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Generalized Bessel functions and exact solutions of partial differential equations

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RIASSUNTO – *In questo articolo si dimostra come le funzioni di Bessel generalizzate possano essere proficuamente impiegate nella ricerca di soluzioni esatte di alcune equazioni differenziali alle derivate parziali. In particolare, le funzioni di Bessel generalizzate con le rispettive estensioni a più variabili costituiscono soluzioni dell'equazione della diffusione in più dimensioni, dell'equazione di Schrödinger, di quella di Klein-Gordon, nonché di altre equazioni differenziali di interesse in applicazioni fisiche.*

ABSTRACT – *In this paper we show that the generalized Bessel functions can be succesfully exploited in the search of exact solutions of some partial differential equations. In particular, we will prove that the generalized Bessel functions and their multivariable extension are the natural solutions of the multidimensional diffusion equation, of the Schrödinger and Klein-Gordon equations as well as of other equations of interest for physical applications.*

KEY WORDS – *Generalized Bessel functions - Partial differential equations - Diffusion equation - Schrödinger equation - Klein-Gordon equation.*

A.M.S. CLASSIFICATION: 33C10 - 33C99 - 35C05 - 35D05

1 – Introduction

The theory of generalized Bessel functions (G.B.F) has been initially motivated by their usefulness to treat physical problems related, e.g., to Compton scattering by intense laser waves, to undulator brightness and to Free Electron Laser [1-7]. The intrinsic mathematical importance of these functions has also been recognized mainly in connection with their relations to the exponentials of trigonometric series and because they provide the natural solutions of some partial differential equations (PDE) (see [8-13] and references therein). It has been, indeed, shown that two-variable G.B.F. satisfy PDE of the Schrödinger and Helmholtz type.

This paper is addressed to a more systematic study of the link between G.B.F. and PDE and we will show that they provide the natural solutions of a large body of the most interesting PDE of mathematical physics (wave propagation, multidimensional diffusion equation, Schrödinger, Klein-Gordon as well many others). Furthermore, we will see that, using the wealth of recurrence properties associated to the G.B.F., an almost infinite set of exactly solvable PDE can be constructed.

The paper is organized in four Sections. In Sec. 2 we introduce G.B.F., fix the notation and review their main properties. Sec. 3 deals with the specific topic of the paper and Sec. 4 is devoted to the concluding remarks.

2 – The generalized Bessel functions

In previous works [6-8], we have presented various types of Generalized Bessel Functions. The starting point of our investigation has been presented by the two-variable, one-index G.B.F. defined by the series

$$(1) \qquad {}^{(m)}J_n(x_1, x_2) = \sum_{l=-\infty}^{\infty} J_{n-ml}(x_1) J_l(x_2)\,, \quad (x_1, x_2) \in \mathbb{R}$$

whose relevance for scattering problems was stressed, several years ago, by Reiss, Brown and Kibble (see [3-5]) in the case $m = 2$.

It is worth noticing that the series in eq. (1) has the same form of the series appearing in the addition theorem for ordinary Bessel functions; this fact assures the convergence of the series and, therefore, the correctness of the definition (1).

The functions defined by eq. (1) can be considered as a particular case of the following two-variable, one-index and one-parameter G.B.F.

specified by the series

$$(2) \qquad {}^{(m)}J_n(x_1, x_1; s) = \sum_{l=-\infty}^{\infty} s^l J_{n-ml}(x_1) J_l(x_2), \quad s \in \mathbb{C}$$

whose convergence properties will be discussed in the following.

It is worth pointing out that functions (2) reduce to the ordinary Bessel functions when $x_1 = 0$ or $x_2 = 0$. In fact, considering that $J_n(0) = \delta_{n,0}$, one gets the results

$$(3a) \qquad {}^{(m)}J_n(0, x_2; s) = \begin{cases} s^{n/m} J_{n/m}(x_2), & \text{for} \quad n = km, \\ 0, & \text{otherwise}, \end{cases}$$

$$(3b) \qquad {}^{(m)}J_n(x_1, 0; s) = J_n(x_1),$$

$$(3c) \qquad {}^{(m)}J_n(0, 0; s) = \delta_{n,0},$$

where $\delta_{i,k}$ is the Krönecker symbol.

Since

$$(4) \qquad |{}^{(m)}J_n(x_1, x_2; s)| \leq \sum_{l=-\infty}^{\infty} |s^l J_{n-ml}(x_1) J_l(x_2)|$$

$$(5) \qquad \leq \sum_{l=-\infty}^{\infty} |s^l J_l(x_2)|,$$

and series in eq. (5) is known to be convergent [14] (see also [9]), then it follows the convergence of the series in eq. (2).

In the following, we shall make use of a result previously derived in [8,9], that we recast in the following form.

LEMMA 1. *Let a G.B.F. be of the form* (2), *then*

$$(6) \qquad \begin{aligned} \frac{\partial {}^{(m)}J_n}{\partial x_1} &= \frac{1}{2}\left[{}^{(m)}J_{n-1} - {}^{(m)}J_{n+1}\right]; \\ \frac{\partial {}^{(m)}J_n}{\partial x_2} &= \frac{1}{2}\left[s\, {}^{(m)}J_{n-m} - \frac{1}{s}{}^{(m)}J_{n+m}\right], \end{aligned}$$

where ${}^{(m)}J_k \equiv {}^{(m)}J_k(x_1, x_2; s)$.

This Lemma is simply proved by the use of the recurrences valid for the ordinary Bessel functions, namely

$$2\frac{\partial J_n}{\partial x} = J_{n-1} - J_{n+1}\,, \tag{7}$$

The definition (2) can be easily extended to the N-variable case. More precisely, for $N = 3$ one has the following G.B.F.

$$ {}^{(m_1,m_2)}J_n(x_1,x_2,x_3;s_1,s_2) = \sum_{l=-\infty}^{\infty} s_2^l \; {}^{(m_1)}J_{n-m_2 l}(x_1,x_2;s_1) J_l(x_3)\,, \tag{8}$$

which, for $x_j = 0$ ($j = 2,3$) reduces to G.B.F. of type (2), as can be easily verified. By iteration, a N-variable G.B.F. is defined as follows

$$\begin{aligned} &{}^{(m_1,m_2,\dots,m_{N-1})}J(x_1,x_2,\dots,x_N;s_1,s_2,\dots,s_{N-1}) = \sum_{l=-\infty}^{\infty} s_{N-1}^l \cdot \\ &\cdot {}^{(m_1,m_2,\dots,m_{N-2})}J_{n-m_{N-1}l}(x_1,x_2,\dots,x_{N-1};s_1,s_2,\dots,s_{N-2}) J_l(x_N)\,, \end{aligned} \tag{9}$$

which, for $x_N = 0$ becomes a $(N-1)$-dimensional G.B.F.

At this point, one needs the following N-dimensional analogous of Lemma 1.

Lemma 2. *Let a* G.B.F. *be of the form* (8), *then*

$$\begin{aligned} &2\frac{\partial\, {}^{(m_1,m_2,\dots,m_{N-1})}J_n}{\partial x_1} = {}^{(m_1,m_2,\dots,m_{N-1})}J_{n-1} - {}^{(m_1,m_2,\dots,m_{N-1})}J_{n+1}\,, \\ &2\frac{\partial\, {}^{(m_1,m_2,\dots,m_{N-1})}J_n}{\partial x_j} = s_{j-1}\, {}^{(m_1,m_2,\dots,m_{N-1})}J_{n-m_{j-1}} + \\ &\qquad - \frac{1}{s_{j-1}}\, {}^{(m_1,m_2,\dots,m_{N-1})}J_{n-m_{j+1}}\,, \end{aligned} \tag{10}$$

where

$$ {}^{(m_1,m_2,\dots,m_{N-1})}J_k \equiv {}^{(m_1,m_2,\dots,m_{N-1})}J_k(x_1,x_2,\dots,x_N;s_1,s_2,\dots,s_{N-1})\,.$$

In order to prove the above statement, one starts from the case $N = 3$ and obtain, by application of Lemma 1, the recurrences occuring in the three-dimensional case. Then, considering the further case $N = 4$ and making use of the previously obtained recurrences, by iteration, up to a generic N, Lemma 2 is verified.

The G.B.F. depending on complex variables can be defined as in the real case:

$$\begin{aligned}(11)\qquad {}^{(m_1,m_2,\dots,m_{N-1})}J_n(z_1,z_2,\dots,z_N;s_1,s_2,\dots,s_{N-1}) = \sum_{l=-\infty}^{\infty} s_{N-1}^l \cdot \\ \cdot\, {}^{(m_1,m_2,\dots,m_{N-2})}J_{n-m_{N-1}l}(z_1,z_2,\dots,z_{N-1};s_1,s_2,\dots,s_{N-2})J_l(z_n),\end{aligned}$$

It is easily verified that Lemma 2 holds also for G.B.F. of type (11).

Finally, it is worth mentioning that, in addition to G.B.F. of the forms (8) and (11), one can also define their modified versions, involving the modified Bessel functions, $I_k(\xi)$. Moreover, a variety of G.B.F. of mixed type, obtained by combining together $J_k(\xi)$ and $I_k(\xi)$ functions can be defined. These topics will be treated in detail in a forecoming paper.

3 – Analytical solutions of some PDE's in terms of generalized Bessel functions

In this section, we present some statements showing that particular cases of multi-variable G.B.F. satisfy exactly some special cases of the following $(N+1)$-dimensional equation

$$(12)\qquad \Delta f + kf + \gamma\frac{\partial f}{\partial x_{N+1}} - \delta^2\frac{\partial^2 f}{\partial {x_{N+1}}^2} = 0,$$

for particular values of the constants $(k,\delta) \in \mathbb{R}$ and γ which can assume real or purely imaginary values. In eq. (12) f is a function of real or purely imaginary variables and the symbol Δ denotes the N-dimensional Laplace operator expressed by

$$(13)\qquad \Delta f = \sum_{j=1}^{N}\frac{\partial^2 f}{\partial {x_j}^2},$$

In the following, we denote by $\alpha_k, k = 1, 2, \dots N+1$, real constants. We start from the following theorem, which refers to the special case of eq. (12) when $\gamma = i\beta$.

It is worth remarking that the proof of the statement, as well as those of the other theorems, is omitted since they are straightforward, as based on simple applications of the Lemmas of Section 2.

THEOREM 1. *A function* $f(x_1, x_2, \dots, x_N, x_{N+1}), N \geq 3$, *of the form* $f(x_1, x_2, \dots, x_N, x_{N+1}) =$

(14a)
$$\overbrace{{}^{(2,2,\dots,2)}}^{N}J_n(\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_N x_N, \alpha_{N+1} x_{N+1}; i, \overbrace{1, 1, \dots, 1}^{N-2}, i) \cdot \cdot \exp(-i\omega^2 \alpha_{N+1} x_{N+1}),$$

(14b)
$$\alpha_2^2 = \sum_{j=3}^{N} \alpha_j^2,$$

is a solution of the equation

$$\Delta f + i\beta \frac{\partial f}{\partial x_{N+1}} = -kf, \tag{15}$$

where

$$\beta = \frac{\alpha_1^2}{2\alpha_{N+1}}, k = \frac{\alpha_1^2}{2}(1 - \omega^2) + \alpha_2^2. \tag{16}$$

We notice that, when $\alpha_{N+1} > 0$ and $x_{N+1} = t, t$ being the time variable, eq. (15) corresponds to the N-dimensional Schrödinger equation with a constant potential. Moreover, in the particular case $N = 1$, Theorem 1 is valid assuming in the second of conditions (16) $\alpha_2 = 0$. In this case, when $\omega^2 = 1$ and $\alpha_{N+1} = \alpha_2 > 0$, eq. (15) is also found in many problems, such as that relevant to Fourier optics, already treated in [15].

It is worth analyzing that, taking in the G.B.F. of eq. (14a) the coordinates $i\alpha_1 x_1$ and $i\alpha_{N+1} x_{N+1}$ instead of $\alpha_1 x_1$ and $\alpha_{N+1} x_{N+1}$, respectively, and $\omega = 0$, one easily obtains from Theorem 1 the corresponding P.D.E.

and related conditions. More precisely, one has the following result where we put $\gamma = \beta$ for the sake of analogy with the cases of Theorem 1, and $k = -k_1^2$.

COROLLARY. *A function* $f(z_1, z_2, \dots, z_N, z_{N+1}), N \geq 3$, *which is a* G.B.F. *of the form*

$$f(z_1, z_2, \dots, z_N, z_{N+1}) = \tag{17a}$$
$$\overbrace{(2,2,\dots,2)}^{N} J_n(i\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_N x_N, i\alpha_{N+1} x_{N+1}; i, \overbrace{1,1,\dots,1}^{N-2}, i),$$

with

$$\alpha_2^2 = \sum_{j=3}^{N} \alpha_j^2, \quad \alpha_1^2 \geq 2\alpha_2^2, \tag{17b}$$

is a solution of the equation

$$\Delta f + \beta \frac{\partial f}{\partial x_{N+1}} - k_1^2 f = 0, \tag{18a}$$

where

$$\beta = -\frac{\alpha_1^2}{2\alpha_{N+1}}, k_1^2 = \frac{1}{2}\alpha_1^2 - \alpha_2^2. \tag{18b}$$

We notice that when $x_{N+1} = t$, and $\alpha_{N+1} > 0$, eq. (18) has the form of the N-dimensional time-dependent diffusion equation.

Moreover, it is worth pointing out that still in the case $x_{N+1} = t$, and $\alpha_1^2 = 2\alpha_2^2 = -2\alpha_{N+1}$, eq. (18a) becomes

$$\Delta f = \frac{\partial f}{\partial t}, \qquad \Delta f + \frac{\partial f}{\partial t} = 0, \tag{19}$$

according as $\alpha_{N+1} > 0$ or $\alpha_{N+1} < 0$ (see eq. (18b)).

We remark that eqs. (19) have the form of the heat conduction equation and of a wave equation, respectively.

We now present the following Theorem concerning the solution of eq. (12) when $k = k_1^2$ and $\gamma = 0$.

THEOREM 2. *A function* $f(ix, x_2, \dots, x_N, x_{N+1})$, $N \geq 3$, *of the form* $f(ix_1, x_2, \dots, x_N, x_{N+1}) =$

$$(20)\qquad {}^{\overbrace{(2,2,\dots,2)}^{N}}J_n(i\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_N x_N, \alpha_{N+1} x_{N+1}; i, \overbrace{1,1,\dots,1}^{N-2}, i)\cdot \cdot \exp(i\alpha_{N+1} x_{N+1}),$$

with

$$(21)\qquad \sum_{j=3}^{N} \alpha_j^2 = \alpha_2^2 - \frac{\alpha_1^2}{4}, \qquad \alpha_2^2 \geq \frac{7}{8}\alpha_1^2,$$

satisfies the equation

$$(22)\qquad \Delta f - \delta^2 \frac{\partial^2 f}{\partial x_{N+1}^2} = -k_1^2 f,$$

where

$$(23)\qquad \delta^2 = \frac{\alpha_1^2}{4\alpha_{N+1}^2}, \quad k_1^2 = \alpha_2^2 - \frac{7}{8}\alpha_1^2.$$

It is to be noticed that when $N = 3$, $\{x_k\}, k = 1, 2, 3$, are spatial coordinates and $x_4 = t$, eq. (22) has the form of the three-dimensional Klein-Gordon equation for a free particle.

Moreover, it is worth stressing that when $x_{N+1} = t$ and $\alpha_2^2 = \frac{7}{8}\alpha_1^2$, one has that $k_1^2 = 0$ and, hence, eq. (22) becomes

$$(24)\qquad \Delta f - \delta^2 \frac{\partial^2 f}{\partial t^2} = 0.$$

which corresponds to the well-known N-dimensional wave propagation equation.

We have thus specified a set of G.B.F. occuring in the solution of subcases of eq. (12), of interest for applications. With reference to these functions, it is worth noticing that they are of the same form in the sense that they have all the N indices $\{m_j\} = 2$ and the same parameters $\{s_j\}$ which are equal to the unity (real or imaginary). This result is in agreement with the fact that the considered G.B.F. are related to the solution of particular subcases of the same equation, namely, eq. (12).

The above multi-index and multi-variable G.B.F. have been considered as the more interesting ones, on the basis of simplicity and homogeneity considerations. In fact, it is worth mentioning that the conclusions of Theorem 1. and 2. remain still valid, with different conditions for the involved coefficients, when the relevant G.B.F. have the first $(N-2)$ indices $\{m_j\} = l$, l integer and $l \neq 2$ and $N \geq 4$.

As for the particular case $l = 1$, it has not been considered here since the relevant G.B.F. reduce to lower-dimension Bessel functions as can be easily seen making use of the following results

$$ {}^{(1)}J_n(x_1, x_2; 1) = J_n(x_1 + x_2)\,, \tag{25} $$

$$ {}^{(1)}J_n(x_1, x_2; i) = e^{in\psi} J_n\left(\sqrt{x_1^2 + x_2^2}\right)\,,\ \tan\psi = \frac{x}{y}\,. \tag{26} $$

Analogously, the choice of the set of unitary parameters, $\{s_j\}$, is not unique and involves different conditions for the coefficients.

Before concluding this Section, we point out that in the case $x_{N+1} = t$, all the functions $f \equiv g_n$, which are solutions of the quoted equations, satisfy initial conditions

$$ g_n|_{t=0} = p\,,\ p = p(x_1, x_2, \dots, x_N)\,, \tag{27} $$

$$ \frac{\partial g_n}{\partial t}|_{t=0} = q\,,\ q = q(x_1, x_2, \dots, x_N)\,. \tag{28} $$

Finally, for the sake of illustration, we show in figs. (1-10) the behaviour of the functions $\mathrm{Re}^{(2)} J_n(x_1, x_2; i)$ and $\mathrm{Im}^{(2)} J_n(x_1, x_2; i), n = 0, 1, 2, 3, 4$ with the relevant contour lines showing the symmetry properties of these functions which, in addition, posses reflection properties with respect to the index n and to the parameter i according to the results

$$ {}^{(2)}J_n(x_1, x_2; i) = (-1)^n\, {}^{(2)}J_{-n}(x_1, x_2; i)\,, \tag{29} $$

$$ {}^{(2)}J_n(x_1, x_2; -i) = (-1)^n\, {}^{(2)}J_n(x_1, x_2; i)\,. \tag{30} $$

The above results have been obtained by means of a procedure already described in [7,9,10], where numerical aspects and results concerning $J_n(x_1, x_2; i)$ have been extensively discussed.

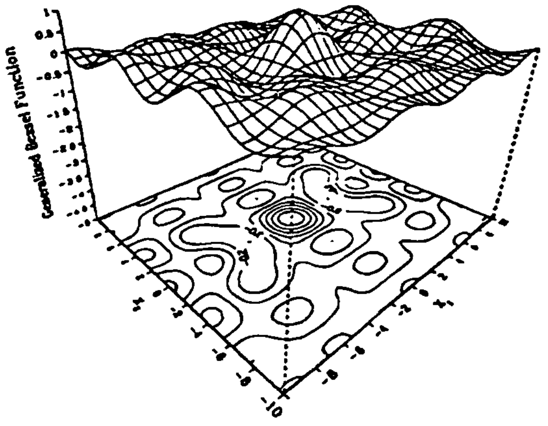

Fig. 1 Behaviour and contour lines of Re $J_0(x_1, x_2; i)$ versus x_1 and x_2.

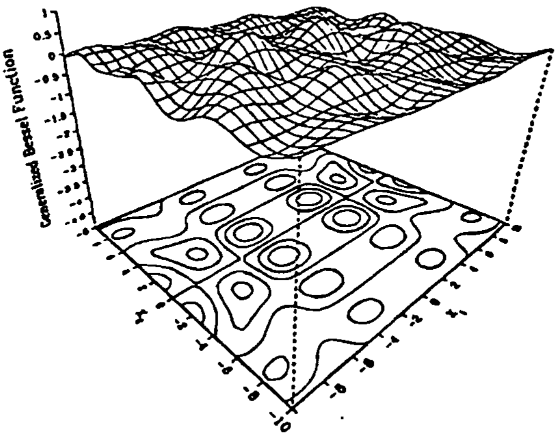

Fig. 2 Behaviour and contour lines of Im $J_0(x_1, x_2; i)$ versus x_1 and x_2.

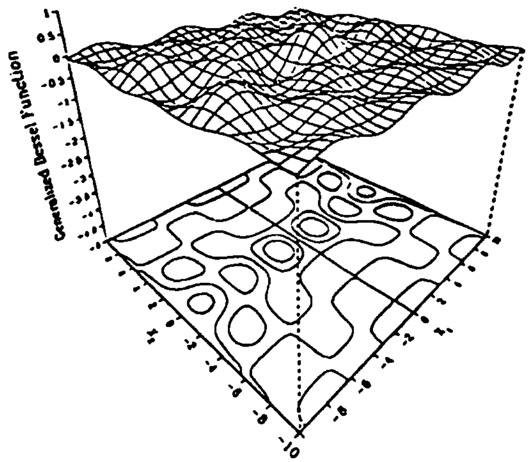

Fig. 3 Behaviour and contour lines of Re $J_1(x_1, x_2; i)$ versus x_1 and x_2.

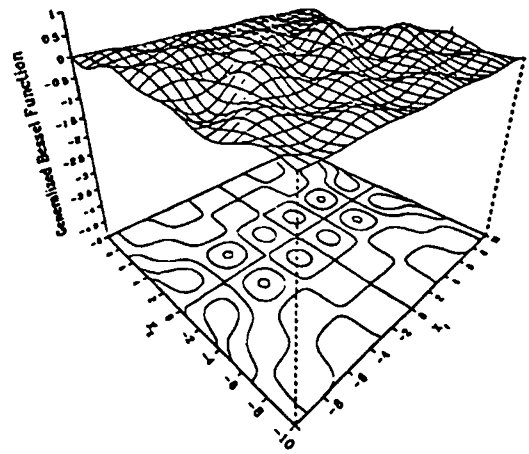

Fig. 4 Behaviour and contour lines of Im $J_1(x_1, x_2; i)$ versus x_1 and x_2.

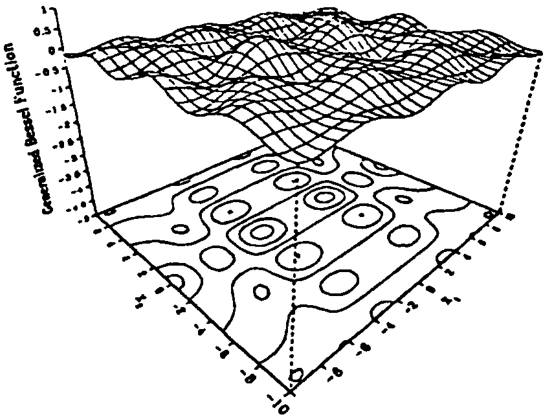

Fig. 5 Behaviour and contour lines of Re $J_2(x_1, x_2; i)$ versus x_1 and x_2.

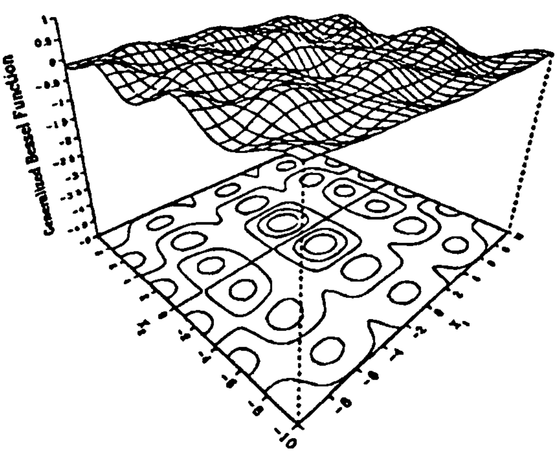

Fig. 6 Behaviour and contour lines of Im $J_2(x_1, x_2; i)$ versus x_1 and x_2.

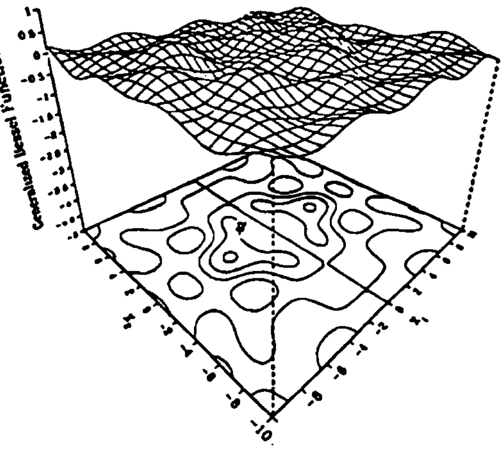

Fig. 7 Behaviour and contour lines of Re $J_3(x_1, x_2; i)$ versus x_1 and x_2.

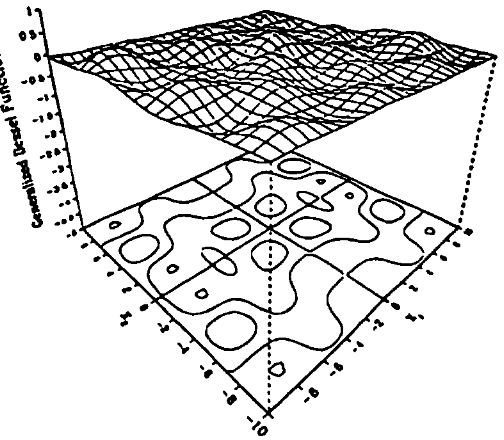

Fig. 8 Behaviour and contour lines of Im $J_3(x_1, x_2; i)$ versus x_1 and x_2.

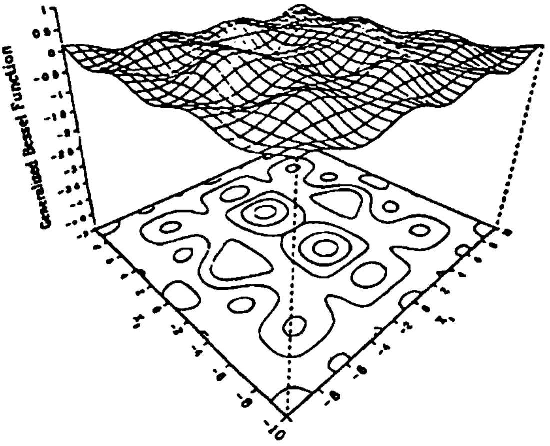

Fig. 9 Behaviour and contour lines of Re $J_4(x_1, x_2; i)$ versus x_1 and x_2.

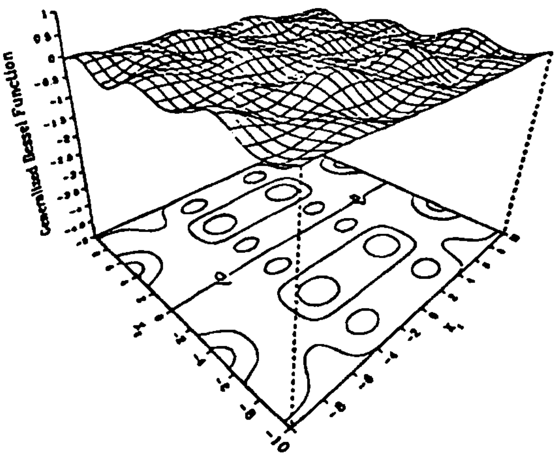

Fig. 10 Behaviour and contour lines of Im $J_4(x_1, x_2; i)$ versus x_1 and x_2.

4 – Concluding remarks

The considerations, developed in the previous Sections have shown that G.B.F. can be exploited in a context much wider than that originally proposed. We can furthermore state that they are not auxiliary functions only, useful just for applications, but they play, together with and within the yet unexplored field of multivariable special functions, a role which should be thouroughly investigated.

We have stated, in the introductory Section, that G.B.F. may be the solutions of an infinite set of PDE, this is not surprising, since, as already shown, there is a striking amount of G.B.F., each one exhibiting a plaethora of recurrence relations. Therefore, it is worth presenting further examples which may offer additional feeling on the relevance of G.B.F. to P.D.E.

In [11-12] we have introduced the function

$$ {}^{(3)}_{(0)}I_n(x,y) = \sum_{l=-\infty}^{\infty} I_{n-3l}(x) I_l(y) , \tag{31} $$

where the subscript "0" stands for odd and, more in general,

$$ {}^{(3)}_{(0)}I_n(x,y,z) = \sum_{l=-\infty}^{\infty} {}^{(3)}_{(0)}I_{n-5l}(x,y) I_l(z) . \tag{32} $$

Function (31) has derivatives which, as easily recognized, satisfy the relations

$$ \begin{aligned} \frac{\partial\, {}^{(3)}_{(0)}I_n(x,y)}{\partial x} &= \frac{1}{2}\left[{}^{(3)}_{(0)}I_{n-1}(x,y) - {}^{(3)}_{(0)}I_{n+1}(x,y) \right] , \\ \frac{\partial\, {}^{(3)}_{(0)}I_n(x,y)}{\partial y} &= \frac{1}{2}\left[{}^{(3)}_{(0)}I_{n-3}(x,y) - {}^{(3)}_{(0)}I_{n+3}(x,y) \right] , \end{aligned} \tag{33} $$

and, therefore, the following PDE

$$ \frac{\partial\, {}^{(3)}_{(0)}I_n(x,y)}{\partial y} = \left(4\frac{\partial^3}{\partial x^3} - 3\frac{\partial}{\partial x} \right) {}^{(3)}_{(0)}I_n(x,y) . \tag{34} $$

In [12] it has also been discussed the relevance of G.B.F. to identify a class of generalized Hermite polynomials. Within that framework, G.B.F. of the type

$$ {}^{(1)}\widetilde{I}_n(x,y/x,y) = \sum_{l=-\infty}^{\infty} I_{n-l}(x,y) J_l(x,y)\,, \tag{35} $$

play a central role. It can be easily verified that

$$ \begin{aligned} \frac{\partial}{\partial x}\,{}^{(1)}\widetilde{I}_n(x,y/x,y) &= {}^{(1)}\widetilde{I}_{n-1}(x,y/x,y)\,,\\ \frac{\partial}{\partial y}\,{}^{(1)}\widetilde{I}_n(x,y/x,y) &= {}^{(1)}\widetilde{I}_{n-2}(x,y/x,y)\,. \end{aligned} \tag{36} $$

Therefore, getting

$$ \frac{\partial^2}{\partial x^2}\,{}^{(1)}\widetilde{I}_n(x,y/x,y) = \frac{\partial}{\partial y}\,{}^{(1)}\widetilde{I}_n(x,y/x,y)\,, \tag{37} $$

thus providing another particular solution to the monodimensional heat conduction equation (see remarks on Corollary), with different initial conditions.

These last examples confirm the statement that G.B.F. can be exploited within the context of PDE with great flexibility. Moreover, it has to be mentioned that G.B.F. could be a useful tool in solving non-linear differential equations, of interest in radiation problems associated with non-linear oscillations. In fact, we have recently shown [11] that G.B.F. of I-type are solutions of Duffing's equation for the anharmonic oscillator. In a forecoming note we will discuss the role of G.B.F. in the field of ordinary differential equations as well as of non-linear PDE's.

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