boundary value problem for a hyperbolic partial differential equation of order 2p that contains the Riquier problem for the said equation. By using the method of G. Fichera, introduced in paper [4] for a hyperbolic equation of second order, the necessary and sufficient conditions for the existence of the solutions are found*

KEY WORDS – *Hyperbolic equation - Linear boundary value problem - Improperly posed problem - Banach spaces - Functional equation.*

A.M.S. CLASSIFICATION: 35L35 - 35G15 - 35R25

1 – G. FICHERA examined (see [4], [5]) the first boundary value problem for a second order hyperbolic partial differential equation. His papers were related with earlier results of M. PICONE (see [12]). In this paper we apply the method of G. FICHERA to a boundary value problem, containing a countepart of the Riquier problem (see [10]), for a certain hyperbolic partial differential equation of order $2p$ that is often called the polyvibrating equation of Mangeron. We also use some results of the

papers [1] of A. BIELECKI and J. KISYNSKI, [2], [3] of A. BORZYMOWSKI and [8], [9] of M. MICHALSKI. Let us note that a boundary value problem for an integro-differential equation of Mangeron, with the boundary conditions different from those in the present paper, was examined by D. MANGERON (see [6]).

2 – Let Y be a Banach space with norm $\|.\|$, $p \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of all positive integers) a fixed number and Ω the rectangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1;\ 0 \le y \le \sigma\},$$

where $0 < \sigma < \infty$.

We consider the system of $2p$ curves $\Gamma_0, \dots, \Gamma_{p-1}$ and $\tilde{\Gamma}_0, \dots, \tilde{\Gamma}_{p-1}$, of equations $y = \alpha_j(x)$ and $y = \beta_j(x)$, respectively, where $\alpha_j, \beta_j : \langle 0, 1\rangle \longrightarrow \langle 0, \sigma\rangle$ for $j = 0, 1, \dots, p-1$.

We shall examine the following problem (P):

Find a solution of the partial differential equation

$$L^p u = F \tag{1}$$

(where $L = \frac{\partial^2}{\partial x \partial y}$ and F is given) in Ω(1) satisfying the boundary conditions

$$\begin{aligned} L^j u[x, \alpha_j(x)] &= M_j(x) \\ L^j u[x, \beta_j(x)] &= N_j(x) \end{aligned} \tag{2}$$

$(x \in \langle 0, 1\rangle;\ j = 0, 1, \dots, p-1)$.

We assume the following

I. The functions α_j and β_j $(j = 0, 1, \dots, p-1)$ are strictly increasing, of class C^{p-j}, respectively, and satisfy the conditions

$$\alpha_j(0) = \beta_j(0) = 0; \quad \alpha_j(1) = \beta_j(1) = \sigma \tag{3}$$

(1) That is a function $u: \Omega \longrightarrow Y$ possessing continuous derivatives $\partial^{|\beta|}/\partial x^{\beta_1} \partial y^{\beta_2}$, where $|\beta| = \beta_1 + \beta_2$; $0 \le \beta_1, \beta_2 \le p$, in Ω and satisfying equation (1) at each point of Ω.

$(j = 0, 1, \dots, p-1)$;

$$\begin{aligned} &\alpha_{\nu-1}(x) \le \alpha_\nu(x); \quad \beta_{\nu-1}(x) \le \beta_\nu(x); \\ &\alpha_{p-1}(x) < \beta_0(x) \end{aligned} \tag{4}$$

$(x \in (0,1); \nu = 1, 2, \dots, p-1)$;

$$\begin{gathered} 0 < \min\left(\alpha_0'(0), \beta_{p-1}'(1)\right); \\ \alpha_{p-1}'(0) < \beta_0'(0); \; \beta_0'(1) < \alpha_{p-1}'(1). \end{gathered} \tag{5}$$

II. The functions M_j and N_j $(M_j, N_j\colon \langle 0,1\rangle \longrightarrow Y;\, j = 0, 1, \dots, p-1)$ are of class C^{p-j}, respectively, and satisfy the conditions

$$M_j(0) = N_j(0); \; M_j(1) = n_j(1) \tag{6}$$

$(j = 0, 1, \dots, p-1)$.

III. The function $F\colon \Omega \longrightarrow Y$ is continuous.

REMARK 1. Let us note that problem (P) was examined by M.N. OGUZTÖRELI (see [11]) and M. MICHALISKI (see, [7], [8]) in the case when $\Gamma_j \underset{j}{\equiv} \Gamma_0$; $\widetilde{\Gamma}_j \underset{j}{\equiv} \widetilde{\Gamma}_0 (j = 0, 1, \dots, p-1)$, under the assumption that the curves Γ_0 and $\widetilde{\Gamma}_0$ intersect only at the point $(0,0)$. Let us also observe that in the aforesaid case and under the present Assumption I, the problem (P) consists in finding a solution u of equation (1) that satisfies conditions (2) on the closed curve $\Gamma_0 \cup \widetilde{\Gamma}_0$, and hence this problem is a counterpart of the Riquier problem known in the theory of elliptic equations (see [10], p.28).

3 – In this section we give some auxiliary theorems.

LEMMA 1. (see [2], [3]). *If $u\colon \Omega \longrightarrow Y$ is of the form*

$$u(x,y) = R_p(x,y) + \sum_{m=1}^{p} \left[(m-1)!\right]^{-1} \cdot \left[y^{m-1}\Phi_m(x) + x^{m-1}\Psi_m(y)\right] \tag{7}$$

$((x,y)\in\Omega)$, *where*

$$(8)\qquad R_p(x,y)=\Big[(p-1)!\Big]^{-2}\int_0^x\int_0^y\Big[(x-\xi)(y-\eta)\Big]^{p-1}F(\xi,\eta)\,d\eta\,d\xi\,,$$

and $\Phi_m:\langle 0,1\rangle\longrightarrow Y$ *and* $\Psi_m:\langle 0,\sigma\rangle\longrightarrow Y$ *are arbitrary functions of class* C^p, *then* u *is a solution of equation (1) in* Ω.

Conversely, if u is a given solution of equation (1) in Ω, then there are functions $\varphi_k:\langle 0,1\rangle\longrightarrow Y$ and $\psi_k:\langle 0,\sigma\rangle\longrightarrow Y$ $(k=0,1,\dots,p-1)$ of class C^{p-k}, respectively, such that

$$(9)\qquad \begin{aligned}\Phi_m(x)&=\delta_{1m}\varphi_0(x)+\frac{1-\delta_{1m}}{(m-2)!}\int_0^x(x-\xi)^{m-2}\varphi_{m-1}(\xi)\,d\xi\,;\\ \Psi_m(y)&=\delta_{1m}\psi_0(y)+\frac{1-\delta_{1m}}{(m-2)!}\int_0^y(y-\eta)^{m-2}\psi_{m-1}(\eta)\,d\eta\,;\end{aligned}$$

$(m=1,2,\dots,p;$ δ_{1m} is the Kronecker delta) and that equality (7) is satisfied by Φ_m and $\Psi_m (m=1,2,\dots,p)$.

Let us observe that if u is a solution of equation (1) in Ω, then, by Lemma 1, we can write

$$(10)\qquad L^j u(x,y)=r_j(x,y)+\sum_{m=j+2}^{p}G_{m,j}(x,y)+R_{p-j}(x,y)$$

$((x,y)\in\Omega;\, j=0,1,\dots,p-1)$, where

$$(11)\qquad G_{m,j}(x,y)=\int_0^x\int_0^y\omega_{m,j}(x,y;\xi\eta)\,r_{m-1}(\xi,\eta)\,d\eta\,d\xi$$

with

$$(12)\qquad \omega_{m,j}(x,y;\xi,\eta)=\Big[(m-j-2)!\Big]^{-2}\Big[(x-\xi)(y-\eta)\Big]^{m-j-2};$$

$$(13)\qquad r_k(x,y)=\varphi_k(x)+\psi_k(y)$$

$(k=0,1,\dots,p-1)$, and $R_{p-j}(x,y)$ is defined by formula (8) with p replaced by $p-j$.

Now, let us introduce the functions

(14) $$\tau_j(y) = \alpha_j \circ \beta_j^{-1}(y)\,;$$

(15) $$\lambda_j(x) = \beta_j^{-1} \circ \alpha_j(x)$$

and

(16) $$\mu_j(x) = \alpha_j^{-1} \circ \beta_j(x)\,,$$

where $(x, y) \in \Omega, j = 0, 1, \dots, p-1$ and $\circ$ is the symbol of composition.

LEMMA 2. *The following relations*

(17) $$\tau_j^n \longrightarrow 0 \;\; on \;\; \langle 0, \sigma); \lambda_j^n \longrightarrow 0 \;\; on \;\; \langle 0, 1)\,;$$

(17') $$\mu_j^n \longrightarrow 1 \;\; on \;\; (0, 1\rangle$$

hold good, when n tends to infinity, with $\longrightarrow$ denoting the almost-uniform convergence.

PROOF. The validity of relation (17) follows from Lemma 3 in [1]. We shall prove relation (17').

To this end let us observe that

$$\mu_j(x) > \alpha_{p-1}^{-1} \circ \beta_j(x) > \alpha_{p-1}^{-1} \circ \alpha_{p-1}(x) = x\,,$$

where $x \in (0, 1)$.

From the above inequalities and relation (16) it follows that

(18) $$\mu_j(x) > x \text{ for } x \in (0, 1); \mu_j(1) = 1$$

$(j = 0, 1, \dots, p-1)$.

Let us also note that we have

(19) $$\mu_j(x_*) \leq \mu_j(x) \leq 1$$

for $x \in \langle x_*, 1\rangle$, where x_* is arbitrarily fixed in $(0, 1)$.

Basing on inequality (18) we easily conclude that the sequence $\{\mu_j^n(x_*)\}$ is non-decreasing, whence and from (19) it follows that there is a number $l_0 \in \langle x_*, 1\rangle$ such that

$$\lim_{n \to \infty} \mu_j^n(x_*) = l_0 . \tag{20}$$

Let us suppose that $l_0 \in \langle x_*, 1)$.

Using the continuity of μ_j, resulting from (16) and Assumption I, we can write the sequence of equalities

$$l_0 = \lim_{n \to \infty} \mu_j\left[\mu_j^{n-1}(x_*)\right] = \mu_j\left[\lim_{n \to \infty} \mu_j^{n-1}(x_*)\right] = \mu_j(l_0) \tag{21}$$

which contradicts relation (18).

Thus, $l_0 = 1$, and as a consequence of this and of (19) relation (17') is valid.

LEMMA 3. *There is a sufficiently small number $\delta \in (0, \min(1, \sigma))$ such that*

$$\begin{aligned} &\min(\lambda_j'(x), \tau_j'(y)) > 0 \\ &\max(\lambda_j'(x), \tau_j'(y)) \le q \end{aligned} \tag{22}$$

for $(x, y) \in (0, \delta)^2$;

$$0 < \mu_j'(x) \le q \tag{23}$$

for $x \in (1 - \delta, 1) (j = 0, 1, \dots, p - 1)$, where q is a number in $(0, 1)$.

PROOF. The proof, being similar in case of inequalities (22), will be given only for (23).

Let

$$q = (1 + \varepsilon_0) \max\left(\frac{\alpha_{p-1}'(0)}{\beta_0'(0)}, \frac{\beta_0'(1)}{\alpha_{p-1}'(1)}\right), \tag{24}$$

where

$$0 < \varepsilon_0 < \min\left(\frac{\beta_0'(0) - \alpha_{p-1}'(0)}{\beta_0'(0) + \alpha_{p-1}'(0)}, \frac{\alpha_{p-1}'(1) - \beta_0'(1)}{\alpha_{p-1}'(1) + \beta_0'(1)}\right). \tag{25}$$

Evidently, $q \in (0,1)$.

By assumption I we can write

$$\left|\mu_j'(x) - \frac{\beta_j'(1)}{\alpha_j'(1)}\right| < \varepsilon_0 \frac{\beta_j'(1)}{\alpha_j'(1)}$$

for $x \in (1-\delta,1), \delta$ being a sufficiently small positive number, whence

$$0 < \mu_j'(x) < (1+\varepsilon_0)\frac{\beta_j'(1)}{\alpha_j'(1)} \leq (1+\varepsilon_0)\frac{\beta_0'(1)}{\alpha_{p-1}'(1)} \leq q,$$

as required.

LEMMA 4. *Let $2 \leq n \in \mathbb{N}$. The inequalities*

$$\begin{aligned} &\min\Big(\frac{d}{dx}\lambda_j^n(x), \frac{d}{dy}\tau_j^n(y)\Big) > 0 \\ &\max\Big(\frac{d}{dx}\lambda_j^n(x), \frac{d}{dy}\tau_j^n(y)\Big) \leq q^n \end{aligned} \tag{26}$$

hold good for $(x,y) \in (0,\delta)^2$, and the inequalities

$$0 < \frac{d}{dx}\mu_j^n(x) \leq q^n \tag{27}$$

are valid for $x \in (1-\delta,1)$ $(j = 0,1,\dots,p-1)$.

PROOF. The validity of inequality (27) follows from the formula

$$\frac{d}{dx}\mu^n(x) = \prod_{\nu=0}^{n-1} \mu_j' \circ \mu_j^\nu(x), \tag{28}$$

inequality (18) and Lemma 3. The proof of (26) is analogous.

Lemma 5. *The following inequality holds good*[2]

$$\max\Big(\sup_{(0,\delta)}\Big|\frac{d^\nu}{dx^\nu}\lambda_j^n(x)\Big|,\sup_{(0,\delta)}\Big|\frac{d^\nu}{dy^\nu}\tau_j^\nu(y)\Big|,\\ \sup_{(1-\delta,1)}\Big|\frac{d^\nu}{dx^\nu}\mu_j^n(x)\Big|\Big)\leq \text{const}\, n^{p(\nu-1)}.q^n \tag{29}$$

$(n\in \mathbb{N},\nu=2,3,\dots,p-j;\ j=0,1,\dots,p-2)$, *where* const *is independent of* n.

Proof. The validity of Lemma 5 follows from Lemma 3 above and the formula (see Remark 1 in [9] and cp. equality (49) in [3])

$$(H\circ z)^{(m)}(x)=\\ =\sum_{i=1}^{m}\sum_{r_1,\dots,r_i\geq 1}\prod_{s=1}^{i}\begin{pmatrix}|\vec{r}_{(s)}| & -1\\ |\vec{r}_{(s)}| & -r_s\end{pmatrix} z^{(r_s)}(x)H^{(i)}\circ z(x) \tag{30}$$

($z\colon D\longrightarrow\mathbb{R}; H\colon z(D)\longrightarrow E$ with $D\subset\mathbb{R}$ and E denoting a Banach space; $z,H\in C^m$; $\vec{r}_{(k)}=(r_1,\dots,r_k)$; $|\vec{r}_{(k)}|=\sum_{\nu=1}^{k} r_\nu$ for $k=i,s$).

We shall end this section with the following remark

Remark 2. It follows from Lemma 2 that for any numbers $y'\in(0,\sigma)$ and $x',x''\in(0,1)$ there is a number $n_0\in N$ such that the relations

$$\tau_j^n(y)\in\langle 0,\delta);\lambda_j^n(x)\in\langle 0,\delta);\mu_j^n(x)\in(1-\delta,1\rangle \tag{31}$$

$(j=0,1,\dots,p-1)$ hold good for $\mathbb{N}\ni n\geq n_0$, and all $y\in\langle 0,y'\rangle$, $x\in\langle 0,x'\rangle$ and $x\in\langle x'',1\rangle$, respectively.

4 – We are going to find necessary and sufficient conditions for the existence of a solution of problem (P).

Imposing on function u (see (7)) the boundary conditions (2) and using relation (10), we obtain the system of integral-functional equations

$$\begin{aligned}\varphi_j(x)+\psi_j\circ\alpha_j(x)&=V_j(x)\\ \varphi_j(x)+\psi_j\circ\beta_j(x)&=W_j(x)\end{aligned} \tag{32}$$

[2] Here and in the sequel, const denotes a positive constant.

($x \in \langle 0, 1\rangle$; $j = 0, 1, \ldots, p-1$), where φ_j and ψ_j are the unknown functions sought in the classes C^{p-j}, respectively, and the functions V_j and W_j are defined by

$$(33)\qquad \begin{aligned} V_j(x) &= V_j^1(x) + V_j^2(x) \\ W_j(x) &= W_j^1(x) + W_j^2(x) \end{aligned}$$

with

$$(34)\qquad \begin{aligned} V_j^1(x) &= M_j(x) - R_{p-j}[x, \alpha_j(x)] \\ W_j^1(x) &= N_j(x) - R_{p-j}[x, \beta_j(x)] \end{aligned}$$

and

$$(35)\qquad \begin{aligned} V_j^2(x) &= -\sum_{m=j+2}^{p} G_{m,j}[x, \alpha_j(x)] \\ W_j^2(x) &= -\sum_{m=j+2}^{p} G_{m,j}[x, \beta_j(x)] \end{aligned}$$

($j = 0, 1, \ldots, p-1$); $\sum_{m=m_1}^{m_2} a_m := 0$ for $m_1 > m_2$).

REMARK 3. Evidently, the functions V_j and W_j depend on $r_{j+1}, \ldots, r_p$ (see (13)) for $j = 0, 1, \ldots, p-2$, while V_{p-1} and W_{p-1} are given by

$$(36)\qquad V_{p-1}(x) = V_{p-1}^1(x); W_{p-1}(x) = W_{p-1}^1(x).$$

We shall examine system (32) by using a method analogous to that in [4].

Let $\sigma_0 \in (0, \sigma)$ be a fixed number and denote $x_j = \alpha_j^{-1}(\sigma_0)$; $\tilde{x}_j = \beta_j^{-1}(\sigma_0)$ for $j = 0, 1, \ldots, p-1$. We introduce the rectangles $\Omega_j^1 = \langle 0, \tilde{x}_j\rangle \times \langle 0, \sigma_0\rangle$ and $\Omega_j^2 = \langle \tilde{x}_j, 1\rangle \times \langle \sigma_0, \sigma\rangle$, where $j = 0, 1, \ldots, p-1$ (see Fig. 1).

The following proposition is valid

PROPOSITION 1. *Let j be a fixed integer ($0 \le j \le p-1$) and assume for $0 \le j \le p-2$ that φ_ν and ψ_ν ($\nu = j+1, \ldots, p-1$) are known functions* *of class $C^{p-\nu}$ on $\langle 0,1\rangle$ and $\langle 0,\sigma\rangle$, respectively. System* (32), *considered in the domain Ω_j^1, has a solution given by the formulae*

$$\begin{aligned} \varphi_j(x) &= \varphi_j^0(x) := W_j(x) - S_j \circ \beta_j(x)\,; \\ \psi_j(y) &= \psi_j^0(y) := S_j(y) \end{aligned} \tag{37}$$

($(x,y) \in \Omega_j^1$), *where*

$$S_j(y) = \sum_{n=0}^{\infty} Q_j \circ \tau_j^n(y) \tag{38}$$

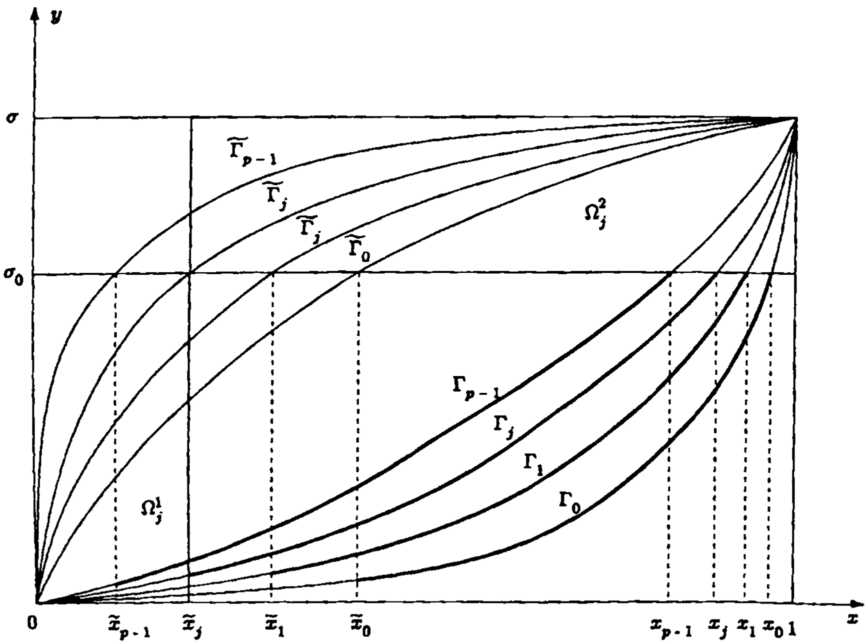

Fig. 1

with

$$Q_j(y) = Q_j^1(y) + Q_j^2(y)\,; \tag{39}$$

(40) $$Q_j^\nu(x) = W_j^\nu \circ \beta_j^{-1}(y) - V_j^\nu \circ \beta_j^{-1}(y)$$

$(\nu = 1, 2)$.

It is the unique solution of the system (32) in the class $\mathcal{K}_j$ of all systems of continuous functions $\varphi_j\colon = \langle 0, \tilde{x}_j\rangle \longrightarrow Y$ and $\psi_j\colon \langle 0, \sigma_0\rangle \longrightarrow Y$, such that $\varphi_j(0) = \psi_j(0) = 0$.

Moreover, the functions φ_j^0 and ψ_j^0 given by formulae (37) are of class C^{p-j}.

PROOF. We shall prove the uniform convergence of the series (38). Of course it is sufficient to consider the case when $N \ni n \geq n_0$ (see Remark 2 with $y' = \sigma_0$ and $x' = x'' = \tilde{x}_j$).

Let us observe (see (6), (34), and (40)) that the equality

(41) $$Q_j^1(0) = 0$$

is valid, whence and by (26) we have

(42) $$\|Q_j^1 \circ \tau_j^n(y)\| \leq \text{const } q^n .$$

Furthermore, it is clear (cp. (10)-(12), (35) and (40)) that as r_m are continuous in Ω_j^1 for $m = j+1, \ldots, p-1$; $0 \leq j \leq p-2$, and $Q_{p-1}^2 \equiv 0$, we have the equality

(43) $$Q_j^2(0) = 0,$$

which, together with (26) yields

(44) $$\|Q_j^2 \circ \tau_j^n(y)\| \leq \text{const } q^n .$$

Thus, by (43) and (44), the series $S_j(y)$ is uniformly convergent for $y \in \langle 0, \sigma_0\rangle$ and the functions φ_j^0 and ψ_j^0 given by (37) are continuous for $x \in \langle 0, \tilde{x}_j\rangle$ and $y \in \langle 0, \sigma_0\rangle$, respectively.

A similar argument can be also used for the series $\sum\limits_{n=0}^{\infty} \dfrac{d^\nu}{dy^\nu}[Q_j \circ \tau_j^n(y))]$, where $\nu = 1, 2, \ldots, p-j$,

To this end we base on formula (30) and we apply inequalities (26), (27) and (29), getting the estimate

(45) $$\|(Q_j \circ \tau_j^n)^{(\nu)}(y)\| \leq \text{const}\, n^{p^2} q^n$$

($\mathbb{N} \ni n \geq n_0; \nu = 1, 2, \ldots, p-j$).

Hence, we can assert that the functions φ_j^0 and ψ_j^0 are of class C^{p-j} for $x \in \langle 0, \bar{x}_j \rangle$; $y \in \langle 0, \sigma_0 \rangle$ and (cp. (41) and (43)) that the system φ_j^0, ψ_j^0 belongs to the class $\mathcal{K}_j$.

As the remaining parts of the thesis of Proposition 1 are easily verified, the proof of this Proposition is completed.

Now, let us consider system (32) (for fixed j) in Ω_j^2, with $x = \alpha_j^{-1}(y)$ in the first equation and $x = \beta_j^{-1}(y)$ in the second one:

$$\begin{aligned} \varphi_j \circ \alpha_j^{-1}(y) + \psi_j(y) &= V_j \circ \alpha_j^{-1}(y) \\ \varphi_j \circ \beta_j^{-1}(y) + \psi_j(y) &= W_j \circ \beta_j^{-1}(y) \end{aligned} \tag{46}$$

We can formulate the following proposition.

PROPOSITION 2. *Under the assumptions of Proposition 1, system* (46), *considered in the domain* Ω_j^2, *has a solution given by the formulae*

$$\begin{aligned} \varphi_j(x) = \tilde{\varphi}_j(x) &:= \widetilde{S}_j(x) + C_j\,; \\ \psi_j(y) = \tilde{\psi}_j(y) &:= W_j \circ \beta_j^1(y) - \widetilde{S}_j \circ \beta_j^{-1}(y) - C_j \end{aligned} \tag{47}$$

($(x, y) \in \Omega_j^2$), *where*

$$\widetilde{S}_j(x) = \sum_{n=0}^{\infty} P_j \circ \mu_j^n(x) \tag{48}$$

with

$$P_j(x) = P_j^1(x) + P_j^2(x)\,; \tag{49}$$

$$P_j^\nu(x) = W_j^\nu(x) - V_j^\nu \circ \mu_j(x) \tag{50}$$

($\nu = 1, 2$), *and* C_j *is an arbitrary constant.*

The functions $\tilde{\varphi}_j$ and $\tilde{\psi}_j$ given by formulae (47) are of class C^{p-j}.

Proof is analogous to that of Proposition 1 and bases on the relations (cp. (3), (5), (34), (35) and (50))

$$P_j^1(1) = 0; \tag{51}$$

$$P_j^2(1) = 0, \tag{52}$$

by which and (27) we have

$$\|P_j \circ \mu_j^n(x)\| \leq \text{const}\, q^n \tag{53}$$

($N \ni n \geq n_0$), and on the inequality

$$\left\|(P_j \circ \mu_j^n)^{(\nu)}(x)\right\| \leq \text{const}\, n^{p^2} \cdot q^n \tag{54}$$

($N \ni n \geq n_0$; $\nu = 1, 2, \dots, p-j$) that is a consequence of estimates (27) and (29), and formula (30).

Thus, in virtue of Propositions 1 and 2, we have

$$\varphi_j(x) = \begin{cases} \varphi_j^0(x) \text{ for } 0 \leq x \leq \tilde{x}_j \\ \tilde{\varphi}_j(x) \text{ for } \tilde{x}_j \leq x \leq 1; \end{cases} \tag{55}$$

$$\psi_j(y) = \begin{cases} \psi_j^0(y) \text{ for } 0 \leq y \leq \sigma_0 \\ \tilde{\psi}_j(y) \text{ for } \sigma_0 \leq y \leq \sigma, \end{cases} \tag{56}$$

and hence φ_j and ψ_j are continuous if and only if

$$\varphi_j^0(\tilde{x}_j) = \tilde{\varphi}_j(\tilde{x}_j); \; \psi_j^0(\sigma_0) = \tilde{\psi}_j(\sigma_0). \tag{57}$$

Using the first of equalities (57)(3) together with (37) and (47), we get

$$C_j = W_j(\tilde{x}_j) - S_j(\sigma_0) - \widetilde{S}_j(\tilde{x}_j). \tag{58}$$

(3) The second of these equalities yields the same result.

We still have to require the first of equations (32) to be satisfied for $x \in \langle \tilde{x}_j, x_j \rangle$ (i.e. the first of conditions (2) to be fulfilled on the part of Γ_j marked on Fig. 1).

By (32), (37), (47), (55) and (56) we can write

$$\widetilde{S}_j(x) + S_j \circ \alpha_j(x) + C_j = V_j(x), \tag{59}$$

whence and from (58) we obtain

$$\widetilde{S}_j(x) + S_j \circ \alpha_j(x) - S_j(\sigma_0) - \widetilde{S}_j(\tilde{x}_j) = V_j(x) - W_j(\tilde{x}_j) \tag{60}$$

$(x \in \langle \tilde{x}, x_j \rangle)$.

On substituting (38) and (48) into (60), and using (34), (39), (40), (49) and (50), we obtain after some rearrangements (cp. [4], p. 362) the following equality

$$E_j(x) + \sum_{n=0}^{\infty} \left\{ \int_{\lambda_j^{n+1}(x_j)}^{\lambda_j^{n+1}(x)} E_j'(t)dt + \int_{\mu_j^{n+1}(\tilde{x}_j)}^{\mu_j^{n+1}(x)} E_j'(t)dt \right\} = 0 \tag{61}$$

$(x \in \langle \tilde{x}_j, x_j \rangle)$ in which

(62)
$$E_j(x) = N_j(x) - M_j(x) + R_{p-j}[x, \alpha_j(x)] - R_{p-j}[x, \beta_j(x)] + W_j^2(x) - V_j^2(x),$$

where the functions V_j^2 and W_j^2 are given by (35) and (11) - (13), with (cp. (37) and (47))

$$r_k(\xi, \eta) = \begin{cases} a_k(\xi, \eta) \text{ for } \xi \in \langle 0, \tilde{x}_k \rangle, \eta \in \langle 0, \sigma_0 \rangle \\ b_k(\xi, \eta) \text{ for } \xi \in \langle 0, \tilde{x}_k \rangle, \eta \in \langle \sigma_0, \sigma \rangle \\ c_k(\xi, \eta) \text{ for } \xi \in \langle \tilde{x}_k, 1 \rangle, \eta \in \langle 0, \sigma_0 \rangle \\ d_k(\xi, \eta) \text{ for } \xi \in \langle \tilde{x}_k, 1 \rangle, \eta \in \langle \sigma_0, \sigma \rangle, \end{cases} \tag{63}$$

a_k, b_k, c_k and d_k being defined by (cp. (37), (47), (55) and (56))

$$\begin{aligned} a_k(\xi,\eta) &= W_k(\xi) - S_k \circ \beta_k(\xi) + S_k(\eta) \\ b_k(\xi,\eta) &= W_k(\xi) + W_k \circ \beta_k^{-1}(\eta) - S_k \circ \beta_k(\xi) - \widetilde{S}_k \circ \beta_k^{-1}(\eta) - C_k \\ c_k(\xi,\eta) &= \widetilde{S}_k(\xi) + S_k(\eta) + C_k \\ d_k(\xi,\eta) &= W_k \circ \beta_k^{-1}(\eta) + \widetilde{S}_k(\xi) - \widetilde{S}_k \circ \beta_k^{-1}(\eta) \end{aligned} \tag{64}$$

(for the meaning of C_k see (58)).

Equality (61), obtained on the basis of (57), is a necessary condition for the existence of a continuous solution φ_j, ψ_j of system (32).

However, φ_j and ψ_j schould belong to C^{p-j} and hence we have to add the conditions

$$\varphi_j^{0(\nu)}(\tilde{x}_j) = \tilde{\varphi}_j^{(\nu)}(\tilde{x}_j);\ \psi_j^{0(\nu)}(\sigma_0) = \tilde{\psi}_j^{(\nu)}(\sigma_0) \tag{65}$$

$(\nu = 1, 2, \dots, p-j)$.

Basing on the second of relations (65), and using (37), (46) and (47), we obtain

$$\Big[\frac{d^\nu}{dy^\nu} S_j(y)\Big]_{y=\sigma_0} = \Big[\frac{d^\nu}{dy^\nu}(V_j \circ \alpha_j^{-1}(y) - \tilde{S}_j \circ \alpha_j^{-1}(y) - C_j)\Big]_{y=\sigma_0} \tag{66}$$

Set $x = \alpha_j^{-1}(y)$, whence $\dfrac{d^\nu}{dy^\nu} = (\alpha_j'(x))^{-\nu}\dfrac{d^\nu}{dx^\nu}$.

Equality (66) together with (33), (34), (38)-(40), (48)-(50) and (58) yields, after repeating the argument used in the derivation of (61), the following relation

$$E_j^{(\nu)}(x_j) + \sum_{n=0}^{\infty}\Big\{\Big[\frac{d^\nu}{dx^\nu}(E_j \circ \lambda_j^{n+1}(x) - E_j \circ \mu_j^{n+1}(x))\Big]_{x=x_j}\Big\} = 0$$

$(\nu = 1, 2, \dots, p-j)$.

By a similar argument, based on the first of relations (65), and on (32), (47)-(50) and (58), we get

$$E_j^{(\nu)}(\tilde{x}_j) + \sum_{n=0}^{\infty}\Big\{\Big[\frac{d^\nu}{dx^\nu}(E_j \circ \lambda_j^{n+1}(x) - E_j \circ \mu_j^{n+1}(x))\Big]_{x=\tilde{x}_j}\Big\} = 0$$

$(\nu = 1, 2, \dots, p-j)$.

Thus, we have obtained the following relation

$$E_j^{(\nu)}(z_j) + \sum_{n=0}^{\infty}\Big\{\Big[\frac{d^\nu}{dx^\nu}(E_j \circ \lambda_j^{n+1}(x) - E_j \circ \mu_j^{n+1}(x))\Big]_{x=z_j}\Big\} = 0 \tag{67}$$

$(z_j = x_j, \tilde{x}_j;\ \nu = 1, 2, \dots, p-j)$.

Equalities (61) and (67) should be considered for $j = p-1, p-2, \ldots, 1, 0$, successively (cp. Remark 3 and Propositions 1 and 2).

It is clear (cp. (11)-(13), (33), (35), (37)-(40), (47)-(50), (62) and (63)) that the said equalities establish relations between the functions $M_j, M_{j+1}, \ldots, M_{p-1}, N_j, N_{j+1}, \ldots, N_{p-1}, F$, and $M_j^{(k)}, M_{j+1}^{(k)}, \ldots, M_{p-1}^{(k)}, N_j^{(k)}, N_{j+1}^{(k)}, \ldots, N_{p-1}^{(k)}, F$ (where $k = 1, 2, \ldots, p-j$), respectively,[4] which are necessary for the existence of a solution $\varphi_j, \psi_j \in C^{p-j}$ $(j = 0, 1, \ldots, p-1)$ of the system of integral-functional equations (32) and hence (cp. Lemma 1 and relations (1), (2) and (10)) of a solution u of problem (P).

It is also easily seen (cp. Lemma 1 and Propositions 1 and 2) that the said equalities are sufficient for the existence of these solutions.

Thus, we can formulate the following theorem

THEOREM. *If Assumptions I-III are satisfied, then conditions* (61) *and* (67) *are necessary and sufficient for the existence of a solution of problem (P).*

REMARK 4. If $p = 1$ and $F \equiv 0$, then conditions (61) and (67) are identical with relations (25) and (26_s) ($s = 1, 2$), respectively, in paper [4].

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(4) Let us note that (61) and (67) are satisfied if

$$M_j(x) = N_j(x) + R_{p-j}[x, \alpha_j(x)] - R_{p-j}[x, \beta_j(x)] + W_j^2(x) - V_j^2(x).$$

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Lavoro pervenuto alla redazione il 8 aprile 1992
ed accettato per la pubblicazione il 14 luglio 1992
su parere favorevole di G. Fichera e di F. Rosati

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