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The general inverse quartic interaction

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RIASSUNTO – *Si vogliono ottenere espressioni analitiche degli autovalori e delle autofunzioni dell'equazione di Schrödinger con un potenziale che è l'inverso di un polinomio del quarto ordine. A tale scopo si utilizza la soluzione esplicita di una equazione differenziale lineare del secondo ordine le cui due singolarità sono entrambe irregolari del secondo tipo.*

ABSTRACT – *The explicit solution of a second order linear differential equation the only two singularities of which are both irregular of the second type, is used to obtain analytic expressions for the eigenvalues and eigenfunctions of the Schrödinger equation in which the potential consists of the general inverse quartic polynomial.*

KEY WORDS – *Confluent Heun.*

A.M.S. CLASSIFICATION: 33E20 - 33E30 - 34A30 - 81Q05

1 – Introduction

The Schrödinger equation with a special inverse fourth-power potential has been studied by BÜHRING [1], where it was shown to be related to Mathieu's equation. This present study is concerned with the same type of differential equation with the general inverse quartic potential for a particle of mass μ, namely

$$u'' + 2\mu n^{-2}[W - V(x)]u = 0\,. \tag{1.1}$$

The potential $V(x)$ is given by

$$V(x) = \frac{\hbar^2}{2\mu}\left(A_1x^{-1} + A_2x^{-2} + A_3x^{-3} + A_4x^{-4}\right), A_4 \neq 0, \tag{1.2}$$

and which may be associated with a type of double-well potential infinite at the origin. From the form of this potential we consider $0 \leq x \leq \infty$.

On putting

$$u = x^{b/2}\exp[ax/2 - c/(2x)]y, \tag{1.3}$$

(1.1) takes the form

$$x^2y'' + (ax^2 + bx + c)y' + (dx + f)y = 0, \tag{1.4}$$

where

$$\begin{aligned} &a = \pm\, 2\sqrt{(-2\mu W)/\hbar}, \qquad b = 2 + 2A_3/c, c = \pm 2\sqrt{A_4}, \\ &d = ab/2 - A_1 \quad \text{and} \quad f = ab/2 + b/2(b/2 - 1) - A_2. \end{aligned} \tag{1.5}$$

Since, in a physical context, (1.1) is in the real domain, it follows that a and c are either real or purely imaginary, although the parameters b, d and f may be complex. If $A_4 = 0$, the following analysis is not applicable and a separate discussion is necessary.

The differential equation (1.4) is the doubly confluent Heun equation characterised by the Ince symbol $[0, 0, 1_2]$ having only two singularities both of which are irregular of the second type. From the point of view of the Frobenius method of tacking linear differential equations, such a case would present considerable difficulty. It would normally be expected that convergent series representations could not be obtained by that method.

LEAVER [4] has indicated the rather suprising existence of convergent series representations of solutions of the doubly confluent Heun equation in a slightly different form in a cosmological context. In the course of a systematic investigation of all the confluent forms of Heun's equation, EXTON [3] has obtained explicit forms of such solutions.

2 – Solutions of (1.4)

The Laplace transform of (1.4) may easily be shown to be

$$s(1-s)u'' + \left[2 - d/a - (4-b)s\right]u' = [2 + f - b - acs]u = 0\,, \tag{2.1}$$

where $s = -t/a$ and

$$y = \int_\Gamma e^{xt} u(t)dt\,. \tag{2.2}$$

The form of the contour of integration Γ will be indicated below. This differential equation is a generalised Mathieu equation with Ince symbol $[0, 2, 1_1]$. EXTON [2] has be given the following explicit solution of (2.1):

$$\begin{aligned} &CH(A,B;C;K;s) = \\ &= \sum_{q=0}^{\infty} \frac{K^q}{4^q q!(C/2+1/2,q)} \sum A_{m_0}, \dots, m_q(A,B;C) s^{m_0+\cdot\cdot+m_q+2q}\,. \end{aligned} \tag{2.3}$$

The symbol $A_{m_0,\dots,m_q}(A,B;C)$ denotes the coefficient

$$\begin{aligned} &\frac{(A,m_0)(B,m_0)(C+1,m_0)(2,m_0)(A+2,m_0+m_1)(B+2,m_0+m_1)(C+3,m_0+m_1)(4,m_0+m_1)}{(A+2,m_0)(B+2,m_0)(C,m_0)(1,m_0)(A+4,m_0+m_1)(B+4,m_0+m_1)(C+2,m_0+m_1)(3,m_0+m_1)} \\ &\times \dots \times \\ &\frac{(A+2q-2,m_0+\cdot\cdot+m_{q-1})(B+2q-2,m_0+\cdot\cdot+m_{q-1})(C+2q-1,m_0+\cdot\cdot+m_{q-1})(2q,m_0+\cdot\cdot+m_{q-1})}{(A+2q,m_0+\cdot\cdot+m_{q-1})(B+2q,m_0+\cdot\cdot+m_{q-1})(C+2q-2,m_0+\cdot\cdot+m_{q-1})(2q-1,m_0+\cdot\cdot+m_{q-1})} \\ &\times \frac{(A+2q,m_0+\cdot\cdot+m_q)(B+2q,m_0+\cdot\cdot+m_q)}{(C+2q,m_0+\cdot\cdot+m_q)(1+2q,m_0+\cdot\cdot+m_q)} \end{aligned} \tag{2.4}$$

and the Pochhammer symbol (a, m) denotes the product

$$a(a+1)(a+2)\dots(a+m-1) \tag{2.5}$$

Here and in what follows, the summation sign $\sum$ without further qualification applies for each of the indices of summation $m_0, m_1, \dots, m_q$ for all non-negative integer values. The parameters A, B, C and K are given by $A + B = 3 - b, AB = 2 + f - b, C = 2 - d/a$ and $K = -ac$.

By means of a simple transformation of the generalised Mathieu equation, the function

$$s^{1-C}(1-s)^{C-A-B} CH(1-A, 1-B; 2-C; K; s) \tag{2.6}$$

is also seen to be a solution of (2.1), and this is more convenient than (2.3) for the present purposes. The required solution of (1.4) then follows if (2.6) is inserted into (2.2), the contour of integration Γ being a Pochhammer double-loop slung around the points 0 and 1 in the s-plane. This solution is

$$\begin{aligned} &DC(a,b,c,d,f;x) = \\ &= \sum_{q=0}^{\infty} \frac{K^q}{4^q q!(3/2-C/2,q)} \sum A_{m_0},\dots,m_q(1-A,1-B;2-C)\cdot \\ (2.7) \quad &\cdot \frac{(2-C,m_0+\dots+m_q+2q)}{(3-A-B,m_0+\dots+m_q+2q)}\cdot \\ &{}_1F_1\left(\begin{matrix} 2-C+m_0+\dots+m_q+2q\,; \\ 3-A-B+m_0+\dots+m_q+2q\,; \end{matrix} -ax\right). \end{aligned}$$

See EXTON [2].

If y is replaced by $\exp(c/x)y$ in (1.4), it follows that

$$\exp(c/x)x^{2-b}DC(a,4-b,-c,d+2a-ab,f-ac-b+2;x) \tag{2.8}$$

is an independent solution of the doubly confluent Heun equation.

3 – The asymptotic behaviour of $DC(a,b,c,d,f;x)$ for large real values of x

Before discussing the eigenvalues of the system under consideration, the behaviour of the doubly confluent Heun function must be investigated for large real values of x. This is carried out by noting that the inner confluent hypergeometric function of (2.7) may be written asymptotically as

$$\begin{aligned} &\frac{\Gamma(3-A-B+m_0\cdot+m_q+2q)}{\Gamma(2-C+m_0+\cdot+m_q+2-q)}\exp(-ax)(-ax)^{A+B-C-1}+ \\ (3.1) \quad &+\frac{\Gamma(3-A-B+m_0+\cdot+m_q+2q)}{\Gamma(C-A-B-1)}(ax)^{C-2-m_0-,m_q-2q}. \end{aligned}$$

See SLATER [5] page 60 for example.

If this expression is inserted into (2.7), we have, after a little reduction,

$$DC(a,b,c,d,f;x) \sim$$
$$\sim \frac{\Gamma(b)}{\Gamma(d/a)}\exp(-ax)(-ax)^{d/a-b}CH\Big(\frac{b-1=\sqrt{[(b-1)^2-4f]}}{2},$$
$$\frac{b-1-\sqrt{[(b-1)^2-4f]}}{2};d/a;-ac;1\Big)+\frac{\Gamma(b)}{\Gamma(b-d/a)}(ax)^{d/a},$$

for large real values of x.
The function $CH(\frac{b-1+\sqrt{[(b-1)^2-4f]}}{2},\ \frac{b-1-\sqrt{[(b-1)^2-4f]}}{2};$ $d/a;-ac;1)/\Gamma(d/a)$ is defined for all values of its parameters provided that $\mathrm{Re}(d/a-b+1)\le 0$ when its series representation does not converge.

4 – The eigenvalues of the system

Various cases need to be considered separately, beginning with the behaviour of the wave function at the origin. If $\mathrm{Re}(c)>0$, we take

$$u(x)=x^{b/2}\exp(ax/2-c/(2x))DC(a,b,c,d,f;x), \tag{4.1}$$

using (2.7). The function $u(x)$ is then finite at the origin.

By means of (3.2), for large values of x

(4.2)
$$u(x)\sim\Gamma(b)/\Gamma(d/a)\exp(-ax/2)(-a)^{d/a-b}x^{d/a-b/2}\cdot$$
$$\cdot CH\Big(\frac{b-1+\sqrt{[(b-1)^2-4f]}}{2},\frac{b-1-\sqrt{[(b-1)^2-4f]}}{2};d/a;-ac;1\Big)+$$
$$+\Gamma(b)/\Gamma(b-d/a)\exp(ax/2)z^{-d/a}x^{b/2-d/a}.$$

The generalised Mathieu function of unit argument converges if $\mathrm{Re}(d/a-b)>-1$, or $A_3\sqrt{A_4}>\hbar A_1\sqrt{(-2W)}$, which must be taken into account if the associated term of (4.2) is not recessive.

On putting $\mathrm{Re}(a)>0$, the first term of (4.2) is recessive, and the second term vanishes if

$$b-d/a=-N, \tag{4.3}$$

where N is a non-negative integer. In terms of the original parameters of the system, (4.3) becomes

$$W = -\frac{-h^2 A_1^2}{8\mu[N+1+A_3/(2\sqrt{A_4})]} \tag{4.4}$$

thereby furnishing one set of eigenvalues.

On the other hand, if $\mathrm{Re}(a) > 0$, the second term of (4.2) is recessive, so that the eigenvalues are given by the negative real zeros of the function

$$CH\left(\frac{b-1+\sqrt{([b-1]^2-4f)}}{2}, \frac{b-1-\sqrt{([b-1]^2-4f)}}{2}; d/a; ac; 1\right). \tag{4.5}$$

For convenience, the original parameters of the system are not restored to (4.5) or to similar expression below, while it is recalled that a, f and d are functions of the eigenvalues.

When the eigenvalues are non-negative, $\mathrm{Re}(a) = 0$. Neither term of (4.2) is then recessive, and if the eigenfunctions are to be finite at infinity, $\mathrm{Re}(b/2-d/a)$ must vanish, since $b-d/a$ is now complex. The convergence condition $\mathrm{Re}(d/a-b) > -1$ in this case implies that $b < 2$ or that $A_3 < 0$. If this condition is met, then a continuous positive spectrum results as would would be expected on grounds of intuition.

From (2.8), eigenvalues of the type

$$\begin{aligned} u(x) =& x^{2-b/2}\exp(ax/2+c/(2x))\cdot \\ &\cdot DC(a, 4-b, -c, d+2a-ab, f-ac-b+2; x), \end{aligned} \tag{4.6}$$

arise which are appropriate of $\mathrm{Re}(c) < 0$. From (3.2), for large values of x,

$$\begin{aligned} u(x) \sim\ & \Gamma(4-b)/\Gamma(2-b+d/a)\exp(-ax/2)(-a)^{d/a-2}x^{d/a-b/2}\cdot \\ &\cdot CH\left(\frac{2-b+\sqrt{([b-1]^2-4f+4ac)}}{2}, \right. \\ &\left. \frac{2-b-\sqrt{([b-1]^2-4f+4ac)}}{2}; 2+d/a-b; ac; 1\right)+ \\ &+\Gamma(4-b)/\Gamma(2-d/a)\exp(ax/2)a^{b-d/a-2}x^{b/2-d/a}. \end{aligned} \tag{4.7}$$

The generalised Mathieu function converges of $\mathrm{Re}(d/a) > 0$.

When $\mathrm{Re}(a) > 0$, the first term of (4.7) is recessive, and the corresponding eigenvalues are given by

$$d/a = N - 2. \tag{4.8}$$

If $\mathrm{Re}(a) < 0$, the second term of (4.7) is recessive, and the eigenvalues are given by the zeros of

$$CH\Big(\frac{2-b+\sqrt{([b-1]^2-4f+4ac)}}{2}, \frac{2-b-\sqrt{([b-1]^2-4f+4ac)}}{2}; 2+d/a-b; ac; 1\Big) \tag{4.9}$$

as a function of the eigenvalues.

As before, if positive eigenvalues are considered, a continuous spectrum arises if $\mathrm{Re}(d/a) = 1 - A_3/\sqrt{A_4} > 0$.

5 – Appendix: Computational Aspects

Difficulties arise in directly computing the generalised Mathieu function of unit argument. This is tackled by considering an analytic continuation formula which may be deduced from solutions of the associated differential equation (4.1) valid near the point $x = 1$. Such solutions are given in EXTON [2] and the result in question may be written as

(5.1)

$$\begin{aligned} CH(A,B;C;K;x) = \\ = H_1 CH\Big(&\frac{A+B+\sqrt{([A_B]^2-4K)}}{2}, \frac{A+B-\sqrt{([A-B]^2-4K)}}{2}; \\ &A+B+1-C; -K; 1-x\Big) + \\ +H_2(1-x)^{C-A-B} CH\Big(&\frac{2C-A-B+\sqrt{([A-B]^2-4K)}}{2}, \\ &\frac{2C-A-B-\sqrt{([A-B]^2-4K)}}{2}; 1+C-A-B; -K; 1-x\Big). \end{aligned}$$

The constants H_1 and H_2, which are functions of A, B, C and K but independent of x can be determined by computing the functions of (5.1) for two convenient values of x, for example 0.45 and 0.55. Such an approach is necessary because the series representation of the generalised Mathieu function converges very slowly on its circle of convergence and is, in fact, equal to $H_1(A, B; C; K)$.

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