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Derivation of partial flocks of quadratic cones

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RIASSUNTO – *Il metodo di Bader, Lunardon e Thas per la costruzione di q flocks di un cono quadratico in $PG(3,q)$ a partire da un flock assegnato, è esteso alla costruzione di t flocks parziali a partire da un flock parziale di t coni. Ai flock parziali con $q-1$ coniche sono associati $q-1$ piani di traslazioni, questa relazione consente di produrre nuovi esempi di flock a partire da classi note.*

ABSTRACT – *In this paper, the method of Bader, Lunardon and Thas for the construction of q flocks of a quadratic cone in $PG(3,q)$, q odd, from a given flock is extended to include the construction of t partial flocks from a given partial flock of t conics. For partial flocks with $q-1$ conics, there are $q-1$ associated translation planes. Also, certain of the known classes of flocks are shown to produce new flocks using the connections with translation planes.*

KEY WORDS – *Partial flock - Translation plane.*

A.M.S. CLASSIFICATION: 51E

1 – Introduction

In [9], GEVAERT and JOHNSON discuss the interconnections between flocks of quadratic cones in $PG(3,q)$, generalized quadrangles of type (q^2,q), and translation planes of order q^2 and kernel containing a field K isomorphic to $GF(q)$ whose spread is defined by the union of q reguli in $PG(3,K)$ that mutually share a line. Furthermore, the known flocks, associated generalized quadrangles, and translation planes are listed. Including the linear flock associated with the Desarguesian plane, there are exactly eight infinite classes of flocks. (There are seven classes if the even and odd order flocks arising from the derived Barriga/Cohen-Ganley planes of even and odd order are considered to be within the same family.)

Recently, BADER, LUNARDON and THAS [2] have developed a procedure which they call *derivation* for the construction of q flocks of order q from a given flock in $PG(3,q)$. This is not to be confused with derivation of an affine plane. In particular, the spreads of the associated translation planes of order q^2 admit q reguli that share a line. The associated translation nets may be *derived* in this sense producing different translation planes. However, intrinsic to the q reguli defining the spread is an elation group of the translation plane whose component orbits union the axis of the group define the reguli in the projective space. In these derived translation planes obtained by the derivation of one of the reguli, the elation group becomes a group which fixes a Baer subplane pointwise; a Baer group.

For this reason, we shall make a point to distinguish between the derivation of the flock of the cone which then produces a possibly new translation plane from the original plane associated with the flock and the derivation of a regulus of the spread of the original translation plane.

When referring to the translation plane obtained via a derivation of the flock associated with it, we shall use the term *s-inverted translation plane* (for a reason that will become clear).

1.1 – Derivation(s) of Flocks

In BADER, LUNARDON, and THAS [2], the derivation of a flock is determined as follows: Let q be odd and let $\Sigma_3 \cong PG(3,q)$ and contained in $\Sigma_4 \cong PG(4,q)$ in such a way so that there is a quadratic Q_4 in Σ_4 such that $\Sigma_3 \cap Q_4$ is a quadratic cone Q_3 with vertex p_0. Now let $\{C_i | i = 1,2,\dots q\}$ be a flock of Q_3 in Σ_3 which we further denote by H_0. Let π_1 denote the plane of H_0 which contains the conic C_i for $i = 1,2,\dots q$. Let $\perp$ denote the polarity of Σ_4 associated with Q_4. Then $\pi_i^\perp = p_0 + p_i$ (where p_0 denotes the 1-dimensional supspaces and $+$ is vector addition) where $p_i \in Q_4$. Now form $p_i^\perp = H_i$ for $i = 1,2,\dots q$, so that H_i is a 3-dimensional projective space. Then, for each $i = 1,2,\dots q$, $\{H_i \cap H_j | j = 1,2,\dots q$ for $j \neq i\} \cup \{\pi_i\}$ is a flock of the quadratic cone $Q_4 \cap H_i$ with vertex p_i in H_i

(BADER, LUNARDON, THAS [2]). Thus, there are q such flocks associated with the original flock which are called *derived flocks or derivations*.

In this article, it is shown that the derivation process described above is more generally valid for partial flocks of quadratic cones of odd order and/or for a partial spread consisting of a set of reguli that share a line. In particular, the derivation process applies to partial flocks in $PG(3,q), q$ odd, of $q-1$ mutually disjoint conics of a quadratic cone (deficiency one partial flock).

In [14], the author shows how to relate partial flocks of deficiency one of a quadratic cone in $PG(3,q)$ with certain translation planes of order q^2.

THEOREM (JOHNSON [14] (Theorem C)).

(1) A translation plane π_F of order q^2 and kernel $K \cong GF(q)$ which admits a Baer collineation group B_q of order q in its translation complement is equivalent to a partial flock F of $q-1$ conics of a quadratic cone in $PG(3,q)$.

(2) The partial flock F is maximal (cannot be extended to a flock) if and only if the fixed point subplane of Q_q does not lie in a derivable net of π_F.

Actually, the above theorem may be stated in a more general manner which we shall do in the course of this article.

Our main result for translation planes admitting Baer groups is a corollary to the derivation of partial flocks of quadratic cones. In particular, we prove:

COROLLARY (see (2.10)(2)). *Let π be a translation plane of odd order q^2 and kernel $GF(q)$ which admits a Baer collineation group B_q in the translation complement. Then there exist $q-1$ associated translation planes (the s-inverted planes) also admitting a Baer group of order q.*

One of the eight known classes of flocks of quadratic cones may be constructed from the likeable translation planes of characteristic 5 of Kantor (see e.g. the table in [10]). Bader, Lunardon, and Thas show that there is a new infinite class of flocks which are derived from these flocks.

In [10], GEVAERT and JOHNSON show that two flocks are isomorphic if and only if the associated translation planes are isomorphic. However, even though there is an algebraic construction of generalized quadrangles of type (q^2, q) from either the flock or the translation plane, there is no isomorphism theorem relating generalized quadrangles and planes or generalized quadrangles and flocks.

There are also the flocks originating from the translation planes of Ganley of characteristic 3 and from the translation planes of Barriga/Cohen-Ganley of order q^2 for $q \equiv \pm 2 \bmod 5$.

The three translation planes mentioned above are more generally examples of translation planes which are (i) likeable, (ii) semifield, and (iii) translation planes that admit an autotopism group which fixes a regulus and acts transitively on the components not in the fixed regulus. In this article, we consider the s-inverted planes of these three general classes. Our main results are as follows: In section 4, we consider the family of $q+1$ translation planes associated with a given likeable plane of order q^2 by derivation or s-inversion union the plane itself (this family is called the *skeleton* of the plane) and we prove:

(4.3) THEOREM. *Let π be a likeable plane of odd order q^2 which is not Walker and such that the 0-inverted plane of the skeleton of π is isomorphic to a plane of the skeleton of one of the known families of flock planes. Then π is the Kantor characteristic 5 likeable plane; a new likeable plane produces a flock plane which is distinct from the known flock planes or their s-inversions.*

Note that we obtain the result of BADER-LUNARDON-THAS [2] on the Kantor characteristic 5 planes as a corollary.

COROLLARY (BADER-LUNARDON-THAS [2]). *Let π denote the Kantor likeable plane of order 5^{2r}, for $r > 1$. Then the 0−inversion is not isomorphic to any of the known flock planes or their inversions.*

Note that a Kantor likeable plane of order 5^2 is Walker (see e.g. GEVAERT and JOHNSON [10]).

In section 5, we consider semifield skeletons and prove:

(5.1) THEOREM. *Let π denote a semifield plane of odd order q^2 and kernel $K \cong GF(q)$ which corresponds to a flock of a quadratic cone. If* *the autotopism group modulo the kernel homology group does not admit a group of order divisible by* $(q^{1/3}-1)/2$ *then the* 0*-inversion of* π *is not isomorphic to* π *(we are not assuming that* $q^{1/3}$ *is an integer).*

From this result, we are also able to prove:

(5.4) COROLLARY. *Let* π *denote the Ganley semifield plane of order* 3^{2r} *for* $r > 4$. *Then the* 0*-inversion is not isomorphic to* π. *Furthermore, the* 0*-inversion is not isomorphic to any plane of the skeleton of any other known flock plane.*

In section 6, we consider translation planes with large autotopism groups and prove:

THEOREM (see (6.1),(6.3),(6.4)). *Let* π *denote a translation plane of odd order* q^2 *where if* $q = -1 \bmod 4$ *then* q *is not a Mersenne prime and if* $q \equiv 1 \bmod 4$ *then* $(q+1)/2$ *is not a prime power, and kernel* $K \cong GF(q)$ *that admits a linear collineation group* G *which fixes a regulus* R_0, *fixes at least two Baer subplanes of* R_0 *incident with the zero vector and acts transitively on the components not in* R_0. *Then*

(1) there are constants in $K(\alpha, \beta, k)$ such the spread for the translation plane may be represented in the following form:

$$x = 0, y = x\begin{bmatrix} u + \alpha t^{1+k}, & \beta t^{1+2k} \\ t & , u \end{bmatrix} \text{ for all } u, t \in K$$

where x, y are 2-vectors over K.

And,

(2) If $q - 1 > (1+2k)^2$ then either

(i) π is a Walker plane,

(ii) a Knuth semifield plane of flock type or

(iii) not all of the planes of the skeleton of π are isomorphic.

Using this result, we are able to show the following:

THEOREM (see (6.6)). *If* $(q+1)/2$ *is not a prime power then there is a derivation of the flock of order* q *associated with the Barriga/Cohen-Ganley plane of order* q^2 *(for* $q \equiv \pm 2 \bmod 5$*) which is not isomorphic to any known flock.*

In section 2, we extend the derivation construction of Bader, Lunardon, and Thas to include the partial flock situation. Also in section 2, we show how to utilize the construction combined with an partial spread extension procedure to construct a tremendous number of partial spreads from a given partial spread and further prove (2.10) (2) listed above. In section 3, we consider the connections between flocks and planes and give some geometric propertities which will be used in the subsequent sections.

The ideas for many of the arguments in this paper were motivated by the arguments of BADER, LUNARDON, and THAS [2].

2 – Partial derivation and growing in flocks

We recall the main result for partial flocks on quadratic cones from [10] and [14].

(2.1) THEOREM (GEVAERT and JOHNSON [10] and JOHNSON [15]).

(1) Let P_E^t for $1 \leq t \leq q$ be a partial spread of cardinality $qt + 1$ in $PG(3, q)$ consisting of t reguli with share exactly one line. Then either P_E^t is a maximal partial spread or P_E^t may be extended to a partial spread of type P_E^{t+1}

(2) P_E^t admits an elation group E of order q which fixes the common line of the t reguli. Conversely, any partial spread which admits an elation group of order q one of whose orbits union the axis of the group is a regulus, is of type P_E^t for some integer t.

(3) The partial spread P_E^t is equivalent to a partial flock of a quadratic cone in $PG(3, q)$. Furthermore, the partial flock is maximal if and only if the partial spread P_E^t is maximal.

We also consider a procedure of *growing* a partial spread from a partial spread of type P_E^t.

2.1 – Growing in flocks

Let V_4 denote the 4-dimensional vector space associated with the projective space in which the partial spread P_E^t is defined and let K denote a field $\cong GF(q)$. Consider the corresponding translation net V_E^t

of degree $1+qt$. Let $\{T_i | i = 1, 2, \ldots q+1\}$ denote the set of 1-spaces over K of the fixed point subspace of the elation group E (see (2.1)). Let $V_4 - V_E^t$ denote the set of vectors of V_4 over K which do not lie on a component of V_E^t.

THEOREM (see JOHNSON [14],[15]). *Using the notation above,*

(1) for each T_i of $\mathrm{Fix}\, E$*, there exist exactly q E-orbits of 1-spaces of $V_4 - \mathrm{Fix}\, E$ such that the subspace generated is a 2-dimensional K-space containing T_i.*

(2) There are exactly $q-t$ such orbits in $V_4 - V_E^t$. For each $T_i, i = 1, 2, \ldots q+1$, choose any such E-orbit of 1-spaces and denote the set by D. Then $(V_E^t - \mathrm{Fix}\, E) \cup D = V_E^t(D)$ is a translation net of degree $1 + qt + q = 1 + q(t+1)$ which contains $\mathrm{Fix}\, E$ *as a Baer subplane and admits E as a Baer collineation group.*

(2.3) DEFINITION AND NOTES.

(1) The net $V_E^t(D)$ is said to be grown from V_E^t. Note that $V_E^t(D)$ does not contain $\mathrm{Fix}\, E$ *as a component.*

(2) Let $P_E^t(D)$ denote the partial spread corresponding to $V_E^t(D)$. Then $P_E^t(D)$ is maximal if $P^t(E)$ is maximal.

(3) The number of partial spreads grown from P_E^t is $(q-t)^{(q+1)}$.

(4) In any translation net V_E^T, let R denote one of the regulus nets and let $\overline{R}$ denote the opposite regulus net. Then $(V_E^t - R) \cup \overline{R}$ is a translation net of the same cardinality as V_E^t and the net is maximal if and only if the net V_E^t is maximal.

PROOF. (2) Let L denote a line of $PG(3, K)$ which is not in $P_E^t(D)$ and does not intersect any line of the partial spread. Since $\mathrm{Fix}\, E$ is a Baer subplane of $P_E^t(D)$ then L does not intersect $\mathrm{Fix}\, E$ so that L does not intersect any line of the partial spread P_E^t so that P_E^t is not maximal.

PROOF. (3) For each 1-space T of $\mathrm{Fix}\, E$, there are $(q-t)$ choices of a 1-space such that the E-orbit of this 1-space generates a 2-space containing T.

The proof of (4) is straightforward and is left to the reader.

As an application of (2.3), we construct:

2.2 – Some maximal partial spreads of cardinalities $4 \cdot 8 + 1$ and $5 \cdot 8 + 1$ in $PG(3, q)$

By GEVAERT and JOHNSON [10], the following is a maximal partial spread:

Let $K \cong GF(8) = GF(2)[\delta]$ where $\delta^3 + \delta + 1 = 0$. Let V_4 denote the vector space of rank 4 over K. Let $V_4 = \{(x_1, x_2, y_1, y_2) | x_i, y_i \in K, i = 1, 2\}, x = (x_1, x_2), y = (y_1, y_2), 0 = (0, 0)$. Then V_E^4 defined by the following set of four reguli is a maximal partial spread of degree $4 \cdot 8 + 1$: $x = 0, y = x\begin{bmatrix} u + t^2 & t^3 \\ t & u \end{bmatrix}$ for $t = 0, 1, \delta \in K$ and $y = x\begin{bmatrix} u + \delta^6 & \delta \\ \delta^3 & u \end{bmatrix}$ for all $u \in K$.

(2.4) (1). *There is a derived maximal partial spread given as follows:* $\left\{x = 0, y = x\begin{bmatrix} -ab^{-1}, & f - ab^{-1}g \\ b^{-1}, & b^{-1}g \end{bmatrix}, y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}\right\}$ *for all* $u \in K$ *and where* $y = x\begin{bmatrix} a & f \\ b & g \end{bmatrix}$ *is a component of* P_E^4.

Hence, the derived maximal partial spread has the form

$$y = x\begin{bmatrix} (u + t^2)t^{-1}, & t^3 + (u + t^2)t^{-1} \\ t^{-1} & , t^{-1}u \end{bmatrix} \textit{for } t = 1, \delta \textit{ and } y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix},$$
$$y = x\begin{bmatrix} (u + \delta^6)\delta^{-3}, & \delta + (u + \delta^6)\delta^{-3} \\ \delta^{-3}, & \delta^{-3}u \end{bmatrix} \textit{for all } u \in K.$$

(2) There are 4^9 *maximal partial spreads* $V_E^4(D)$ *constructed from* V_E^4 *as in* (2.3).

PROOF. See JOHNSON [19] (section II, p.25) to see that the form of the derived maximal partial spread may be taken as stated.

By (2.1) above, to construct partial flocks of quadratic cones, it suffices to construct partial spreads to the type V_E^4 or equivalently, translation nets in V_4 over $K \cong GF(q)$ which admit an elation group of order q one of whose component orbits union the axis is a regulus in $PG(3, q)$. Actually, we also may characterize the translation nets equivalent to partial flocks in another way using Baer groups.

(2.5) THEOREM. *Let W denote a translation net in a 4-dimensional vector space V_4 over $K \cong GF(q)$ which contains a Baer suplane π_0. If B is a collineation group of W of order q which fixes π_0 pointwise then there is a corresponding partial flock of a quadratic cone P^W corresponding to the set of orbits of B of length q union $\mathrm{Fix}\, B = \pi_0$. Conversely, a partial flock of a quadratic cone gives rise to a translation net admitting a Baer group of order q. If the partial flock is not a flock then the translation net may be grown from the partial spread of type $V_E^{\pm}$ obtained from the partial flock.*

PROOF. The reader is referred to the author's articles [14] and [15] to verify this. In particular, see section 2 of [14].

2.3 – s-Inversion

Let W be a translation net in V_4 over $K \cong GF(q)$ of the type mentioned above in (2.5) admitting a Baer group B of order q. By the work of FOULSER [8], we may let the Baer subplane pointwise fixed by $B = \{(0, x_2, 0, y_2) | x_2, y_2 \in K\}$ and the group $B = \left\{ \begin{bmatrix} 1 & u & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix} | u \in K \right\}$.
And, the net of degree $q + 1$ containing the Baer subplane may be represented by $x = 0$, $y = x\begin{bmatrix} u & m(u) \\ 0 & u \end{bmatrix}$ for all $u \in K$ for some funciton m on K (also see JHA-JOHNSON [12] for this set-up).

Now change bases by $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to transform the fixed point space to $x = 0$, and the group B into $\left\{ \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} | u \in K \right\}$. Now the translation net $W - \{\text{the net containing } \mathrm{Fix}\, B\} \cup (\mathrm{Fix}\, B)$ admits B as an elation net and the form of B insures that each component orbit union $\mathrm{Fix}\, B$ is a regulus in $PG(3, K)$ and thus corresonds to a partial flock of a quadratic cone.

Choose any two of the components of one of the orbits of B to be represented as $y = 0$, and $y = x$. Then, it easily follows that the form of the regulus R defined is $y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ (note that we are not trying to say that the function m defined above is identically zero as this regulus net is completely distinct from the original net containing $\mathrm{Fix}\, B$ as a Baer subplane).

We now assert that $(W - \{R\}) \cup \overline{R}$ is a union of k reguli R_t for $t \in \Omega \subset K$ such that $|\Omega| = k$ which may be represented in the form: $R_t = \{x = 0, y = x\begin{bmatrix} u + g(t), & f(t) \\ t, & u \end{bmatrix}$ for each fixed $t \in \Omega$ and for all $u \in K\}$, where the regulus net R chosen above is R_0 and where g and f are functions from Ω to K such that the differences of the distinct pairs of the indicated matrices are nonsingular.

PROOF. It is certainly clear that the partial spread may be represented in the form $x = 0, y = 0$, and $y = xM$ where M is a nonsingular 2×2 matrix over K and such that the differences of the matrices are all nonsingular. Furthermore, as this net admits B in the form indicated and since each component orbit union $x = 0(\mathrm{Fix}\, B)$ defines each of the regulus nets, it follows that each net has the form $\{x = 0, y = x\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} + uI_2\right\}$ for fixed constants $a, b, c, d \in K$ and for all $u \in K\}$. Note that we may choose a representative component for the regulus so that $d = 0$. Hence, a and b are defined uniquely by c. Thus, it follows that no two regulus nets have the same constant c in the (2,1)-entry. Since there are k regulus nets, there is a subset (containing 0) Ω of K such that $c \in \Omega$ defines the (2,1)-entry of the regulus net and hence the net. Clearly, there are functions $f, g \colon \Omega \longrightarrow K$ such that $g(c) = a$, and $f(c) = b$. Note it also follows that f is $1 - 1$ into K.

(2.6) PROPOSITION. *Assume q is odd. Then the differences of the matrices defining the net $(W - R) \cup \overline{R}$ are nonsingular if and only if $(g(t) - g(s))^2/4 + (t - s)(f(t) - f(s))$ is a nonsquare in K for all distinct pairs $t, s \in \Omega$.*

PROOF. $\begin{bmatrix} u + g(t), & f(t) \\ t, & u \end{bmatrix} - \begin{bmatrix} v + g(s), & f(s) \\ s, & v \end{bmatrix}$ is nonsingular unless $u = v$ and $t = s$. If $t = s$ then the nonsingularity implies that $u = v$. If $t \neq s$ then these differences are nonsingular if and only if $w^2 + w(g(t) - g(s)) -$

$(t-s)(f(t)-f(s) \neq 0$ for all distinct pairs $t, s \in \Omega$ and for all $w \in K$. Since q is odd this is equivalent to $((g(t)-g(s))^2+4(t-s)(f(t)-f(s))$ is nonsquare in K for all distinct pairs $t, s \in \Omega$ which is equivalent to $((g(t)-g(s))^2/4+(t-s)(f(t)-f(s))$ is nonsquare for all distinct pairs t, s.

(2.7) DEFINITION s-INVERTED PARTIAL SPREAD.

Let $\left\{x=0, y=x\begin{bmatrix} u+g(t), & f(t) \\ t, & u \end{bmatrix}\right\}$ *denote a partial spread* P_E^k *in* V_4 *over* $K \cong GF(q)$ *of degree* $1+qk$ *where* f, g *are functions from a set* $\Omega \longrightarrow K$ *of cardinality* k *and* $0 \in \Omega$ *and for all* $u \in K$ *and* $t \in \Omega$. *Let* $M_t = \begin{bmatrix} g(t)/2, & f(t) \\ t & -g(t)/2 \end{bmatrix}$ *be the matrix obtained by taking* $u=-g(t)/2$.

The s-inverted partial spread $(P_E^k)^{-s}$ *is defined to be* $\{x=0, y = x((M_t-M_s)^{-1}+uI_2)\}$ *for all* $t \in \Omega$ *for fixed* $s \in \Omega$ *and for all* $u \in K$ *and where* $(M_t-M_t)^{-1}$ *is taken to be* 0 *by definition.*

(2.8) Note: Consider the 0-inverted structure $(P_E^k)^{-0}$. If this set is a partial spread then the s-inverted structure $(P_E^k)^{-s}$ is also a partial spread for each $s \in \Omega$.

PROOF. Change bases by $\sigma = \begin{bmatrix} I_2 & \begin{matrix} -g(s), & -f(s) \\ -s, & 0 \end{matrix} \\ O_2 & I_2 \end{bmatrix}$. This basis change commutes with B in the form given and the set of images $(P_E^k)\sigma$ contains $y=0$. Hence the set of images form a partial spread given by matrices whose pairs of distinct differences are nonsingular. Thus, it is clear that the s-inverted structures are partial spreads if and only if the 0-inverted structures are partial spreads. Also, note that the s-inverted structures become 0-inverted structures with new defining functions.

(2.9) THEOREM.

(1) The s-inverted structure $(P_E^k)^{-s}$ *is a partial spread for any* $s \in \Omega$.

(2) There are k *associated partial flocks of a quadratic cone associated with each partial spread of type* P_E^k.

PROOF. To prove (1), it suffices to prove this for $s = 0$ (see (2.8)). (2) follows from (1) and (2.1)(3).

Thus, given the condition

(*) $((g(t) - g(r))^2/4 + (t - r)(f(t) - f(r))$ is a nonsquare for each distinct pair $t, r \in \Omega$, we must show that $\{x = 0, y = x(M_t^{-1} + uI_2)\}$ for all $t \in \Omega$ and for all $u \in K$, is a partial spread.

Let δ_t denote the determinant of M_t which is $-\{g(t))^2/4 + tf(t)\}$ for t not 0 and is, by definition, say 1 if $t = 0$. Then the above set is a partial spread if and only if $(M_t^{-1} + uI_2) - (M_r^{-1} + vI_2)$ is either nonsingular or the zero matrix for all $t, r \in \Omega$ and for all $u, v \in K$. $M_t^1 = \begin{bmatrix} g(t)/2\delta_t, & f(t)/\delta_t \\ t/\delta_t, & -g(t)/2\delta \end{bmatrix}$. Now letting $u = w + g(t)/(2\delta_t)$, the asserted partial spread has the form $\left\{x = 0, y = x\begin{bmatrix} w + g(t)/\delta_t, & f(t)/\delta_t \\ t/\delta_t, & w \end{bmatrix}\right\}$ for all $t \in \Omega$ and for all $w \in K$. That is, we have replaced the functions $g(t)$ and $f(t)$ by the functions $g(t)/\delta_t$ and $f(t)/\delta_t$ respectively.

Now the 0-inverted structure is a partial spread if and only if we obtain the condition

(**) $(g(t)/\delta_t - g(r)/\delta_r)^2/4 + (t/\delta_t - r/\delta_r)(f(t)/\delta_t - f(r)/\delta_r)$ is a nonsquare for all distinct pairs $t, r \in \Omega$.

First assume that $r = 0$. Then (**) is clearly valid using (*) multiplied by $(\delta_t)^2$.

Assuming t and r are both nonzero, multiply the equation in (**) by $(\delta_t\delta_r)^2 = (-\delta_t)(-\delta_r)^2$. Recalling that $\delta_t = -\{(g(t))^2/4 + tf(t)\}$, so that $(-\delta_t)$ is nonsquare, we obtain condition (***) equivalent to (**):

(***) $(g(t)\delta_r - g(r)\delta_t)^2/4 + (t\delta_r - r\delta_t)(f(t)\delta_r - f(t)\delta_t)$ is nonsquare for all distinct pairs $t, r \in \Omega$ both not zero.

However, $(g(t)\delta_r - g(r)\delta_t)^2/4 + (t\delta_r - r\delta_t)(f(t)\delta_r - f(r)\delta_t) = (g(t))^2/4 + tf(t))\delta_r^2 + ((g(r)^2/4 + rf(r))\delta_t^2 - \{g(t)g(r)/2 + rf(t) + tf(r)\}\delta_r\delta_t$ which, in turn, is equal to $-\delta_t\delta_r^2 - \delta_r\delta_t^2 - \{g(t)g(r)/2 + rf(t) + tf(r)\}\delta_r\delta_t = \delta_r\delta_t\{-\delta_r - \delta_t - g(t)g(r)/2 - rf(t) - tf(r) = \delta_r\delta_t\{g(r))^2/4 + rf(r) + (g(t))^2/4 + tf(t) - g(t)g(r)/2 - rf(t) - tf(r)\} = \delta_r\delta_t\{(g(t) - g(r))^2/4 + (t - r)(f(t) - f(r))\}$. Since $\delta_r\delta_t = (-\delta_r)(-\delta_t)$ is square, we obtain (***) using (*).

Note that this argument is essentially the same as one given in BADER, LUNARDON, and THAS [2] (section 1) but phrased in terms of a partial spread instead of a flock of a quadratic cone.

(2.10) COROLLARY.

(1) Let π *be a translation plane of odd order* q^2 *and kernel* $K \cong GF(q)$*. If* π *admits a Baer collineation group* B *of order* q *in the translation complement then there are* $(q-1)$ *associated translation planes of order* q^2 *and kernel* K *which admit the Baer group* B *of order* q *as a collineation group. These are the unique translation planes which may be grown from the s-inverted partial spreads* $(P_B^{(q-1)})^{-s}$ *where* $P_B^{(q-1)}$ *denotes the partial spread consisting of the components of* π *which are not fixed by* B *union the Baer subplane* Fix B *pointwise fixed by* B.

(2) Corresponding to the translation plane π *of (1) are* $(q-1)$ *partial flocks of a quadratic cone. These partial flocks are the structures corresponding to the s-inverted partial spreads. Furthermore, one of the partial flocks may be completed to a flock if and only if the translation plane grown from the corresponding s-inverted partial spread contains a derivable net containing the Baer subplane pointwise fixed by the Baer group.*

Note that there are no known partial flocks of deficiency one which cannot be extended to flocks ore equivalently, there are no translation planes of order q^2 and kernel $GF(q)$ admitting a Baer group of order q where the net defined by the Baer subplane pointwise fixed by the Baer group is not derivable.

3 – Derived flocks and s-inverted translation planes

In this section, we consider the derivations of the known flocks or equivalently the s-inversions of the corresponding translation planes. More generally, we may apply the results of GEVAERT and JOHNSON [10] and of section 2 to obtain algebraic relations fo the s-inversions and derivations of flocks.

We simply note

(3.1) THEOREM. *The s-derivations or s-inversions produce flocks or translation planes.*

PROOF. Apply (2.10). In [2], BADER, LUNARDON and THAS study the derivations of the eight classes of known flocks and prove the following result:

(3.2) THEOREM (Bader, Lunardon, Thas). *The 0-derivation of the Kantor likeable flock of characteristic is new.*

The reader is referred to the table in GEVAERT and JOHNSON [9] describing the functions $f(t), g(t)$ for the known flocks or translation planes of the type under investigation (see (2.7) applied to spreads).

3.1 – Geometric properties

In order to discuss whether derivations or inversion are new, we shall require some geometry as the new functions do not appear to lend themselves to easy calculation (see (2.7)).

In BADER, LUNARDON and THAS, the argument that the 0-derivations of the Kantor flocks are new is based on the original geometric construction mentioned in section 1.

In this section, we also study the s-derivations or s-inversions using this geometry and for this, we shall require a few observations. For this part, we utilize the construction and notation of section 1.

(3.3) NOTES ON THE CONSTRUCTIONS

It is not obvious that the s-inverted translation planes correspond to the derivations of the associated flocks. To see that this is, in fact, the case, the reader is referred to BADER, LUNARDON, THAS [2] (secton 1). In fact, the geometric construction also works for partial flocks and the s-inverted partial spreads correspond to the partial flocks that can be obtained via the construction using the 4-dimensional projective space.

To aid in working in the partial spreads and partial flocks, it is helpful to list some of the connections.

When we are concerned with the partial spread P_E^k and the k corresponding s-inverted partial spreads, we always consider one of the reguli in standard form: $x = 0, y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for all $u \in GF(q)$. This does not mean that we consider this regulus is contained in each partial spread but merely that we are recoordinatizing so as to obtain this form. However, we do consider that the axis $x = 0$ of the elation group E is common to all of the partial spreads. Now embedd into the 5-dimensional projective space Σ_5 using the Klein correspondence and take the associated polar structures which produce the corresponding partial flocks. We see that there is a common point on the Klein quadric of all of the partial ovoids corresponding to the $k+1$ partial spreads which, in turn, implies that all of the partial flocks are contained in a common 4-dimensional projective space Σ_4.

Note that the geometric construction of k partial spreads from a given partial spread involves k points p_i for $i = 1, 2, \dots, k$ which, in turn, define, by the polar spaces with respect to Σ_4, the associated 3-dimensional spaces in which the partial focks are defined. If we use one 3-dimensional space and the partial flock therein as a reference to construct the remaining k partial flocks, there will be exactly one conic from the reference partial flock in each of the other partial flocks. In the algebraic construction of s-inversion, the conic in the reference partial flock appearing in the constructed partial flock will appear within the associated partial spread as the regulus in standard form.

(3.4) PROPOSITION. *Let F be a flock of a quadratic cone and π_F the corresponding translation plane. Let F^s and π_F^s denote the s-derivations and s-inversions of F and π_F respectively. Then the set of $\bar{s}$-derivations or $\bar{s}$-inversions of F^s, or π_F^s is* $\{F, F^t | t \neq s\}, \{\pi_F, \pi_F^t | t \neq s\}$.

PROOF. Consider the derivations of the flock F^1 which is a derivation of F. Without loss of generality let H_1 denote the corresponding 3-dimensional projective space in Σ_4 (the use of the integer 1 is different in F^1 and H_1) then $\{H_1 \cap H_i | i \neq 1\} \cup \{\pi_1\}$ is or defines the flock of the quadratic cone $Q_3 \cap H_1$ with vertex p_1. The derivations of the flock F^1 come from the plane $H_1 \cap H_i$ as follows: $(H_1 \cap H_i)^\perp = p_1 + p_i$ (note that $H_i = p_i^\perp$ so that $p_i \subset H_i^\perp$), then form $p_i^\perp$ for $i \neq 1$ and $p_0^\perp = H_0$. The derivation of F^1 defined in either H_i for $i \neq 1$ or in H_0 is defined by $\{H_i \cap H_j | j \neq i; j = 1, 2, \dots q\} \cup ((H_1 \cap H_0) = \pi_1)\}$. Thus, F^1 reconstructs F and all of the derivations of F not equal to F^1.

(3.5) DEFINITION AND NOTES.

(1) We shall call the set of flocks derived from a flock F union F the skeleton $S(F)$ of F. Similarly, the set of translation planes corresponding to the skeleton of F is called the skeleton $S(\pi_F)$ of π_F. So, if $\overline{F} \in S(F)$ then $S(\overline{F}) = S(F)$.

(2) If two translation planes π_1, π_2 are isomorphic then each plane in the skeleton of π_1 is isomorphic to a plane in the skeleton of π_2.

PROOF. Under the present assumptions, an isomorphism $f \in \Gamma L(4, K)$. If we realize both spreads within the same projective 3-space over K and use the connections with the flock of the cone in $PG(3, q)$, there is an induced element $\bar{f}$ in $P\Gamma L(4, q)$ acting on the projective space H_0 in which the flock is defined and without loss of generality, we may assume that $\bar{f}$ leaves the cone invariant. Further, there is an extension to $P\Gamma L(5, q)$ which leaves the quadric Q_4 invariant and whose restriction to H_0 is f. Note that this assertion follows easily from BADER, LUNARDON, and THAS [2] Theorem 2 p.13.

(3.6) THEOREM. *Let F be a nonlinear flock of a quadratic cone of odd order q and π_F the corresponding translation plane. Let $\{p_0, p_1, \cdots, p_q\}$ denote the $q+1$ points in 4-dimensional projective space over $GF(q)$ that defines the skeleton of F. Then the full translation complement of π_F modulo the collineation group which fixes each of the q reguli of π_F sharing a fixed component is isomorphic to the stabilizer of p_0 in $P\Gamma L(5, q)_{\{p_0, p_1, \cdots, p_q\}, Q_4} = G$.*

PROOF. This is almost immediate from Theorem 2 of BADER, LUNARDON, and THAS [2]. Note that a collineation of π_F must permute the reguli (GEVAERT, JOHNSON, THAS [10]).

Furthermore, by studying the construction of derivation in the projective 4-space, extending to the 5-space and using the Thas-Walker construction via the Klein correspondence to produce the translation plane, it follows that the points p_i for $i \neq 0$ correspond to the reguli in π_F (see the remarks in (3.3)). And, the group acting on the reguli is permutation isomorphic to the group acting on the p_i. Note that in the 3-space $H_i = p_i^{\perp}$, the planes π_i are such that $\pi_i^{\perp} = p_0 + p_i$ and the π_i correspond to the opposite reguli in the translation plane π_F. Suppose $g \in G$ and maps p_i onto p_j so that π_i maps onto π_j and leaves H_0 invariant. The restriction to H_0, induces an element in the 5-dimensional space which leaves the Klein quadric invariant (see the statement of Corollary 3 of BADER, LUNARDON and THAS [2]). There is an induced group acting on the polar planes of the π_i with respect to the Klein quadric. But the polar planes correspond under the Klein correspondence to the reguli in the translation plane.

(3.7) NOTATION. *In* (3.6), *above, let* $\pi_j, j = 0, 1, 2, \ldots, q$ *denote the translation planes in the skeleton of* $\pi_F = \pi_0$ *and corresponding to the points* p_j *for* $j = 0, 1, 2, \ldots, q$.

(3.8) THEOREM. *Suppose that two translation planes of a given skeleton both admit collineation groups that act transitively on the components not equal to the common line of the reguli in each spread. Then all planes of the skeleton are isomorphic and the corresponding group* G *in* $P\Gamma L(5, q)$ *acts doubly transitively on the points* $\{p_0, p_1, p_2, \cdots, p_q\}$.

PROOF. Without loss of generality, let the two planes be denoted by π_0 and π_1.

G_{p_0} is transitive on the the points $\{p_1, p_2, \cdots, p_q\}$ since the group induced from translation plane π_0 must act transitively on the reguli. Furthermore, G_{p_1} must act transitively on the points $\{p_0, p_2, \ldots p_q\}$ since the skeleton of π_1 is the skeleton of π_0 and thus there is but one corresponding set of points in the associated projective 4-space.

Since, $q > 2$, the orbit length of any point under the group $\langle G_{p_0}, G_{p_1} \rangle$ is $q + 1$. Hence, G must act doubly transitively on the set of $q + 1$ points.

(3.9) THEOREM. *Let* π *be a non Desarguesian translation plane of odd order* q^2 *that corresponds to a flock of a quadratic cone. If there exists a linear collineation group which acts transitively on the* q *reguli then* π *is either likeable or a semifield plane.*

PROOF. Clearly, there must admit a collineation p-group G for $q = p^r$ which acts transitively on the components of the plane not equal to the common component and fixes this common component. Assume the order of the group is strictly larger than q^2. Then, since the group is linear, there must be a Baer p-element. However, since there is an elation group of order q acting transitively on the components not equal to the fixed component of each regulus, we have a contradiction by FOULSER [7]. The result now follows directly from the main results of FINK, JOHNSON and

WILKE (2.11) [6] and JHA, JOHNSON, WILKE (corollary 2.3 of [13] - note that desirable planes are likeable under our assumptions).

4 – Likeable skeletons

In this section, we utilize (3.9) to first consider likeable translation planes and their s-inversions. We first note:

(4.1) THEOREM. *Let π_F denote a likeable translation plane of odd order q^2 which is not a Walker plane. Then the 0-inversion is not isomorphic to π_F.*

PROOF. Deny! By (3.9), there is a collineation group G which acts doubly transitively on the set of points $\{p_0, p_1, \ldots, p_q\}$.

By the classification theorem of simple groups, there is a minimal normal subgroup N of G such that N is elementary Abelian or non-Abelian simple.

$q+1$ is a prime power only if q is a prime (since q^2-1 does not admit a p-primitive divisor for $q=p^r$ (see KALLAHER [21] for this idea)). However, in this case, any such likeable plane is Walker by JOHNSON and WILKE [20] (7.1).

Thus, we may assume that N is non-Abelian simple.

Since q is odd, we have the following posibilities for N (see CAMERON [3] pp. 8 and 9):

(i) A_{q+1},
(ii) $PSL(d,h)$ such that $(h^d-1)/(h-1)=q+1$,
(iii) $PSU(3,h)$ such that $h^3+1=q+1$, or
(iv) ${}^2G_2(h)$ (Ree) such that $h^3+1=q+1, h=3^s$ for s an integer ≥ 1.

Note that since N is a non-Abelian simple subgroup of $P\Gamma L(5,q)_{\{p_0,p_1,\ldots p_q\},Q_4}$, is follows that N is a subgroup of $PGL(5,q)$.

Let F denote that linear collineation group of $\pi(\pi_0)$ and M the subgroup which fixes each regulus of π. Then N_{p_0} is a subgroup of F/M. Before we continue the proof, we shall require two lemmas on likeable planes.

Lemma 1. If π is likeable then M is the direct product of the elation group of order q and the kernel homology group of order $q-1$.

PROOF. Let $g \in M$. Since M contains the elation group or order q and M fixes each regulus, we may assume that g leaves $x = 0$, and $y = 0$ invariant. Now using FINK, JOHNSON and WILKE [6] (3.3), we may assume that there is a collineation which fixes each regulus of the form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b^2 & 0_3 \\ 0 & 0 & 0 & b \end{bmatrix} = A_b$ for some $b \in K \cong GF(q)$ where the matrix representation of the spread is given by $x = 0$, $y = x\begin{bmatrix} u - a^2, & -(1/3)a^3 + l(a) \\ a, & u \end{bmatrix}$ $= M_{u,a}$ where l is the corresponding likeable function. The reguli are defined by $x = 0$ and the matrices with fixed $a \in K$. A_b maps $y = xM_{0,a}$ onto $y = x\begin{bmatrix} -(ab)^2, & -(1/3)(ab)^3 + l(a)b^3 \\ (ab), & 0 \end{bmatrix}$. Hence, if each (actually two) regulus is left invariant by A_b then $b = 1$.

LEMMA 2. *If π is likeable and $(|F/M|, p-1) > 2$ for $q = p^r$ then π is Walker.*

PROOF. $M = EH$ where E denotes the elation group of order q and H denotes the kernel homology group of order $q-1$.

Since $(F/H)/(EH/H) \cong F/M$, we have that $(F/H, p-1) > 2$. Hence, we may apply FINK, JOHNSON and WILKE [6] (theorem (4.2) and JOHNSON, WILKE [20] (7.1) to obtain that the function l is identically zero; π is Walker.

Case (i) $N \cong A_{q+1}$. Then $|N_{p_0}| = q!$ so that by Lemma 2, π is Walker.

Case (ii) $N \cong PSL(d, h)$ such that $1 + h + h^2 + \dots h^{d-1} = q + 1$.

Let $h = n^s$ for n a prime. Then n must divide q so that n must be p. Therefore, $(q^2 - 1, h^d - 1) \geq q + 1$. $(q^2 - 1, h^d - 1) = (p^{2r} - 1, p^{sd} - 1) = p^{(2r, sd)} - 1$ so that it must be that $(2r, sd) > r$ and hence $(2r, sd) = 2r$. Hence, $(h^d - 1) = q^2 - 1$. Thus, $(q^2 - 1)/(h - 1) = q + 1$ implies that $h = q$ so that $d = 2$.

In this latter case, $N_{p_0} \cong PSL(2, q)$ so that $|N_{P_0}| = q(q-1)/2$. $(q - 1/2, p - 1) > 2$ unless possibly $p = 5 (p \neq 3$ by [20]). So case (ii) cannot occur unless $q = 5^r$.

By the lemmas 1 and 2 above, if $|F/M|$ is divisible by $(q-1)/2$ then

$|F/H|$ is also divisible by $(q-1)/2$ so that there exists a collineation group which leaves $y=0$ (and $x=0$) invairant and of order $(q-1)^2/2$ (including the kernel homology group of order $q-1$). By FINK, JOHNSON and WILKE [6] (3.3) there must be $(q-1)/2$ elements A_b for various elements $b \in K$. By the main result of JHA, JOHNSON and WILKE [13], either π is Walker or there is an associated collineation cyclic group of order $(q-1)/2$ represented by A_b for $|b|=(q-1)/2$.

Furthermore, it follows easily that $c^3l(a) = l(ca)$ for all c of order divisible by $(q-1)/2$ and for all $a \in K$. If c and $2c$ both have orders divisible by $(q-1)/2$ then since l is additive, we have that $8c^3l(a) = l(2ca) = 2l(ca) = 2c^3l(a)$ for all $a \in K$ so that l must be identically zero and thus Walker. Thus, if c has order divisible by $(q-1)/2$ then $2c$ and/or $c/2$ does not and conversely. So, in general $l(t) = t^3l(a)$ if t has order divisible by $(q-1)/2$ and otherwise, $l(t) = l(2(t/2)) = 2l(t/2) = 2(t/2)^3l(1) = t^3l(1)/4 = -t^3l(1)$ since the characteristic is 5.

So, let c, d both have orders divisible by $(q-1/2)$ and assume so does their sum $c+d$. Then $(c+d)^3l(1) = (c^3+d^3)l(1)$ so that if $l(1)$ is not zero then $c=-d$.

Let c,d both have orders divisible by $(q-1)/2$ and $c \neq -d$ so that $c+d$ does not have order divisible by $(q-1)/2$. Then $l(c+d) = -(c+d)^3l(1)$. Now assume that $l(1) \neq 0$. Let $c=1$ and $d \neq 1$. Then $d^3-d^2-d+1=0$ for at least $(q-1)/2-1$ elements $d \neq -1$ and since this is also true for -1, it must be that $(q-1)/2 \leq 3$ or rather that $q=5$. But, all likeable planes of order 5^2 are Walker. Hence $l(1)=0$. Since $l(t) = \pm t^3l(1), l$ is identically zero so that in all cases, π is Walker.

Case (iii) $N \cong PSU(3,h)$ such that $h^3=q$. $|N_{p_0}| = q(q^{2/3}-1)/2$ so that $(|F/M|, p-1) \geq ((q^{2/3}-1)/2, p-1)$ so that for $q=p^{3s}$, $((q^{2/3}-1)/2, p-1) \geq (p-1) > 2$ unless $p=3$. However, this characteristic cannot occur in likeable planes.

Case (iv) $N \cong^2 G_2(h)$ such that $h^3=q$. In this case, $|N_{p_0}|$ is divisible by $q^{1/3}-1$ so that again we must obtain a Walker plane in this situation by lemma 2.

This proves (4.1).

(4.2) COROLLARY (BADER, LUNARDON, THAS [2]). *Let π_F denote the Kantor likeable plane of order 5^{2r}, $r>1$. Then the 0-inversion is not isomorphic to π_F.*

More generally, we may prove the following:

(4.3) THEOREM. *Let π be a likeable plane which is not Walker and such that the 0-inverted plane of the skeleton of π is isomorphic to a plane of the skeleton of one of the known families of flock planes. Then π is the Kantor characteristic 5 likeable plane; a new likeable plane produces a flock plane which is distinct from the known flock planes or their s-inversions.*

(4.4) COROLLARY (BADER, LUNARDON, THAS [2]). *The 0-inversion of the Kantor characteristic 5 likeable plane or order > 25 is nonisomorphic to any of the known flock planes or their inversions.*

PROOF. (4.3) Assume that π is not Kantor characteristic 5.

Lemma 1. The collineation group of the 0-inverted plane of a non-Walker likeable plane is the subgroup of the likeable plane which fixes a given regulus.

PROOF. This collineation group may be identified with the stabilizer of a point say p_1 in $\{p_0, p_1, \dots, p_q\}$. If this stabilizer moves p_0 then the generated group is doubly transitive and the argument of (4.1) shows that the plane is Walker. Hence, the stabilizer group must be a subgroup of the collineation group of π and leave the regulus corresponding to p_1 invariant. This proves the lemma.

Recall that the seven families of odd order flock planes are: (1) Desarguesian, (2) Fisher, (3) Kantor characteristic 5 likeable (4) derived Barriga/Cohen-Ganley, (5) Ganley semifield of characteristic 3, (6) Walker, and (7) Knuth semifield (Kantor quadrangle).

By Lemma 1, the 0-inverted plane cannot be in the skeleton of the planes of types (1), (5), and cannot be in the skeletons of the planes of type (7) or (8) since by assumption the plane itself is not either Walker or Kantor characteristic 5 likeable.

By the result of Bader, Lunardon and Thas, the planes of the Fisher skeleton are either Desarguesian or admit a collineation group which induces a q-nest of reguli within the translation plane. EBERT has shown that this type of plane is isomorphic to the plane of Fisher. Hence, (4.5) THEOREM. *The s-inversions of the Fisher plane are either Fisher or Desarguesian.*

We shall show that no plane in the skeleton of the derived Barriga/ Cohen-Ganley plane can admit a collineation group of order q^2. Hence, we cannot have situation (4) and we have the proof to our theorem (4.4).

5 – Semifield skeletons

(5.1) THEOREM. *Let π_F denote a semifield translation plane of odd order q^2 and kernel $GF(q)$ which corresponds to a flock of a quadratic cone. If the autotopism group modulo the kernel homology group does not admit a group of order divisible by $(q^{1/3}-1)/2$ then the 0-inversion of π_F is not isomorphic to π_F (we are not assuming that $q^{1/3}$ is an integer).*

PROOF. We may used the proof of (4.1) to verify that the autotopism group modulo the kernel has order divisible by $(q-1)!$, $(q-1)/2$, $(q^{2/3}-1)$, or $(q^{1/3}-1)$ in the four cases obtained in the proof. Each of these numbers is formally divisible by $(q^{1/3}-1)/2$ so we have the proof to (5.1).

(5.2) THEOREM. *The linear autotopism group of the Ganley semifield plane of order 3^{2r} for $r > 4$ and r odd has order $2(q-1)$ for $r > 4$ and r even has order $4(q-1)$ and includes the kernel homology group.*

PROOF. The Ganley planes are never Desarguesian so by GEVAERT, JOHNSON and THAS [10], the full collineation group permutes the reguli. Hence, the linear autotopism group must leave invariant the regulus containing $y=0$. Represent the spread by $x=0$, $y=x\begin{bmatrix} u+g(t), & f(t) \\ t, & u \end{bmatrix}$ where $u,t \in K \cong GF(q)$. The reguli are represented by the set of matrices with fixed values of t and all $u \in K$. Hence, a typical element of the linear autotopism group g must map $y=x$ onto say $y=x\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$

and thus g must have the form $\begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & av & bv \\ 0 & 0 & cv & dv \end{bmatrix}$ where $ad-bc \neq 0$.

g maps $y=x\begin{bmatrix} g(t), & f(t) \\ t, & 0 \end{bmatrix}$ onto $y=x\begin{bmatrix} \bar{u}+g(\bar{t}), & f(\bar{t}) \\ \bar{t}, & \bar{u} \end{bmatrix}$ where

(1) $\bar{t} = ((-cg(t) + at)av - c^2 f(t)v)/\delta$ where $\delta = ab - cd$,
(2) $\bar{u} = ((-cg(t) + at)bv - cdvf(t))/\delta$,
(3) $\bar{u} + g(\bar{t}) = ((dg(t) - bt)av + dcvf(t))/\delta$,
(4) $f(\bar{t}) = ((dg(t) - bt)bv + d^2 vf(t))/\delta$. Notice that this calculation is valid for any flock plane for different functions.

The functions representing the Ganley plane may be chosen as follows:

$$g(t) = -\alpha t^3, f(t) = n_1 t^9 + n_1 n_2^2 t \text{ where } \alpha^2 = n_1 n_2 .$$

Using (1) and (4) above and using the funcitons listed above, we obtain the following

(5) $(((c\alpha t^3 + at)av - c^2(n_1 t^9 + n_1 n_2 t)v)/\delta)^9 n_1 + (c\alpha t^3 + at)av - c^2(n_1 t^9 + n_1 n_2 t)v)n_1 n_2/\delta = (-d\alpha t^3 - bt)av + d^2(n_1 t^9 + n_1 n_2 t)v)/\delta$ for all $t \in K \cong GF(q)$.

Now assume that $3^r > 3^4$ so that $t^{3^4} \neq t$. Then we have two polynomials in t of the form $l_1 t^{3^2} + l_2 t^{3^2} + l_3 t^{3^4} = m_1 t^3 + m_2 t + m_3 t^{3^2}$ so that l_1, l_3, m_1, m_2 are all 0 and $l_2 = m_3$. The coefficient of t^{3^4} is $(-c^2 n_1^2 v/\delta)^q$ so that $c = 0$. The coefficient of t^3 is $-d\alpha bav/\delta$ so that $dba = 0$ but since $c = 0$, this forces $b = 0$.

(6) $(a^2 vt/\delta)^9 n_1 + a^2 vtn_1 n_2/\delta = d^2(n_1 t^9 + n_1 n_2 t)v/\delta$ for all $t \in K$.

So, $(a^2 v/\delta)^9 n_1 = d^2 n_1 v/\delta$ and $a^2 vn_1 n_2/\delta = d^2 n_1 n_2 v/\delta$.

With $\delta = ad$, it follows that $a^2 = d^2$ and so $v^9 = v$. Putting this information into equations (1) and (3) above forces $\bar{t} = atv/d, \bar{u} = 0$ and $g(atv/d) = vg(t)$ so that $av^2/d = 1$. Also, $f(t/v) = dvf(t)/a$ and since $v^9 = v$, it follows that $dv^2/a = 1$ which in turn shows that $v^4 = 1$ and $(a/d)^2 = 1$. Thus, modulo the kernel, we obtain exactly the autotopism group generated by $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v^2 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & v^3 \end{bmatrix}$ such that $v^4 = 1$. This proves the theorem.

(5.3) Corollary. *If π_F denotes the Ganley semifield plane of order 3^{2r} for $r > 4$ and kernel $GF(3^r)$ the 0-inversion is not isomorphic to π_F.*

Proof. To apply (5.1), we must simply determine the linear autotopism group of the Ganley semifield planes. This is at most 4(modulo the kernel homology group) so since $3^{r/3} - 1 > 4$, we may apply (5.1) (note if $q = 3^6$ then there are only the cases where $(3^6 - 1)/2$, or $3^{6/3} - 1$ to consider and each of these numbers is strictly larger than 4).

Note that (5.3) is probably valid for $r = 2, 3$. However, the autotopism group of the Ganley plane must be worked out in order to apply our arguments. We shall not be concerned with this.

Finally, we consider the possibility that the 0-inversion of the Ganley plane is isomorphic to another plane of a skeleton of a known plane.

(5.4) COROLLARY. *If π denotes the Ganley semifield plane of order 3^{2r} for $r > 4$ then the 0-inversion is not isomorphic to any plane of a skeleton of any of the other known flock planes.*

PROOF. This is straightforward and based upon the fact that if the 0-inversion is isomorphic to a plane of the skeleton of π_1 then the original plane is also isomorphic to a plane of the skeleton of π_1. We shall leave the details to the reader.

6 – Translation planes with large autotopism groups

Although not phrased in this manner, the following is due to HIRAMINE [11]. Actually, the same theorem for planes of prime square order is due (previously) to COHEN and GANLEY[4].

(6.1) THEOREM (HIRAMINE [11], GANLEY [4] for prime square order). *Let π be a translation plane of order q^2 and kernel $K \cong GF(q)$ which admits a linear autotopism group G which has a component orbit of length $q^2 - q$. Then (1) The remaining set of $q + 1$ components define a regulus R in $PG(3, K)$. (2) The matrix spread set for the translation plane derived from π by deriving R may be represented in the form $x = 0$ $y = x\begin{bmatrix} u + \alpha t^{1+k}, & \beta t^{1+2k} \\ t, & u \end{bmatrix}$ for some integer k and constants α, β in K for all $u, t \in K$.*

PROOF. HIRAMINE [11] determines the form of the plane derived from π and that K is a central kernel in the associated quasifield. This equivalent to the corresponding net being a regulus net (i.e. a regulus $PG(3, K)$). By JOHNSON [19] (section III) if $\begin{bmatrix} c & f \\ d & g \end{bmatrix}$ for $d \neq 0$ represents a component in π then $\begin{bmatrix} -cd^{-1}, & f - cd^{-1}g \\ d^{-1}, & d^{-1}g \end{bmatrix}$ represents a component in the translation plane obtained by deriving the regulus net represented in the form $x = 0, y = x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for all $u \in K$. This procedure with the functions determined by Hiramine will produce the form for the spread of the derived plane. The details are left to the reader.

(6.2) NOTES. *In* (6.1) *above,*

(1) if $k = 1, \alpha = -1, \beta = -1/3$ and $q \equiv -1 \bmod 3$, the Walker-Betten planes are obtained (see GEVAERT *and* JOHNSON [9] *table),*

(2) if $k = 2, \alpha = -\rho, \beta = -\gamma$ where $\rho^2 = 5\gamma$, $q = p^r, p \equiv \pm 2 \bmod 5$, the derived Barriga/Cohen-Ganley (also the planes of NARAYANA RAO, SATYANARAYANA, VITHAL RAO [23] *also see* BADER [1]*) are obtained (see* GEVAERT *and* JOHNSON [10] *table).*

(3) If $\alpha = 0$ and $k = (p^i - 1)/2$ where $q = p^r$ then for β a nonsquare a Knuth semifield plane is obtained which corresponds to the generalized quadrangles of Kantor [22].

We shall simply call these planes Knuth semifield planes of flock type.

(6.3) THEOREM. *Let π be a translation plane or order q^2 which is derived from a translation plane of order q^2 and kernel $K \cong GF(q)$ that admits a linear autotopism group with a component orbit of length $q^2 - q$. Represent the plane in the form of* (6.1) *for the constant triple (α, β, k).*

(1) Then the 0-inverted plane $\pi^{-0} = \pi_1$ is isomorphic to π. Further, the s-inverted planes for $s \neq 0$ are all isomorphic.

(2) Assume that $\alpha \neq 0$ and $(q-1) > (1+2k)^2$ then either

(i) the full linear collineation group modulo the kernel homology group is of order $q(q-1)$, fixes one regulus and acts transitively on the $q-1$ remaining reguli or

(ii) π is either Walker or Knuth semifield of flock tupe.

(3) If $\alpha = 0$ then π is a Knuth semifield plane of flock type.

PROOF. (1)

Note that $\begin{bmatrix} \alpha t^{1+k}/2, & \beta t^{1+2k-1} \\ t, & -\alpha t^{1+k}/2 \end{bmatrix} = \begin{bmatrix} \alpha t^{-1k}/2\delta, & \beta t^{-1-2k}/\delta \\ t^{-1}/\delta, & -\alpha t^{-1-k}/2\delta \end{bmatrix}$ where $\delta = \alpha^2/4 + \beta$. It is easy to verify from here that the 0-inverted plane is isomorphic to the original plane.

The plane π admits a group which fixes one regulus R_0 and acts transitively on the components not in the fixed regulus. This group must induce a group in $PGL(5,q)_{\{p_0,p_1,\dots p_q\},Q_4}$ and so permutes the planes of the skeleton of π in orbits of lengths 2 and $q-1$. This proves (1).

PROOF. (2) The full collineation group of the plane must permute the q reguli. If R_0 is moved then there must be a group which acts transitively on the reguli by (1). But, by (3.9), the plane π must be likeable or a semifield plane. If π is likeable then from JHA, JOHNSON, WILKE [12], since there is a collineation group of order $(q-1)$ modulo the kernel homologies, the plane must be Walker. If π is a semifield plane then the functions $g(t) = \alpha t^{1+k}$ and $f(t) = \beta t^{1+2k}$ must be additive. And, since $f(t)$ is $1-1, \beta \neq 0$ so that $t^{1+2k} = t^{p^s}$ for $q = p^r$. Assume that $\alpha \neq 0$. Then $t^{1+k} = t^{p^d}$ implies that $t^{2k} = t^{p^s-1}$ for all t so that $2p^d - p^s - 1 \equiv 0$ modulo $p^r - 1$. We may assume that $0 \leq d \leq r$. So, $1+k \equiv p^d(\mathrm{mod}(p^r-1)$ implies, by an easy calculation, that $1+k = p^d$. If $1+k = p^r$ then $2k \equiv 0 \,\mathrm{mod}(p^r-1)$ so that it follows that the plane must be Desarguesian in this case. So, assume that $d < r$ and $1+k = p^d$. Since p is odd then $2p^d - 1 < p^r$. But, it follows that now $2p^d - 1 = p^s$ so that d or $s = 0$. If $d = 0$ we have the situation as above and the plane is Desarguesian. If $s = 0$ then $2p^d - 1 = 1$ implies $d = 0$. Thus, it must be that $\alpha = 0$.

With $g(t) = 0$, it is easy to check that the planes containing the conics of the flock of the quadratic cone corresponding to π all contain a common point. By a result of THAS [24], the flock must be of Kantor type (see [9])) and the translation plane must be a Knuth semifield plane of flock type. This proves that the situation described in part (ii) of (2) occurs when the regulus R_0 is moved by the collineation group. Furthermore, note that this last remark also proves (3).

So, we may assume that the full collineation group of π leaves the regulus R_0 invariant. We may use the argument in (5.2) above for the collineation g which fixes both $x = 0$, and $y = 0, g = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & av & bv \\ 0 & 0 & cv & dv \end{bmatrix}$ and use equations (1) thru (4). In particular, we obtain $f(((-cg(t)$ $at)av - c^2f(t)v)/\delta) = ((dg(t) - bt)bv + d^2vf(t))/\delta$ where $\delta = ad - bc$. Using $g(t) = \alpha t^{1+k}, f(t) = \beta t^{1+2k}$, we obtain:

(5*) $\beta(((-c\alpha t^{1+k}+at)av-c^2\beta t^{1+2k}v)/\delta)^{1+2k}=((d\alpha t^{1+k}-bt)bv+d^2v\beta t^{1+2k})/\delta$ for all $t\in K$.

We are assuming that $(q-1)>(1+2k)^2$ so that the polynomial in t given in (5*) indicates that $c=0$. But, then this, in turn, implies that $b=0$. Hence,

(6*) $\beta(a^2vt/ad)^{1+2k}=d^2v\beta t^{1+2k}/ad$. Hence, $(av/d)^{1+2k}=dv/a$.

Putting this into the equations (1) thru (4) in (5.2) gives $(av/d)^k=d/a$. Now we may assume that $a=1$ by appropriate multiplication of a kernel homology. Let $dv=c^{1+2k}$ for some c (recall $f(t)=\beta t^{1+2k}$ is $1-1$) so that $d=c^k\lambda$ and thus $v=c^{1+k}/\lambda$ for some $\lambda\in K$. Since now $d^{1+k}=v^k$ from above, it follows directly that $\lambda=1$. Thus, modulo the kernel homology group, we obtain exactly the following collineations:
$\begin{bmatrix}1&0&0&0\\0&c^k&0&0\\0&0&c^{k+1}&0\\0&0&0&c^{1+2k}\end{bmatrix}=\tau_c$ for any nonzero c in K. This proves (2) and thus we have the proof to (6.3).

Note that in RAO et al [23], the full collineation group of the planes of (6.2) (2) is obtained for all orders.

(6.4) THEOREM. *Let π denote a translation plane of odd order q^2 where if $q\equiv -1 \bmod 4$ then q is not a Mersenne prime or if $q\equiv 1 \bmod 4$ then $(q+1)/2$ is not a prime power, and kernel $K\cong GF(q)$ that is derived from a translation plane admitting a linear autotopism group with a component orbit of length q^2-q. Let (α,β,k) denote the constants defining the spread of π. Assume that $q-1>(1+2k)^2$. Then π corresponds to a flock of a quadratic cone and if all of the planes of the skeleton of π are isomorphic then π is either*

(1) Walker,

(2) Knuth semifield plane of flock type.

Note: the hypothesis $q-1>(1+2k)^2$ is probably unnecessary. We use this so as to simplify the determination of the collineation group of the plane.

PROOF. Assume the plane is neither Walker or Knuth semifield plane of flock type.

Consider the $q+1$ points $\{p_0, p_1, \ldots p_q\}$ and assume that 0-inverted plane of π corresponds to p_1. Then by (6.3), the group induced on these points by the collineation group of π has orbits $\{p_0\}$, $\{p_1\}$, and $\{p_2, \ldots p_q\}$.

Consider the partition of the $q+1$ points induced by the element τ_{-1} (listed above in the proof of (6.3)). Without loss of generality, assume that τ_{-1} maps p_i onto p_{i+1} where the indices are taken modulo 2 relative to the mapping. So, the partition is $\{\{p_i, p_{i+1}\} | i = 0, 2, 4, \ldots q-1\}$.

We assert that if the planes of the skeleton are all isomorphic and the plane is not Walker or Knuth semifield plane of flock type then the full collineation group acting on the $q+1$ points and preserving the quadric Q_4 induces a doubly transitive permutation group on the partition.

PF: If all of the planes of the skeleton are isomorphic then the 1-inverted plane admits a collineation group which fixes one regulus and acts transitively on the remaining $q-1$. Since the skeleton of a plane in the skeleton of π is identical to the skeleton of π, it follows that the collineation group of the 1-inverted plane induces a group acting on the $q+1$ points $\{p_0, p_1, \cdots, p_q\}$ which has two orbits of length 1 and one orbit of length $q-1$. At least one the orbits of length 1 is neither p_0 nor p_1. Suppose the stabilizer of say p_2 also leaves p_0 invariant. Then the group generated by the stabilizer of p_0 and the stabilizer of p_2 must act transitively on the q points not equal to p_0 so that the plane π is either Walker or Knuth by previous results of this article. Since all of the planes of the skeleton are isomorphic, the same argument works if the stabilizer of p_2 also fixes p_1.

Hence, the stabilizer of p_2 leaves invariant a unique point $\bar{p}_2$ distinct from p_0 or p_1. The exact same argument shows that the element h in the stabilizer of p_0 which maps p_2 onto $\bar{p}_2$ must map $\bar{p}_2$ onto p_2. This clearly implies that h is induced from τ_{-1} in the collineation group of π. Moreover, it is now clear that the group acting on the $q+1$ points acts transitively on the partition indicated. Further, since the stabilizer of p_0 must also act on the partition, if follows that the group acting on the partition is doubly transitive. The element τ_{-1} acts trivially on the partition so that by (6.3)(2)(i), the collineation group of π induces a group of order $(q-1)/2$ on the partition. Since the collineation group of π induces the stabilizer of p_0, it follows that the full linear group induced on the partition has order either $(q+1)(q-1)/4$ or $(q+1)(q-1)/2$ and acts doubly transitively on the $(q+1)/2$ elements of the partition depending on whether there is an element leaving Q_4 invariant which interchanges p_0 and p_1 while moving some partition element.

By our assumption, $(q+1)/2$ is not a prime power. Since we have a group acting doubly transitively on $(q+1)/2$ elements, we may use the classification of finite simple groups to finish our analysis.

Notice that we have a doubly transitive group of order dividing $(q+1)(q-1)/2$ acting on $(q+1)/2$ elements and which must contain a minimal normal non-Abelian simple subgroup N.

We can have the following possibilities for N:

(i) $N \cong A_{(q+1)/2}$, (ii) $N \cong PSL(d,h)$ such that $h^d - 1/h - 1 = (q+1)/2$, (iii) $N \cong PSU(3,h)$ such that $h^3+1 = (q+1)/2$, (iv) $N \cong {}^2B_2(h)$ (Suzuki) such that $h^2+1 = (q+1)/2$, (v) $N \cong {}^2G_2(h)$ (Ree) such that $h^3+1 = (q+1)/2$ or (vi) $PS_p(2d,2)$ such that $s^{d-1}(2^d \pm 1) = (q+1)/2$.

Since N are acts transitively on the $(q+1)/2$ elements and the stabilizer of an element must have order dividing $(q-1)$, elementary arguments on the orders of the groups and the fact that $(q+1)/2$ is not a prime power show that none of the possibilities can occur. We shall leave these details to the reader.

Hence, not all of the planes of the skeleton of π can be isomorphic unless the plane is Walker of Knuth semifield of flock type.

(6.5) COROLLARY. *If π is a derived Barriga/Cohen-Ganley plane of order q^2 and $(q+1)/2$ is not a prime power then not all of the planes of the skeleton of π are isomorphic.*

PROOF. These planes have k constant equal to 2 (see (6.2)(2)). To apply (6.4), we must show that $(q-1) > (1+2k)^2 = 25$. But, a quick check shows that if $(q+1)/2$ is not a prime power then $q > 26$.

(6.6) COROLLARY. *Let π be a derived Barriga/Cohen-Ganley plane of order q^2 where $(q+1)/2$ is not a prime power. Then the 1-inverted plane in the skeleton of π is not isomorphic to any plane in the skeleton of any other known plane corresponding to a flock of a qudratic cone.*

PROOF. If the 1-inverted plane π_1 is likeable or a semifield plane then all of the planes of the skeleton must be isomorphic since the 2-inverted plane is isomorphic to the 1-inverted plane. But, this would say the plane itself is either likeable or a semifield plane. If π_1 is isomorphic to the 1-derived plane of the Ganley semifield plane (assuming that $q = 3^r$ in this case) since each plane of the skeleton of π_1 is isomorphic to some plane of the skeleton of the Ganley plane, and there are exactly two mutually nonisomorphic planes within the skeleton of π, then the original plane π must be a Ganley semifield plane which, of course, cannot be the case. Since the planes of the Fisher skeleton are either Fisher of Desarguesian (see (4.5)) and since the order q^2 is such that for $q = p^r$ the $p \equiv \pm 2 \bmod 5$, the plane cannot be in the Fisher skeleton nor the Kantor characteristic 5 skeleton.

ADDED IN PROOF: Recently, Payne and Thas (S. Payne and J.A. Thas. Conical flocks, partial flocks, derivation and generalized quadrangles. Preprint) have shown that every partial flock of a quadratic cone in $PG(3,q)$ with $q-1$ conics may be uniquely extended to a flock. Hence, using [14] and (2.10)(2), we obtain the following consequence:

COROLLARY.

(i) Flocks of quadratic cones in $PG(3,q)$ are equivalent to translation planes of order q^2 and kernel $GF(q)$ which admit a Baer collineation group of order q.

(ii) If π is a translation plane of odd order q^2 and kernel $GF(q)$ which admits a Baer collineation group of order q, then the net of degree $q+1$ containing the Baer axis corresponds to a regulus and there are q corresponding flocks of quadratic cones in $PG(3,q)$ obtained by first deriving the regulus net in question and then using the Bader, Lunardon, Thas construction.

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