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Moment-preserving approximations: a monospline approach

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RIASSUNTO – *Si affronta il problema di approssimare una funzione con splines defettive, che conservino quanti più possibile momenti della funzione stessa; tale problema viene risolto utilizzando particolari proprietà delle monosplines a nodi multipli.*

ABSTRACT – *Some properties of a class of monosplines with multiple knots are used for the solution of certain problems concerning the approximation of a function f, by defective spline functions, which preserve as many moments of f as possible.*

KEY WORDS – *Splines - Moment-preserving - Monosplines - Quadrature Rules.*

A.M.S. CLASSIFICATION: 41A15 - 65D32 - 33A65

1 – Introduction

In a series of papers [2-5, 18, 19] initially motivated by some computational questions of plasma physics, several authors considered the problem of finding a polynomial spline, approximating a given function f, and preserving as many moments of f as possible. Among other things, the approximation of spherically symmetric distribution in R^d by linear combination of Heaviside step functions or Dirac delta functions has been obtained in [4].

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The approximation considered in the above papers concerns the case of splines with simple knots, or splines with knots having the same odd multiplicity. The problem was treated from different points of view, depending on the assumptions on f. One of the salient aspects of this analysis consists in the fact that the existence and uniqueness of the requisite spline can be related to the existence and uniqueness of suitable quadrature rules.

In a nice recent paper [18] another approach to the problem, arising from the close connection between quadrature rules and monosplines [21], is presented. The results of [18] concern the recovery of a function on a finite interval by splines with simple knots, and provide several extentions of the original problem mentioned above.

In this paper, the construction of moment-preserving approximations is obtained by splines with multiple knot (the multiplicity being possibly different from one knot to another) which satisfy three different types of conditions.

Following the ideas of [18], the corresponding problems are put into a unified framework and treated using some properties of monosplines, with multiple knots, which have recently been obtained in a paper of ZHENSYKBAEV [26].

Furthermore, we establish a convergence theorem for Gaussian quadrature rules with multiple knots, from which the convergence of the sequence of approximating splines, when the number of knots tend to infinity, can be derived. It is possible to apply this theorem also to the sequences of splines given in [3], where the convergence question is not addressed. Therefore, in this sense, we complete the results presented there.

2 – Preliminaries and statement of the problem

A defective spline function on [0,1] of degree m ($m \geq 2$), with n distinct knots $x_1, x_2, \dots, x_n$, respectively of multiplicity $k_1+1, k_2+1, \dots, k_n+1$, has the form

$$(2.1) \qquad s_{nm}(x) = p_m(x) + \sum_{i=1}^{n}\sum_{j=0}^{k_i} a_{ij}(x - x_i)_+^{m-j}, \quad 0 \leq x \leq 1,$$

where a_{ij} are real numbers, p_m is a polynomial of degree $\leq m$ and

$$0 < x_1 < x_2 < \ldots < x_n < 1,$$
$$0 < k_i \leq m, \quad i = 1(1)n,$$

$$\begin{cases} (x - x_j)_+ : = \max((x - x_i), 0), \\ (x - x_i)_+^j = [(x - x_i)_+]^j, \qquad i = 1(1)n, \qquad j \in \mathbb{N}. \end{cases}$$

We shall call k_i the defect of x_i, $i = 1(1)n$; the case $k_i = 0$ for each i, gives rise to ordinary splines.

A monospline M of degree $m + 1$ with multiple knots x_i, $i = 1(1)n$, is a function of the form

$$M(x) = C\frac{x^{m+1}}{(m+1)!} + s_{nm}(x), \tag{2.2}$$

where C is a positive constant, see [22, p.330, p.403]. A more general definition of monospline, given in [18], is obtained replacing the first term of (2.2) by $F(x)$, where $F \in C^{m+1}[0, 1]$ with nonvanishing $(m+1)$-st derivative on $(0, 1)$.

It is well known that a typical example of monospline is given by the Peano kernel of a quadrature rule $Q(f)$, which is exact for $f \in P_{m-1}$. Indeed, the remainder of such a rule may be expressed, for any $f \in C^m[0, 1]$, by

$$\int_0^1 R_t\Big((x - t)_+^{m-1}\Big) f^{(m)}(x)dx,$$

where $R_t((x - t)_+^{m-1})$, the Peano kernel, is a polynomial spline [14,21].

More generally, the remainder $R(f)$ of a quadrature rule, exact for the solutions $f \in C^m[0, 1]$ of a homogeneous linear differential equation of order m, can be represented in the form

$$R(f) = \int_0^1 \Phi(x) f^{(m)}(x)dx,$$

where $\Phi(x)$ is a monospline constructed by L-splines [22], as one can prove by the use of the Green-Lagrange identity [6].

In the sequel, we shall be concerned with the case of knots of odd multiplicity, that is,

$$k_i = 2\sigma_i, \quad i = 1(1)n, \tag{2.3}$$

where

$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n).$$

is a vector of given nonnegative integers.

We will denote by Ω_{nm} the linear space of defective splines (2.1), with defects (2.3), and we also set

$$N^* = 2\Big[\sum_{i=1}^{n}(\sigma_i + 1)\Big]. \tag{2.4}$$

We consider the following problems:

A - Determine $s_{nm} \in \Omega_{nm}$ such that

$$\int_0^1 t^j s_{nm}(t)dt = \int_0^1 t^j f(t)dt, \qquad j = 0(1)N^* + m. \tag{2.5}$$

B - Determine $s_{nm} \in \Omega_{nm}$ such that

$$\int_0^1 t^j s_{nm}(t)dt = \int_0^1 t^j f(t)dt, \qquad j = 0(1)N^* - 1, \tag{2.6}$$

$$s_{nm}^{(k)}(1) = f^{(k)}(1), \qquad k = 0(1)m. \tag{2.7}$$

C - Determine $s_{nm} \in \Omega_{nm}$ such that

$$\int_0^1 t^j s_{nm}(t)dt = \int_0^1 t^j f(t)dt, \qquad j = 0(1)r, \tag{2.8}$$

$$s_{nm}^{(k)}(1) = f^{(k)}(1), \qquad k = 0(1)l, \quad l \le m. \tag{2.9}$$

with $l + r + 1 = N^* + m$.

Problem C is a generalization of problems A and B, the latter being obtained by letting $l = -1, r = N^* + m$ and $l = m, r = N^* - 1$, respectively.

The case $\sigma_i = 0$, $i = 1(1)n$, has been considered in [2, 18] and some results concerning the case $\sigma_i = s, i = 1(1)n$, were obtained in [3].

3 – Solution of the problems

The solution of the above problems can be analyzed in terms of certain properties of monosplines. The properties we need are contained in some recent results of ZHENSYKBAEV, [26], which we summarize in the following theorem.

THEOREM Z. *Let the points* $0 \le t_1 < \ldots < t_k \le 1$ *and the integers* ρ_i, $i = 1(1)k$, *be given where*

$$1 \le \rho_1 \le \mu + 1, \sum_{i=1}^{k} \rho_i = \mu + N^*, \rho_1 \le \mu + \operatorname{sgn} t_1, \rho_k \le \mu + \operatorname{sgn}(1 - t_k).$$

There exists a unique monospline of degree μ *with zeros at the points* t_i *of multiplicities* $\rho_i, i = 1(1)k$, *if and only if the conditions*

$$(3.1)\quad 2\sum_{j=1}^{i}(\sigma_j+1)-1 \le \sum_{j=1}^{\rho_i} \rho_j \le \mu-(2\sigma_i+1)+2\sum_{j=1}^{i}(\sigma_j+1), \quad i = 1(1)n,$$

hold for some indices $\rho_1 \le \rho_2 \le \cdots \le \rho_n$, *where* $2\sigma_i + 1$, $i = 1(1)n$, *are the multiplicities of monospline knots.*

Now, assume that the function f satisfies the conditions

$$(3.2)\qquad f \in C^{m+1}[0,1], \quad f^{(m+1)} \neq 0, \qquad \text{a.e. in } [0,1],$$

and denote by μ the following integer depending on N,

$$(3.3)\qquad \mu = N + m + 1,$$

and by $K_h(x, \xi)$ the kernels

$$(3.4)\qquad K_h(x,\xi) = \frac{(x-\xi)_+^h}{h!}, \quad h = 0, 1, \ldots, N-1.$$

THEOREM 1. *Under assumptions* (3.2) *on* f, *problem* A *has a unique solution.*

PROOF. For every spline s_{nm}, of the form (2.1), we introduce the monospline M

$$M(x) = \int_0^1 K_{N-1}(x,t)\Big[f(t) - s_{nm}(t)\Big]dt. \tag{3.5}$$

Assuming

$$N = N^* + m + 1, \tag{3.6}$$

M is a monospline of degree μ having a zero of multiplicity N at $x = 0$. Moreover, it is easy to derive from (3.5) the following relations

$$M^{(N-k)}(1) = \sum_{h=0}^{k-1}\binom{k-1}{h}\frac{(-1)^h}{(k-1)!}\int_0^1 t^h\Big[f(t) - s_{nm}(t)\Big]dt, \qquad k = 1(1)N. \tag{3.7}$$

Formula (3.7) allows us to translate condition (2.5) into the following equivalent conditions on M:

$$\begin{cases} M^{(j)}(0) = 0, & j = 0(1)N-1, \\ M^{(j)}(1) = 0, & j = 0(1)N-1. \end{cases} \tag{3.8}$$

Now, existence and uniqueness of a monospline M, fulfilling (3.8), can be derived from Theorem Z. In fact, putting $\rho = \max \rho_i, \tau = \max(2\sigma_i + 1), i = 1(1)n$, yields

$$\rho + \tau \leq \mu + 2,$$

This condition, as remarked in [26], is sufficient in order that (3.1) holds. □

Concerning the remaining problems, we can prove the following Theorem.

THEOREM 2. *Under assumptions* (3.2) *on* f, *problems* B *and* C *have a unique solution.*

PROOF. We need consider only problem C, since problem B reduces to C assuming $r = N^* - 1, l = m$. For problem C we introduce the monospline

$$L(x) = \int_0^1 K_r(x,t)\Big[f(t) - s_{nm}(t)\Big]dt. \tag{3.9}$$

L is a monospline of degree $r + m + 2$ and conditions (2.8), (2.9) yield

$$\begin{cases} L^{(j)}(0) = 0, & j = 0(1)N - l - 2, \\ L^{(j)}(1) = 0, & j = 0(1)l + r + 1, \end{cases} \tag{3.10}$$

where N is given by (3.6). For $l = m$ and $r = N^* - 1$ (3.10) are equivalent to conditions (2.6), (2.7) of problem B.

As for condition (3.1), we remark that here we have

$$\sum_{i=1}^{k} \rho_i = N + r + 1, \quad k = 2, \; t_1 = 0, \; t_2 = 1.$$

Moreover for every $i = 1(1)n, \rho_i = 1$ and

$$2\sum_{j=1}^{k}(\sigma_j + 1) - 1 \leq N - l - 1 \leq r + m + 2 - (2\sigma_i + 1) + 2\sum_{j=1}^{i}(\sigma_j + 1),$$

thus (3.1) holds, which implies the claim. □

Now, we turn to the evaluation of the splines, which solve our problems. For this purpose, it is convenient to stress the connection between monosplines and quadrature rules.

Concerning the case A, we first note that from (3.5) it follows

$$M^{(\mu)}(x) = f^{(m+1)}(x), \tag{3.11}$$

and so successive integration by parts yields, recalling formula (3.3),

$$\int_0^1 g(x)f^{(m+1)}(x)dx = \sum_{j=0}^{\mu-1}(-1)^{\mu-j}M^{(j)}(0)g^{(\mu-j-1)}(0)+$$

$$(3.12)\qquad +\sum_{j=0}^{\mu-1}(-1)^{\mu-j-1}M^{(j)}(1)g^{(\mu-j-1)}(1)+$$

$$+\sum_{j=1}^{n}\sum_{j=0}^{k_i}(-1)^j\Big[M^{(\mu-j-1)}(x_i^-)-M^{(\mu-j-1)}(x_i^+)g^{(j)}x_i+$$

$$+\int_0^1 M(x)dg^{(\mu-1)}(x).$$

Therefore, letting

$$A_j=(-1)^{m-j+1}M^{(N+j)}(0),\quad B_j=(-1)^{m-j}M^{(N+j)}(1),$$
$$C_{ij}=(-1)^j\Big[M^{(\mu-j-1)}(x_i^-)-M^{(\mu-j-1)}(x_i^+)\Big],$$

and using (3.8), we obtain the following Lobatto quadrature rule with multiple knots

$$(3.13)\qquad \int_0^1 g(x)f^{(m+1)}(x)dx \cong \sum_{h=0}^{m}A_h g^{(m-h)}(0)+$$
$$+\sum_{h=0}^{m}B_h g^{(m-h)}(1)+\sum_{i=1}^{n}\sum_{j=0}^{k_i}C_{ij}g^{(j)}(x_i).$$

It is possible to give a more precise evalutation of the knots and the coefficients of s_{nm} in terms of the quadrature rule (3.13). In fact, taking into account (2.1), one gets

$$a_{ij}=\frac{(-1)^j}{(m-j)!}C_{ij},\quad i=1(1)n,\quad j=0(1)k_i,$$

and

$$p_m^{(m-k)}(0)=f^{(m-k)}(0)+(-1)^k A_{m-k},\quad k=0(1)m.$$

Moreover the knots x_i are the zeros of the polynomial of degree n, σ-orthogonal in $[0,1]$ with respect to the weight function

$$w(x)=\gamma x^{m+1}(1-x)^{m+1}f^{(m+1)}(x),$$

(γ is a constant such that $w(x) > 0$ a.e. on [0,1]).

An useful method for the calculation of such zeros is treated, for instance, in [8], while the connection between the coefficients of (3.13) and a related Turán-type quadrature rule is developed in [9] and [10].

In the case of problems B and C, a reasoning similar to that given above, now using the monospline $L(x)$ introduced in Theorem 2, establishes a connection between the splines of these cases and certain quadrature rules.

Specifically, putting $\nu = r + m + 2$, one has

$$L^{(\nu)}(x) = f^{(m+1)}(x),$$

and, since $L(x)$ fulfils (3.10), the quadrature rule in this case is

$$\begin{aligned}\int_0^1 g(x)f^{(m+1)}(x)dx &\cong \sum_{h=0}^{m} E_h g^{(m-h)}(0)+ \\ &+\sum_{h=0}^{m-l-1} F_h g^{(m-l-h-1)}(1) + \sum_{i=1}^{n}\sum_{j=0}^{k_i} G_{ij} g^{(h)}(x_i).\end{aligned} \tag{3.14}$$

The degree of precision of (3.14) is ν, and one has

$$E_j = (-1)^{\nu-j+1}L^{(j)}(0), \quad F_j = (-1)^{\nu-j}L^{(j)}(1),$$
$$G_{ij} = (-1)^{j-1}\left[L^{(\nu-j-1)}(x_i^-) - L^{(\nu-j-1)}(x_i^+)\right].$$

where $F_k = 0$ for $k > m - l - 1$.

Here, one has

$$a_{ij} = \frac{(-1)^j}{(m-j)!}G_{ij}, \quad i = 1(1)n, \quad j = 0(1)k_i,$$

and

$$p_m^{(m-k)}(0) = f^{(m-k)}(0) + (-1)^k E_{m-k}, \quad k = 0(1)m,$$

The knots x_i are the zeros of the polynomial of degree n, σ-orthogonal in $[0, 1]$ with respect to the weight function

$$\chi(x) = \gamma x^{m+1}(1-x)^{m-l} f^{(m+1)}(x).$$

(γ is a constant such that $w(x) > 0$ a.e. on $[0,1]$).

We remark that in the case B, quadrature rule (3.14) reduces to a Radau formula, while for $l < m$, (3.14) is a quadrature formula with multiple knots of STANCU type [25].

4 – A convergence result

In this section, we wonder whether or not the sequences of the splines s_{nm} obtained above converge to f when n tends to infinity. The answer to this question is affirmative and its proof will be based on Theorem 3 below.

For this purpose, we recall that

$$f(x) - s_{nm}(x) = R_{nm}(\rho_x; d\sigma_m),$$

where $R_{nm}(\rho_x; d\sigma_m)$ is the remainder term of quadrature formulas (3.13) (or (3.14)) and

$$\begin{aligned} &\rho_x(t) = (x-t)_+^m, \qquad && t \in [0,1], \\ &d\sigma_m = w(x)dx, && (\text{or } d\sigma_m = \chi(x)dx). \end{aligned}$$

The above quadrature rules have the preassigned nodes $y_1 = 0$ with multiplicity α_1 and $y_2 = 1$ with multiplicity α_2. They are particular cases of the class of quadrature rules with preassigned nodes y_j with multiplicity $\alpha_j, j = 1(1)h$, considered in [9,11], where α_j is even if $y_j \in (0,1)$. To state our result we find the following notation more convenient

$$\prod(x) = (-1)^{\alpha_h} \prod_{j=1}^{h} (x - y_j)^{\alpha_j};$$

$$\begin{cases} t_j &= \alpha_j - 1, \quad j = 1(1)h, \\ \tau &= \max(k_1, k_2, \dots, k_n, t_1, t_2, \dots, t_h); \quad T = \sum_{i=1}^{h} \alpha_i. \end{cases}$$

Let $d\alpha(x)$ be a (positive) Stieltjes measure, with $\alpha(x)$ having infinite number of points of increase and $d\alpha(x)$ having all finite moments.

Moreover if f is sufficiently smooth, we can define the following linear functional

$$F_n(f) = \sum_{j=1}^{h} \sum_{k=0}^{t_j} B_{k_j} f^{(k)}(y_j) + \sum_{i=1}^{n} \sum_{h=0}^{k_i} A_{hi} f^{(k)}(x_{in}),$$

such that

$$F_n(f) = \int_0^1 f(x) d\alpha(x) =: I(f),$$

for every $f \in P_Q$, where $Q = N^* + T - 1$ and P_Q is the space of polynomials of degree $\leq Q$.

The nodes x_{in} coincide with the zeros of polynomials $P_n(d\beta, x)$ of n degree, σ-orthogonal in $[0,1]$ with respect to the measure

$$d\beta(x) = \prod(x) d\alpha(x).$$

Let g be the function

$$g(x) = [f(x) - H_f(x)] / \prod(x).$$

where $H_f(x)$ is the interpolating Lagrange-Hermite polynomial of the function f, related to the preassigned nodes $y_i, j = 1(1)h$.

We prove the following Theorem 3 below.

THEOREM 3. *Let $f \in C^r[0,1]$. We have*

$$\lim_{n \to \infty} F_n(f) = I(f), \tag{4.1}$$

and, furthermore, the operator

$$R_n(f) = I(f) - F_n(f),$$

is such that

$$R_n(f) = 0[(1/n)^r],$$

if the Gaussian nodes satisfy the conditions

$$|x_{i+1,n} - x_{in}| < Cn^{-1}, \qquad i = 1(1)n.$$

where C is independent of n and i.

PROOF. The proof follows the same line of the reasoning of Theorem 1 in [11]. Specifically, we write

$$I(f) = I(H_f) + I(f - H_f) = I(H_f) + \int_0^1 g(x) d\beta(x) .$$

and remark that it is possible to prove, as in above mentioned Theorem 1, that $g \in C^\nu[0,1]$, where $\nu = \max(k_1, k_2, \dots, k_n)$. By Theorem 2 in [11] the convergence (4.1) is assured. □

We remark that Theorem 3 holds even if $f \in C^{\tau-1}[0,1], f^{(\tau)}(x)$ exists and is continuous at least in a neighborhood of the points where it is required.

An approach to the solution of problems A and B in terms of functional moments is contained in [12].

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