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Radon-Nikodym theorems for vector-valued finitely additive measures

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RIASSUNTO – *Fissata una misura finitamente additiva ed esaustiva m, definita su una σ-algebra $\mathcal{A}$ e a valori in uno spazio localmente convesso $\mathcal{X}$, si danno condizioni per l'esistenza di una derivata alla Dunford-Radon-Nikodym di m rispetto ad una misura (finitamente additiva) di controllo λ. Se ne deducono teoremi di esistenza, per un analogo tipo di derivata, nel caso in cui $\mathcal{X}$ sia uno spazio di Banach con duale separabile.*

ABSTRACT – *Given an s-bounded finitely additive measure m, defined on a σ-algebra $\mathcal{A}$ and taking values in a locally convex linear space $\mathcal{X}$, we find conditions ensuring the existence of a Dunford-Radon-Nikodym derivative of m with respect to a (finitely additive) control λ. We then deduce existence theorems for such a derivative when $\mathcal{X}$ is a Banach space with separable dual.*

KEY WORDS – *Dunford-type integral - Radon-Nikodym derivatives - Finitely additive measures.*

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1 – Introduction

In the finitely additive case it is impossible to obtain exact Radon-Nikodym theorems without some further assumption. Some necessary and sufficient conditions have been given by Greco ([7]) and by Maynard ([11]) in the scalar case, and by Hagood ([8]) and by Martellotti and

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Sambucini ([9], [10]) in the vector case: nevertheless, those conditions seem not to have an easy geometrical meaning.

However it is possible to get a Radon-Nikodym theorem when the range of the pair of the involved finitely additive measures (f.a.m.'s) is closed ([4], [2]); the closedness hypothesis is in general just a sufficient one, but it can be weakened in such a way that it turns out to be even necessary.

In this paper we investigate the existence of a density function for a vector-valued finitely additive measure in the Dunford sense.
More precisely, given a f.a.m. $m : \mathcal{A} \to \mathcal{X}$, where $\mathcal{A}$ is a σ-algebra on a space Ω and $\mathcal{X}$ is a locally convex vector space with separable dual, and $\lambda : \mathcal{A} \longrightarrow \mathbb{R}_0^+$ is a control for m, we seek a "measurable" function f, ranging on $\mathcal{X}''$, such that

$$< y, m > (A) = \int_A < f, y > d\lambda$$

for all $A \in \mathcal{A}$ and $y \in \mathcal{X}'$.

Here, by "measurability" for f we mean that $< f, y >$ is measurable for all $y \in \mathcal{X}'$ and that the property f *is bounded* is λ-exhaustive.
The following condition:
there exists a λ-exhaustion $(\Omega_n)_n$ such that

$$S(\Omega_n) = \left\{\frac{m(F)}{\lambda(F)}, F \subseteq \Omega_n, \lambda(F) > 0\right\}$$

is bounded for every n
characterizes those f.a.m.'s admitting a Dunford-density among those admitting weak Radon-Nikodym derivatives w.r.t. λ.

We remark here that, contrary to what intuition suggests, the existence of "weak" derivatives $\dfrac{d < y, m >}{d\lambda}$ for all $y \in \mathcal{X}'$ does not yield an "easy" construction of a Dunford derivative, even if the whole of $S(\Omega)$ is bounded.

The general result enables us to investigate the case of Banach-valued f.a.m.'s by making use of the geometry of the range, and in the finite-dimensional case the strictly connected condition of the existence of Hahn decompositions.

2 – Preliminaries

We begin with some notations and assumptions, and keep them valid till the end of this chapter.

Let $(\Omega, \mathcal{A})$ be any measurable space, and λ any positive finitely additive measure (f.a.m.) on $\mathcal{A}$. Let $\mathcal{M}$ denote the linear space of all bounded f.a.m.'s $\mu : \mathcal{A} \to \mathbb{R}$ such that $\mu \ll \lambda$, and such that there exists $\frac{d\mu}{d\lambda}$: i.e. we assume that a measurable, λ-integrable function $f : \Omega \to \mathbb{R}$ exists, satisfying

$$\mu(A) = \int_A f d\lambda$$

for all $A \in \mathcal{A}$.

Next, let Y be any real, separable normed space.

THEOREM 2.1. *Assume that $T : Y \to \mathcal{M}$ is a linear operator, satisfying the following condition:*

$$|T(y)| \leq \|y\|\lambda \quad \textit{for every} \quad y \in Y,$$

where $|T(y)|$ is the total variation of the f.a.m. $T(y)$.

Then there exists a function $f : \Omega \to Y'$ satisfying:

$$< f(\cdot), y >= \frac{dT(y)}{d\lambda} \tag{2.1.1}$$

λ-almost everywhere for all $y \in Y$.

PROOF. Pick any linearly independent total sequence $(e_n)_n$ in Y, such that $\|e_n\| = 1$ for every n. For each n, let's set $Z_n = span\{e_1, \ldots, e_n\}$, and $Z = \bigcup Z_n$. Thus, Z is dense in Y.

For $n = 1$, we write

$$g_1(\omega) = h(\omega, e_1) = \frac{dT(e_1)}{d\lambda}(\omega)$$

for all $\omega \in \Omega$. Without loss of generality, we can suppose

$$|h(\omega, e_1)| \leq 2$$

for each ω: this is because $|T(e_1)| \leq \lambda$ by assumption.

It's easy now to define $h(\omega, y)$ for every y in Z_1, in such a way that $h(\omega, \cdot)$ is linear on Z_1, and $h(\cdot, y) = \frac{dT(y)}{d\lambda}$, and finally $|h(\omega, y)| \leq 2\|y\|$ for every $y \in Z_1$ and $\omega \in \Omega$.

From now on we shall assume that a Radon-Nikodym derivative $g_n = \frac{dT(e_n)}{d\lambda}$ has been fixed, for every $n \in \mathbb{N}$, and that $|g_n(\omega)| \leq 2$ for all ω.

Assume now that $h(\omega, y)$ has been defined for every $\omega \in \Omega$ and for every $y \in Z_n$ in such a way that

a) $h(\cdot, y) = \frac{dT(y)}{d\lambda}$ for every $y \in Z_n$;

b) $h(\omega, \cdot) \in Z_n'$ and $\|h(\omega, \cdot)\| \leq 2$ for every ω.

We shall define $h(\omega, y)$ for all ω and for $y \in Z_{n+1}$, in such a way that a) and b) above are still valid with n replaced by $n+1$; this is much harder than one might think because the natural linear extension by means of g_{n+1} may fail to satisfy the norm condition in b).
As Z_{n+1} has finite dimension, we can choose a compact set $H \subseteq Z_n$ and a positive number $k > 1$ in such a way that

$$\{z \in Z_{n+1}, \|z\| \leq 1\} \subset \{y + s e_{n+1} : y \in H, |s| \leq k\}. \tag{1}$$

Fix $\varepsilon > 0, \varepsilon < \frac{1}{4+k}$, and choose $t_1, \ldots, t_s$ in H, satisfying

$$H \subset \bigcup_{1 \leq \rho \leq s} B(t_\rho, \frac{\varepsilon}{2}).$$

As g_{n+1} is bounded, there exists a decomposition $\{A_1, ..., A_N\}$ of Ω, consisting of pairwise disjoint, measurable sets, and N corresponding numbers $\eta_1, ..., \eta_N$ in $[-2, 2]$ satisfying

$$|g_{n+1}(\omega) - \eta_i| < \frac{\varepsilon}{2} \tag{2}$$

for every $\omega \in A_i$, and for $i = 1, \ldots, N$.

Similarly, for each $\rho = 1, \ldots, s$ there exists a decomposition of Ω, say $\{B_1(\rho), ..., B_{N(\rho)}(\rho)\}$, with corresponding real numbers $\theta_1(\rho), ..., \theta_{N(\rho)}(\rho)$ satisfying

$$|h(\omega, t_\rho) - \theta_j(\rho)| < \frac{\varepsilon}{2} \tag{3}$$

for every $\omega \in B_j(\rho)$ and for $j = 1, \dots, N(\rho)$.

We denote by $\mathcal{F}$ the algebra generated by the sets $A_1, \dots, A_N$ and $B_1(\rho), \dots, B_{N(\rho)}(\rho)$, $\rho = 1, \dots, s$; also, denote with C the union of all λ-null atoms of $\mathcal{F}$. Of course, $\lambda(C) = 0$, because $\mathcal{F}$ is finite. We shall suitably modify g_{n+1} in C.

For each $\omega \in \Omega$ it holds:

$$\sup_{t \in Z_n}\{h(\omega,t) - 2\|t - e_{n+1}\|\} \leq \inf_{y \in Z_n}\{2\|y + e_{n+1}\| - h(\omega,y)\}$$

by virtue of condition b) above, and the "trick" $t - y = (t - e_{n+1}) + (e_{n+1} - y)$.

Then we modify g_{n+1} into $h(\cdot, e_{n+1})$ as follows:

$$h(\omega, e_{n+1}) = \begin{cases} g_{n+1}(\omega) & \omega \notin C, \\ \sup_{t \in Z_n}\{h(\omega,t) - 2\|t - e_{n+1}\|\}, & \omega \in C. \end{cases}$$

Again we find $h(\cdot, e_{n+1}) = \dfrac{dT(e_{n+1})}{d\lambda}$, and $|h(\omega, e_{n+1})| \leq 2$ for all ω.

Now we define linearly $h(\omega, z)$ for all $z \in Z_{n+1}$: of course condition a) is satisfied in Z_{n+1}.

Let's prove also b). Fix $\omega \in C$. Fix $z \in Z_{n+1}$, and assume that $z = t + e_{n+1}$ for some $t \in Z_n$. We find

$$h(\omega,z) = h(\omega,t) + h(\omega,e_{n+1}) \leq h(\omega,t) + 2\|t + e_{n+1}\| - h(\omega,t) = 2\|z\|.$$

Similarly, if $z = t - e_{n+1}$, we find $h(\omega, z) \leq 2\|z\|$ and therefore b) is proved in case $\omega \in C$ and z of the form $t + e_{n+1}$ for some $t \in Z_n$.

It is now easy to prove b) for $\omega \in C$ and *any* $z \in Z_{n+1}$. So we have now to consider the case $\omega \notin C$.

We observe that C^c is a finite union of non-zero atoms of $\mathcal{F}$. Let A be one of such atoms, and fix $\rho = 1, \dots, s$. Then we must have $A \subset B_j(\rho) \cap A_i$ for some $j = 1, \dots, N(\rho)$ and some $i = 1, \dots, N$. We have

$$\int_A |h(\cdot, t_\rho) + g_{n+1}| d\lambda \leq \|t_\rho + e_{n+1}\| \lambda(A)$$

because of a).

Now, from $A \subset B_j(\rho) \cap A_i$ and from (2), (3) we deduce

$$|h(\omega, t_\rho) - \theta_j(\rho)| < \frac{\varepsilon}{2}$$

and

$$|g_{n+1}(\omega) - \eta_i| < \frac{\varepsilon}{2},$$

for all $\omega \in A$, whence

$$\left| \int_A |h(\cdot, t_\rho) + g_{n+1}| d\lambda - |\theta_j(\rho) + \eta_i| \lambda(A) \right| < \varepsilon \lambda(A).$$

Then we obtain

$$|\theta_j(\rho) + \eta_i| \lambda(A) \leq \varepsilon \lambda(A) + \|t_\rho + e_{n+1}\| \lambda(A).$$

Dividing by $\lambda(A)$ we get

$$(4) \qquad |h(\omega, t_\rho) + h(\omega, e_{n+1})| = |h(\omega, t_\rho) + g_{n+1}(\omega)| \leq 2\varepsilon + \|t_\rho + e_{n+1}\|.$$

Now, if we fix $t \in H$ there exists ρ such that $\|t - t_\rho\| < \frac{\varepsilon}{2}$ and therefore

$$|h(\omega, t) - h(\omega, t_\rho)| < \varepsilon;$$

moreover, for the same t and t_ρ we find

$$\|t_\rho + e_{n+1}\| - \|t + e_{n+1}\| \leq \|t - t_\rho\| < \frac{\varepsilon}{2},$$

from which (4) gives

$$|h(\omega, t) + h(\omega, e_{n+1})| \leq 4\varepsilon + \|t + e_{n+1}\|,$$

for $\omega \in A$.

Similarly, for $\omega \in A$ and $|s| \leq k$, we find

$$|sh(\omega, e_{n+1}) - s\eta_i| \leq |s| \frac{\varepsilon}{2} \leq k \frac{\varepsilon}{2}.$$

As a consequence, if $z \in Z_n, \|z\| \leq 1$, we get $|h(\omega, z)| = |h(\omega, t) + h(\omega, se_{n+1})|$ for suitable $t \in H$ and $s \in [-k, k]$, because of (1).

So we can deduce $|h(\omega, z)| \leq 1 + (4 + k)\varepsilon \leq 2$, whenever $z \in Z_n$, $\|z\| \leq 1$, and for $\omega \in A$.

As A is arbitrary, we conclude that b) holds for Z_{n+1}.

Now we proceed by induction, thus defining $h(\omega, \cdot)$ on the whole of Z, in such a way that a) and b) are satisfied.

By density, $h(\omega, \cdot)$ can be (uniquely) defined on Y, there satisfying b). However, we are still to prove that $h(\cdot, y) = \dfrac{dT(y)}{d\lambda}$.

This is true, for $y \in Z$. If $y \in Y$, pick any sequence (z_n) in Z, norm-converging to y. Then we have

$$|h(\omega, z_n) - h(\omega, y)| \leq 2\|z_n - y\|$$

for all n, and all ω.

This implies that $h(\cdot, z_n)$ converges uniformly to $h(\cdot, y)$. Therefore h is measurable, and

$$\lim_{n \to \infty} \int_E h(\cdot, z_n) d\lambda = \int_E h(\cdot, y) d\lambda$$

for each set $E \in \mathcal{A}$.

As $\int_E h(\cdot, z_n) d\lambda = T(z_n)(E)$, for every n and for every E, we see that

$$\lim_{n \to \infty} T(z_n)(E) = \int_E h(\cdot, y) d\lambda$$

From the condition $|T(z_n - y)| \leq \|z_n - y\|\lambda$, it follows that $T(z_n)(E)$ converges to $T(y)(E)$, and so

$$T(y)(E) = \int_E h(\cdot, y) d\lambda$$

for all $E \in \mathcal{A}$, which means that property a) is satisfied.

Finally, if we define $f : \Omega \to Y'$ as

$$< f(\omega), y >= h(\omega, y)$$

for every $\omega \in \Omega$ and $y \in Y$, we are finished.

3 – Existence of a density

We are going to deduce some existence theorems of Radon-Nikodym derivatives, for vector-valued finitely additive measures.

We shall deal with a locally convex TVS $\mathcal{X}$, a measurable space $(\Omega, \mathcal{A})$, and dominated finitely additive measures $m : \mathcal{A} \to \mathcal{X}$; our aim is to find a "Dunford-type" derivative of m with respect to its control λ, according with the following definition.

DEFINITION 3.1. Given any f.a.m. $m : \mathcal{A} \to \mathcal{X}$, we say that m is *dominated* if there exists a f.a.m. $\lambda : \mathcal{A} \to \mathbb{R}_0^+$ such that $m \ll \lambda$. (If this is the case λ can be chosen to be equivalent to m - see [5]). In such cases, λ will be said to be a *control* for m.

In dealing with $\mathcal{X}'$, we shall endow it with the "strong" topology, i.e. the topology whose base at 0 is given by the polar sets of all bounded subsets in $\mathcal{X}$; the same we shall do for $\mathcal{X}''$, once $\mathcal{X}'$ has been "strongly" topologized.

When m happens to be dominated, with a control λ, we say that a *Dunford-type* derivative $\frac{dm}{d\lambda}$ exists, if there exists a function $f : \Omega \to \mathcal{X}''$ satisfying the following conditions:

(3.1.1) $< f(\cdot), y >$ is measurable and λ- integrable for every $y \in \mathcal{X}'$.

(3.1.2) There exists an increasing sequence $(\Omega_n)_n$ in $\mathcal{A}$ such that $\lambda(\Omega - \Omega_n) \to 0$, and such that $f|_{\Omega_n}$ is bounded for every n (recall that both $\mathcal{X}'$ and $\mathcal{X}''$ are endowed with the "strong" topology).

(3.1.3) $\int_E < f(\cdot), y > d\lambda =< y, m(E) >$ for every $E \in \mathcal{A}$.

In the next proposition we shall obtain a *bounded* Dunford-type derivative.

PROPOSITION 3.2. *Let a dominated f.a.m. $m : \mathcal{A} \to \mathcal{X}$ be given, and assume that a control λ exists such that a Radon-Nikodym derivative exists for $< y, m >$ with respect to λ, for all $y \in \mathcal{X}'$.*

Assume that $\mathcal{X}'$ is separable, and that the set

$$S = \left\{ \frac{m(A)}{\lambda(A)} : A \in \mathcal{A}, \lambda(A) > 0 \right\}$$

is bounded in $\mathcal{X}$.

Then there exists a bounded Dunford-type derivative f of m with respect to λ.

PROOF. The polar set of S, S^o, is a balanced, convex neighbourhood of 0 in $\mathcal{X}'$. Let q denote the Minkowski functional of S^o.

If $y \in \mathcal{X}$ satisfies $q(y) = 0$, then $y \in tS^o$ for every $t > 0$, and then

$$| < y, m(A) > | \leq \varepsilon\lambda(A)$$

for all $\varepsilon > 0$ and for all $A \in \mathcal{A}$ with $\lambda(A) > 0$: hence $q(y) = 0$ if and only if $< y, m >= 0$.

This allows us to consider the quotient $\mathcal{Y} = \mathcal{X}'/Ker(q)$, endowed with the usual norm deduced from q, and to define on $\mathcal{Y}$ an operator $T : \mathcal{Y} \to \mathcal{M}$ by setting

$$T([y]) =< y, m >$$

for every $y \in \mathcal{X}'$. This operator is well-defined and satisfies the hypothesis of Theorem 2.1: in particular, from $| < y, m > | \leq \lambda$ for every $y \in S^o$, we can deduce that

$$|T([y])| = | < y, m > | = q(y)| < \frac{y}{q(y)}, m > | \leq q(y)\lambda = \|[y]\|\lambda$$

for all $y \in \mathcal{X}'$.

Then, by virtue of Theorem 2.1 we can easily get the required derivative f: to see that f is bounded, we can observe that its range is contained in $2S^{oo}$ (here S^{oo} is thought embedded in $\mathcal{X}''$), and this set is bounded in $\mathcal{X}''$.

We can now deduce a more general existence theorem, and then a necessary and sufficient condition for the existence of a Dunford-type derivative.

THEOREM 3.3. *Let $\mathcal{X}, m, \lambda$ be as in Proposition 3.2. Assume that $\mathcal{X}'$ is separable and that*

(3.3.1) *there exists an increasing sequence $(\Omega_n)_n$ in $\mathcal{A}$, such that $\lambda(\Omega - \Omega_n) \to 0$ and such that the sets*

$$S_n = \left\{\frac{m(A)}{\lambda(A)} : A \in \mathcal{A}, A \subset \Omega_n, \lambda(A) \neq 0\right\}$$

are bounded in $\mathcal{X}$.

Assume also that $\dfrac{d < y, m >}{d\lambda}$ *exists for every* $y \in \mathcal{X}'$. *Then there exists a Dunford-type derivative* $f : \Omega \to \mathcal{X}''$.

PROOF. By applying Proposition 3.2 to the sets

$$\Omega_1, \Omega_2 - \Omega_1, \dots, \Omega_n - \Omega_{n-1}$$

which we shall denote by $F_1, F_2, \dots, F_n, \dots$ respectively, we find bounded functions $f_n : F_n \to \mathcal{X}''$, satisfying

$$\int_A < f_n(\cdot), y > d\lambda =< y, m > (A)$$

for every $A \in \mathcal{A}$, $A \subset F_n$. It is now possible to define f λ-almost everywhere on Ω, by "pasting" together the functions f_n. The Radon-Nikodym property follows from the condition that $\lambda(\Omega - \Omega_n) \to 0$.

Our next result is a necessary and sufficient condition for the existence of a derivative.

THEOREM 3.4. *Let* $\Omega, m, \lambda, \mathcal{X}$ *be as before. Assume also that* $\mathcal{X}'$ *is separable. Then a necessary and sufficient condition for the existence of a Dunford-type derivative* $f : \Omega \to \mathcal{X}''$ *of* m *with respect to* λ *is that* $\dfrac{d < y, m >}{d\lambda}$ *exists for each* $y \in \mathcal{X}'$, *and that an increasing sequence* $(\Omega_n)_n$ *can be found in* $\mathcal{A}$, *satisfying* $\lambda(\Omega - \Omega_n) \to 0$, *and such that the sets*

$$S_n = \Big\{\frac{m(A)}{\lambda(A)} : A \in \mathcal{A}, A \subset \Omega_n, \lambda(A) \neq 0\Big\}$$

are bounded in $\mathcal{X}$ *for all* n.

PROOF. Of course, we just have to prove the "only if" part. Assume therefore that f exists, and let $(\Omega_n)_n$ be the sequence given in (3.1.2): we shall prove that the corresponding sets S_n are bounded in $\mathcal{X}$.

Fix n, and choose any balanced, convex, closed neighbourhood U of 0 in $\mathcal{X}$.

Then the bipolar U^{oo} in $\mathcal{X}''$ is a neighbourhood of 0 in $\mathcal{X}''$, and therefore there exists a positive number $k > 0$ such that $f(\Omega_n) \subset kU^{oo}$. Now, if we pick any element $y \in U^o$, we find

$$\left| < \frac{m(A)}{k}, y > \right| = \left| \int_A < \frac{f(\cdot)}{k}, y > d\lambda \right| \leq \lambda(A)$$

for all $A \in \mathcal{A}$, $A \subset \Omega_n$.

This implies that $\dfrac{m(A)}{\lambda(A)}$ is in $kU^{oo} \cap \mathcal{X}$, for all $A \in \mathcal{A}$, $A \subset \Omega_n$, $\lambda(A) \neq 0$. According with the Bipolar Theorem, $U^{oo} \cap \mathcal{X} = U$, and therefore $S_n \subset kU$. As U is arbitrary, the theorem is proved.

4 – Dunford-type derivatives for Banach-valued f.a.m.'s

We shall now find some existence theorems for Radon-Nikodym derivatives, when the measures are Banach-valued. We start with the finite-dimensional case.

As usual, let $(\Omega, \mathcal{A})$ be any measurable space, and let μ, ν be two non-negative, finitely additive measures on $\mathcal{A}$, satisfying $\mu \ll \nu$. In [7] it is proved that a Radon-Nikodym derivative $\dfrac{d\mu}{d\nu}$ does exists if and only if

(4.1.1) for each real number $r > 0$ the scalar f.a.m. $\mu - r\nu$ has a Hahn decomposition.

Given any scalar *bounded* f.a.m. m on $\mathcal{A}$, we shall denote with $|m|$ its variation.

Definition 4.1 Let $\mathcal{X}$ be a Banach space, and $m : \mathcal{A} \to \mathcal{X}$ any finitely additive measure. We say that m is *s- bounded* if $\lim_n m(A_n) = 0$, for every sequence $(A_n)_n$ of pairwise disjoint sets in $\mathcal{A}$.

Moreover, given a scalar f.a.m. $\lambda : \mathcal{A} \to \mathbb{R}_0^+$ we say that λ is a *control* for m if $m \ll \lambda$ and $\lambda \ll m$, namely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following implications hold, for every $A \in \mathcal{A}$:

$$\lambda(A) < \delta \Longrightarrow \sup\{\|m(B \cap A)\|, B \in \mathcal{A}\} < \varepsilon,$$

and

$$\sup\{\|m(D \cap A)\|, D \in \mathcal{A}\} < \delta \Longrightarrow \lambda(A) < \varepsilon.$$

If λ is a control for m, we say that it is a *Rybakov* control if $\lambda = |<y, m>|$ for some $y \in \mathcal{X}'$.

In [6], [12] the following result is proved.

THEOREM 4.2. *Given any finitely additive measure* $m : \mathcal{A} \to \mathcal{X}$ *then the following are equivalent:*

i) *m is s-bounded.*
ii) *There exists a control for m.*
iii) *There exists a Rybakov control for m.*

So, our aim is to find "mild" conditions which ensure the existence of a Dunford-type derivative for an exhaustive $\mathcal{X}$-valued f.a.m. m with respect to *any* Rybakov control λ.

LEMMA 4.3. *Let μ, ν be two bounded f.a.m.'s on $\mathcal{A}$, and assume that $\mu \ll |\nu|$. If the f.a.m. $a\mu + b\nu$ has a Hahn decomposition for all $a, b \in \mathbb{R}$, then there exists a Radon-Nikodym derivative* $\dfrac{d\mu}{d|\nu|}$.

PROOF. From the assumptions, it follows that both μ and ν admit a Hahn decomposition. We denote with (Q, M) a Hahn decomposition for μ and with (P, N) a Hahn decomposition for ν. Then, for all $A \in \mathcal{A}$

$$\nu^+(A) = \nu(A \cap P),\ \nu^-(A) = -\nu(A \cap N),$$
$$\mu^+(A) = \mu(A \cap Q),\ \mu^-(A) = -\mu(A \cap M).$$

We now restrict ourselves to the space Q, endowed with the σ-algebra $\mathcal{A} \cap Q$, and show that μ^+ admits a Radon-Nikodym derivative with respect to $|\nu|$.

From (4.1.1) it is enough to prove that $\mu^+ - r|\nu|$ has a Hahn decomposition for every $r > 0$. So, let $r > 0$ be fixed, and let (A_r, A_r^c) and (B_r, B_r^c) be Hahn decompositions for $\mu - r\nu$ and $\mu + r\nu$ respectively. Then we have $\mu(E) \geq r\nu(E)$ for every $E \in \mathcal{A}$, $E \subset A_r$ and $\mu(E) \geq -r\nu(E)$ for every $E \in \mathcal{A}$, $E \subset B_r$, and the reverse inequalities hold in the complements of A_r and B_r respectively. We set

$$D_r = Q \cap [(A_r \cap P) \cup (B_r \cap N)].$$

If $E \in \mathcal{A}$, $E \subset D_r$, we put $E_1 = E \cap A_r \cap P$, $E_2 = E \cap B_r \cap N$: then $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$. Furthermore

$$\mu(E_1) \geq r\nu(E_1), \; \mu(E_2) \geq -r\nu(E_2)$$

from which we get $\mu^+(E) \geq r|\nu|(E)$. We now notice that

$$Q - D_r = Q \cap [(A_r^c \cap P) \cup (B_r^c \cap N)];$$

hence, if $F \in \mathcal{A}$, $F \subset Q - D_r$, one can proceed as above and see that $\mu^+(F) \leq r|\nu|(F)$.

Thus $(D_r, Q - D_r)$ is a Hahn decomposition for $\mu^+ - r|\nu|$ in Q. This allows us to deduce that there exists a Radon- Nikodym derivative $\frac{d\mu^+}{d|\nu|}$ in Q. As μ^+ vanishes outside Q, the derivative exists globally. In a similar way it can be proved that μ^- has a derivative with respect to $|\nu|$. This concludes the proof.

THEOREM 4.4. *Let $m : \mathcal{A} \to \mathbb{R}^n$ be any bounded f.a.m., and assume that every linear combination of its components has a Hahn decomposition. Then m has a Radon-Nikodym derivative with respect to any Rybakov control λ.*

PROOF. Indeed, let m_i be any component of m, and the control ν be the variation measure of a linear combination σ of the components of m. Then the f.a.m. $a\sigma + bm_i$ has a Hahn decomposition by hypothesis. So, according with Lemma 4.3, there exists $\frac{dm_i}{d\nu}$. As i is arbitrary, the proof is finished.

COROLLARY 4.5. *Assume that $m : \mathcal{A} \to \mathbb{R}^n$ is a bounded f.a.m. with* closed *range. Then there exists a Radon-Nikodym derivative of m with respect to any Rybakov control.*

PROOF. It's enough to prove that any combination of the components has closed range. But the latter is the continuous image of a compact set, namely the range of m. Therefore the existence of a Radon-Nikodym derivative is a consequence of Lemma 4.3.

We now turn to Banach-valued f.a.m.'s.

THEOREM 4.6. *Let $\mathcal{X}$ be any Banach space with dual $\mathcal{X}'$ having the RNP, and let $m : \mathcal{A} \to \mathcal{X}$ be an s-bounded f.a.m. with separable range. Assume that the scalar f.a.m. $< y, m >$ has a Hahn decomposition for each element $y \in \mathcal{X}'$. Then, for every Rybakov control λ for m, there exists a Dunford-type derivative $\frac{dm}{d\lambda}$ if and only if condition* (3.3.1) *holds.*

PROOF. According with Lemma 4.3, there exists a Radon-Nikodym derivative $\frac{d < y, m >}{d\lambda}$ for all $y \in \mathcal{X}'$. Moreover, by a theorem of Stegall [13], the subspace of $\mathcal{X}$ generated by $m(\mathcal{A})$ has separable dual. So we can readily apply the results in Section 3 and get the theorem.

COROLLARY 4.7. *Let $\mathcal{X}$ and $m : \mathcal{A} \to \mathcal{X}$ be as in Theorem 4.6, and assume also that m has weakly closed range. Then for every Rybakov control λ, a Dunford-type derivative $\frac{dm}{d\lambda}$ exists, if and only if condition* (3.3.1) *holds.*

PROOF. According with [6] the range R of m is weakly relatively compact because of s-boundedness. As R is weakly closed by hypothesis, it is weakly compact. Now, if $y \in \mathcal{X}'$ is fixed, the range of $< y, m >$ is closed in $\mathbb{R}$ because it is the image of R under the (weakly) continuous map y. This implies that $< y, m >$ has a Hahn decomposition for each $y \in \mathcal{X}'$, and therefore we can apply Theorem 4.6 to get the result.

REMARK 4.8. We mention that results concerning the topological properties of the range of a Banach-valued f.a.m., in particular the closedness, can be found also in [3], [1].

REMARK 4.9 We observe that, when $\mathcal{X}$ is a reflexive Banach space, by the Dunford-Pettis-Phillips Theorem, the assumptions on $\mathcal{X}'$ and m in Theorem 4.6 are both fulfilled. Moreover, the Dunford density in this case is in fact a Pettis density.

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