Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 849-967

Conformal changes of almost cosymplectic manifolds

D. CHINEA - J.C. MARRERO(*)

RIASSUNTO – *Si studiano varietà localmente conformemente quasi cosimplettiche. Si danno alcuni esempi.*

ABSTRACT – *In this paper, conformal changes of metrics on almost cosymplectic metric manifolds are studied and some examples are given.*

KEY WORDS – *Almost contact manifolds - almost cosymplectic manifolds - conformal changes - semi-invariant submanifolds - locally conformal Kähler manifolds.*

A.M.S. CLASSIFICATION: 53C15 - 53C55 - 53C40

An almost Hermitian manifold M^{2n} is called locally conformal Kähler (1.c.K) if its metric is conformally related to an Kähler metric in some neighbourhood of every point of M^{2n}. Such manifolds have been studied by various authors (see, for instance, [11], [15], [16], [18], [7]). Examples of l.c.K. manifolds are provided by the Hopf manifolds which have a locally conformal kähler metric while it is known that they admit no Kähler metric (see [15]).

On the other hand, if M^{2n+1} is a differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g), a conformal change of the metric g leads to a metric which is no more compatible with the almost contact structure (φ, ξ, η). This can be corrected by a convenient change

(*)Supported by the "Consejería de Educación del Gobierno de Canarias".

of ξ and η which implies rather strong restrictions. Such a definition is given by I. VAISMAN in [17]. Moreover, he characterize new types of almost contact manifolds and discuss some examples. On the same line, in [6] we introduce a tensor field μ which is a conformal invariant for almost contact metric manifolds. Then, if U is a class of almost contact metric manifolds, by using the tensor field μ, we determine the class U' of all manifolds locally conformally related to manifolds in U.

The aim of this paper is to study the conformal changes on almost cosymplectic manifolds. In section 1 we give some results on almost contact metric manifolds. In section 2, we obtain characterizations for the locally conformal almost cosymplectic and cosymplectic manifolds (see theorem 2.1 and 2.2). Moreover, we prove that if (M,φ,ξ,η,g) is a locally conformal almost cosymplectic manifold then the leaves of the foliation $\eta = 0$ carry and induced locally conformal almost Kähler structure. In section 3 we study semi-invariant submanifolds of a locally conformal Kähler manifold and we prove, under certain conditions, that they are locally conformal cosymplectic submanifolds with the induced almost contact metric structure. Finally, in section 4 we give some examples of locally conformal almost cosymplectic and cosymplectic manifolds. Moreover, we obtain an example of foliation locally conformal cosymplectic.

1 – Preliminaries

Let M be a C^∞ an almost contac metric manifold with metric g and almost contact structure (φ,ξ,η). Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. Then we have,

$$\varphi^2 = -I + \eta\otimes\xi \qquad \eta(\xi) = 1\,,$$
$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)\,,$$

for $X,Y \in \mathfrak{X}(M)$, where I denotes the identity transformation.

An almost contact structure (φ,ξ,η) is said to be normal if the almost complex structure J on $M\times\mathbb{R}$ given by $J(X,a\frac{d}{dt})= (\varphi X - a\xi, \eta(X)\frac{d}{dt})$, where a is a C^∞ function on $M\times\mathbb{R}$, is integrable, which is equivalent to the condition $N_\varphi + 2d\eta\otimes\xi = 0$, where N_φ denotes the Nijenhuis torsion of φ, that is, $N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]$.

We put $N(X,Y) = (N_\varphi + 2d\eta \otimes \xi)(X,Y)$, and we also denote by the same N the tensor field of type $(0,3)$ geometrically equivalent to N, i.e.,

$$N(X,Y,Z) = g(X, N(Y,Z)) . \tag{1.1}$$

The **fundamental 2-form** Φ of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by $\Phi(X,Y) = g(X, \varphi Y)$. If ∇ is the Levi-Civita connection of g and δ is the coderivative operator, it is easy to prove that,

$$(\nabla_X \Phi)(Y,Z) = g(Y, (\nabla_X \varphi)Z) = -g((\nabla_X \varphi)Y, Z) , \tag{1.2}$$

$$3d\Phi(X,Y,Z) = \underset{X,Y,Z}{\mathfrak{S}} (\nabla_X \Phi)(Y,Z) , \tag{1.3}$$

$$2d\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X , \tag{1.4}$$

$$\delta\eta = -\sum_{i=1}^{n} (\nabla_{X_i} \eta)X_i + (\nabla_{\varphi X_i} \eta)\varphi X_i , \tag{1.5}$$

$$\begin{aligned} \delta\Phi(X) = -\sum_{i=1}^{n} \Big\{ &(\nabla_{X_i} \Phi)(X_i, X) + \\ &+ (\nabla_{\varphi X_i} \Phi)(\varphi X_i, X) \Big\} - (\nabla_\xi \Phi)(\xi, X) \end{aligned} \tag{1.6}$$

$$\begin{aligned} g(N_\varphi(X,Y), Z) = {} & (\nabla_X \Phi)(\varphi Y, Z) - (\nabla_{\varphi Y} \Phi)(X, Z) + \\ & + (\nabla_{\varphi X} \Phi)(Y,Z) - (\nabla_Y \Phi)(\varphi X, Z) - \\ & - \eta(Z) \Big[(\nabla_X \Phi)(\xi, \varphi Y) - (\nabla_Y \Phi)(\xi, \varphi X) \Big] + \\ & + \eta(X)(\nabla_Y \Phi)(\xi, \varphi Z) - \eta(Y)(\nabla_X \Phi)(\xi, \varphi Z) , \end{aligned} \tag{1.7}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, where $\{X_1, \ldots, X_n, \varphi X_1, \ldots, \varphi X_n, \xi\}$ is a local orthonormal φ-basis on M.

For an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, the 1-form ω defined by

$$\omega(X) = -\frac{1}{(2n-1)}\delta\Phi(\varphi X) + \frac{\delta\eta}{2n}\eta(X), \tag{1.8}$$

is called **Lee form** of M.

An almost contact metric structure (φ, ξ, η, g) is said to be **Almost quasi Sasakian** if $d\Phi = 0$; **Almost cosymplectic** if $d\eta = d\Phi = 0$; **Cosymplectic** if it is almost cosymplectic and normal; α-**Kenmotsu** (see [9]) if it is normal and $d\Phi = 2\alpha\eta \wedge \Phi, d\eta = 0$ (with $\alpha =$ const.).

The covariant derivative $\nabla\Phi$ of the fundamental 2-form Φ is a covariant tensor of degree 3 which has various symmetry properties. We denote by $C(V)$ the vector space of the tensors with the same symmetries that $\nabla\Phi$, i.e.,

$$\begin{aligned} C(V) =\Big\{\alpha \in \otimes_3^0 V/\alpha(x,y,z) &= -\alpha(x,z,y) = -\alpha(x,\varphi y,\varphi z)+ \\ &+\eta(y)\alpha(x,\xi,z) + \eta(z)\alpha(x,y,\xi)\Big\}. \end{aligned}$$

Here, V denotes a real vector space of dimension $2n+1$ with an almost contact structure (φ, ξ, η) and a compatible inner product $<,>$.

In [5] they have obtained a decomposition of $C(V)$ into twelve components $C_i(V)$ which are mutually orthogonal, irreducible and invariant subspaces under the action of $U(n) \times 1$. Then, it is possible to form 2^{12} different invariant subspaces from these twelve, corresponding to each invariant subspace a class of almost contact metric manifolds. For example, $\{0\}$ corresponds to the class of cosymplectic manifolds (C), C_5 to the class of α-Kenmotsu manifolds (being α a function), C_6 to the class of α-Sasakian manifolds, $C_2 \oplus C_9$ to the class of almost cosymplectic manifolds, $C_5 \oplus C_6$ to the class of trans-Sasakian manifolds, $C_3 \oplus C_4 \oplus C_5 \oplus C_6 \oplus C_7 \oplus C_8$ to the class of normal manifolds ... (for an extensive study of these manifolds we refer to [5]).

We recall the explicit definition of three subspaces of this decomposition,

$$C_4(V) = \left\{ \begin{array}{c} \alpha \in C(V)/\alpha(x,y,z) = \frac{1}{2(n-1)}\Big[(\langle x,y\rangle - \\ -\eta(x)\eta(y))c_{12}\alpha(z) - \Big(\langle x,z\rangle - \eta(x)\eta(z)\Big)c_{12}\alpha(y) - \\ -\langle x,\varphi y\rangle c_{12}\alpha(\varphi z) + \langle x,\varphi z\rangle c_{12}\alpha(\varphi y)\Big],\ c_{12}\alpha(\xi) = 0 \end{array} \right\}$$

$$C_5(V) = \Big\{\alpha \in C(V)/\alpha(x,y,z) = \frac{1}{2n}\Big[\langle x,\varphi z\rangle\eta(y)\bar{c}_{12}\alpha(\xi) - \\ - \langle x,\varphi y\rangle\eta(z)\bar{c}_{12}\alpha(\xi)\Big]\Big\}$$

$$C_{12}(V) = \Big\{\alpha \in C(V)/\alpha(x,y,z) = \\ = \eta(x)\eta(y)\alpha(\xi,\xi,z) + \eta(x)\eta(z)\alpha(\xi,y,\xi)\Big\},$$

where $x, y, z \in V$, $c_{12}\alpha(x) = \sum \alpha(e_i, e_i, x)$, and $\bar{c}_{12}\alpha(\xi) = \sum \alpha(e_i, \varphi e_i, \xi)$ for any arbitrary orthonormal basis $\{e_i\}, i = 1, \ldots, 2n+1$.

2 – Locally conformal almost cosymplectic manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. A conformal change of the almost contact metric structure on M is a change of the form

$$\varphi' = \varphi, \quad \xi' = e^{-\sigma}\xi, \quad \eta' = e^{\sigma}\eta, \quad g' = e^{2\sigma}g$$

where σ is a differentiable function on M. It is clear that $(\varphi', \xi', \eta', g')$ is also an almost contact metric structure on M. We said that $(M, \varphi, \xi, \eta, g)$ is **locally conformal (almost) cosymplectic** (l.c.(A.)C.) if every point $x \in M$ has an open neighbourhood U such that $(U, \varphi', \xi', \eta', g')$ is (almost) cosymplectic for suitable σ. If $U = M$, then we said that $(M, \varphi, \xi, \eta, g)$ is **globally conformal (almost) cosymplectic**. We note that if (φ, ξ, η, g) is l.c.A.C. then the pair (Φ, η) defines a locally conformal cosymplectic structure (in the sense of the definition given in [13]) i.e., for each $x \in M$ there is an open neighbourhood U such that

$$d(e^{\sigma}\eta) = d(e^{2\sigma}\Phi) = 0,$$

for some function $\sigma: U \longrightarrow \mathbb{R}$ (see [13]). Note that in [12] Libermann defines a cosymplectic structure as a pair (Φ, η) where Φ is a closed 2-form and η is a closed 1-form satisfying, moreover, $\eta \wedge \Phi^n \neq 0$. Thus, if

$(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold of class almost cosymplectic then it is cosymplectic in the sense of [12].

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with Lee form ω. By using theorem 1.1, chapter III, of [13] and the definition of cosymplectic structure, the locally conformal (almost) cosymplectic manifolds are characterized as follows (see also [17]),

THEOREM 2.1. *$(M, \varphi, \xi, \eta, g)$ is a l.(g.)c.A.C. manifold iff ω is closed (exact) and*

$$d\Phi = -2\Phi \wedge \omega\,, \quad d\eta = \eta \wedge \omega\,,$$

and $(M, \varphi, \xi, \eta, g)$ is l.(g.)c.C. iff, moreover, $N_\varphi = 0$ where N_φ is the Nijenhuis tensor of φ.

Next, we characterize the locally conformal cosymplectic manifolds through Levi-Civita connection.

Let $(M, \varphi, \xi, \eta, g)$ be a l.c.C. manifold with Lee form ω. Then, for suitable functions σ and open neighbourhoods $U, \{U, e^{2\sigma} g_U\}$ is a family of local metrics on M, wich are conformally related over each intersection $U \cap U'$. If ∇ denotes the Levi-Civita connection of g, we put

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X, Y)B\,, \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M)$, being B the vector field on M given by $g(X, B) = \omega(X)$.

$\overline{\nabla}$ is a torsionless linear connection and $\overline{\nabla}_X g = -2\omega(X)g$, for all $X \in \mathfrak{X}(M)$. Then, using that locally $\omega = d\sigma$, it follows $\overline{\nabla}_X(e^{2\sigma} g) = 0$. Thus, $\overline{\nabla}$ is the Levi-Civita connection of the local metrics $e^{2\sigma} g_U$.

Now, let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with Lee form ω. Then, we have

THEOREM 2.2. *$(M, \varphi, \xi, \eta, g)$ is a l.c.C. manifold iff ω is closed and $\overline{\nabla}_X \varphi = 0$ for all $X \in \mathfrak{X}(M)$.*

PROOF. If (M,φ,ξ,η,g) is an almost contact metric manifold, the covariant derivate of φ is given by [2],

$$
\begin{aligned}
2g\Big((\nabla_X\varphi)Y,Z\Big) =& 3d\Phi(X,\varphi Y,\varphi Z) - 3d\Phi(X,Y,Z)+\\
&+g\Big(N^1(Y,Z),\varphi X\Big) + N^2(Y,Z)\eta(X)+\\
&+2d\eta(\varphi Y,Z)\eta(Z) - 2d\eta(\varphi Z,X)\eta(Y)
\end{aligned}
\tag{2.2}
$$

where N^1 and N^2 are the tensors defined by,

$$N^1(X,Y) = N_\varphi(X,Y)+2d\eta(X,Y)\xi \quad N^2(X,Y) = (L_{\varphi X}\eta)Y-(L_{\varphi Y}\eta)X\,,$$

being L the Lie derivate operator.

We note that,

$$N^2(X,Y) = \eta\Big(N^1(\varphi X,Y)\Big) - 2\eta(X)d\eta(\xi,\varphi Y) \tag{2.3}$$

for all $X,Y \in \mathfrak{X}(M)$.

We suppose that (M,φ,ξ,η,g) is l.c.C.. By using theorem 2.1, and relations (2.2) and (2.3) we have

$$
\begin{aligned}
g\Big((\nabla_X\varphi)Y,Z\Big) =& -\Phi(X,\varphi Y)\omega(\varphi Z) - \Phi(\varphi Z,X)\omega(\varphi Y)+\\
&+\Phi(X,Y)\omega(Z) + \Phi(Z,X)\omega(Y)-\\
&-\eta(X)\eta(Z)\omega(\varphi Y) + \eta(X)\eta(Y)\omega(\varphi Z)
\end{aligned}
\tag{2.4}
$$

for all $X,Y,Z \in \mathfrak{X}(M)$.

Now, from (2.1) it follows:

$$
\begin{aligned}
\Big(\overline{\nabla}_X\varphi\Big)Y =& \Big(\nabla_X\varphi\Big)Y + \omega(\varphi Y)X - \omega(Y)\varphi X-\\
&- \Phi(X,Y)B + g(X,Y)\varphi B\,.
\end{aligned}
\tag{2.5}
$$

for all $X,Y \in \mathfrak{X}(M)$.

Thus, from (2.4) and (2.5), we deduce that $\overline{\nabla}_X\varphi = 0$ for all $X \in \mathfrak{X}(M)$.

Conversely, suppose that $\overline{\nabla}_X\varphi = 0$. Then, by (1.2) and (2.5), we have

$$
\begin{aligned}
\Big(\nabla_X\Phi\Big)(Y,Z) =& \omega(Y)\Phi(X,Z) + \omega(\varphi Y)g(X,Z)-\\
&- \omega(Z)\Phi(X,Y) - \omega(\varphi Z)g(X,Y)\,.
\end{aligned}
$$

Now, if we use (1.3), (1.4) and (1.7) we obtain:

$$d\Phi = -2\Phi \wedge \omega\,, \quad d\eta = \eta \wedge \omega\,, \quad N_\varphi = 0\,, \tag{2.6}$$

which shows that M is l.c.C.

REMARK. If $n \geq 2$, from (2.6) we deduce that

$$\Phi \wedge d\omega = 0 \quad \text{and} \quad \eta \wedge d\omega = 0\,,$$

which implies that the Lee form ω is closed. Thus, we can reformulate the previous theorem as follows

THEOREM 2.2′. *For* $\dim M \geq 5$, $(M, \varphi, \xi, \eta, g)$ *is a l.c.C. manifold iff* $\overline{\nabla}_X \varphi = 0$ *for all* $X \in \mathfrak{X}(M)$. *If* $\dim M = 3$ *one must add the condition that* ω *is closed.*

If N is the tensor defined in (1.1), we obtain

PROPOSITION 2.1. *On a l.c.C. manifold*

$$N(X,Y,Z) = \eta(X)\eta(Y)N(\xi,\xi,Z) + \eta(X)\eta(Z)N(\xi,Y,\xi)\,,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

PROOF. From theorem 2.1, if M is l.c.C., $d\eta = \eta \wedge \omega$ and $N_\varphi = 0$. Thus,

$$\begin{aligned} N(X,Y,Z) &= g\big(X, (N_\varphi + 2d\eta \otimes \xi)(Y,Z)\big) = \\ &= \eta(X)\eta(Y)\omega(Z) - \eta(X)\eta(Z)\omega(Y) \end{aligned}$$

whence $N(\xi,\xi,Z) = \omega(Z) - \eta(Z)\omega(\xi)$ and $N(\xi,Y,\xi) = \eta(Y)\omega(\xi) - \omega(Y)$.
The result follows now from the above relations.

Next, let $(M, \varphi, \xi, \eta, g)$ be a l.(g.)c.A.C. manifold. Since the pair (Φ, η) defines a locally (globally) conformal cosymplectic structure, using proposition 1.2, chapter III, of [13], the leaves of the foliation F, given by $\eta = 0$, carry an induced locally (globally) conformal symplectic structure with fundamental 2-form induced by Φ on the leaves. Thus, if we consider the almost Hermitian structure induced by the almost contact metric structure of M on the leaves of F, we deduce

PROPOSITION 2.2. *Let $(M, \varphi, \xi, \eta, g)$ be a l.(g.)c.(A.)C. manifold. Then the leaves of F carry an induced l.(g.)c.(A.)K. structure.*

Finally, we suppose that M is a l.c.A.C. manifold with $\omega \neq 0$ at every point of M and $\omega(\xi) = 0$. We define on the leaves of the foliation F' given by $\eta = \omega = 0$ an almost contact metric structure $(\varphi', \xi', \eta', g)$ by

$$\varphi' X = \varphi X - \frac{\omega(\varphi X)}{\|\omega\|^2} B, \quad \xi' = -\frac{\varphi B}{\|B\|}, \quad \eta' = \frac{\omega \circ \varphi}{\|\omega\|}.$$

Then we obtain

PROPOSITION 2.3. *The structure $(\varphi', \xi', \eta', g)$ is almost quasi Sasakian on the leaves of the foliation F'.*

3 – Semi-invariant submanifolds of Locally conformal Kähler manifolds

Let M a submanifold immersed in an almost Hermitian manifold $\overline{M}$ with almost Hermitian structure $(J, \bar{g})$, and $\overline{U}$ a unit vector field on $\overline{M}$ normal to M.

We say that M is a semi-invariant submanifold of $\overline{M}$ with respect to $\overline{U}$ if

$$JX = \varphi X - \eta(X)\overline{U} \quad \xi = J\overline{U}_M \in \mathfrak{X}(M)$$

where φX is the tangential component to M of JX.

It is well known that any semi-invariant submanifold M of an almost Hermitian manifold $\overline{M}$ with induced structure (φ, ξ, η, g) is an almost contact metric manifold, where g is the induced metric on M [4].

Hereafter, $\overline{M}$ is a $2(n+1)$-dimensional almost Hermitian manifold and M is a $(2r+1)$-dimensional semi-invariant submanifold of $\overline{M}$.

Comparing the fundamental 2-forms Φ and $\overline{\Phi}$ of the structures in M and $\overline{M}$, respectively, we have $\Phi(X, Y) = \overline{\Phi}(X, Y)$, for all $X, Y \in \mathfrak{X}(M)$.

Let $L, \overline{\nabla}$ and ∇ be denote the Lie differentiation and the Levi-Civita connections of the metrics $\bar{g}$ and g, respectively. Then it is not difficult to check the following:

PROPOSITION 3.1. *For all $X, Y, Z, \in \mathfrak{X}(M)$ and $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$ we have,*

$$(L_{\overline{U}}\bar{g})(X,Y) = 2\bar{g}(\overline{\nabla}_X\overline{U}, Y) = 2\bar{g}(\overline{\nabla}_Y\overline{U}, X) \tag{3.1}$$

$$(L_{\overline{U}}\overline{\Phi})(\overline{X},\overline{Y}) = (L_{\overline{U}}\bar{g})(\overline{X}, J\overline{Y}) + g(\overline{X}, (L_{\overline{U}}J)\overline{Y}) \tag{3.2}$$

$$\begin{aligned}(\overline{\nabla}_X\overline{\Phi})(Y,Z) =&(\nabla_X\Phi)(Y,Z) - 1/2\Big(\eta(Z)(L_{\overline{U}}\bar{g})(X,Y) -\\ &-\eta(Y)(L_{\overline{U}}\bar{g})(X,Z)\Big)\end{aligned} \tag{3.3}$$

$$\begin{aligned}(\overline{\nabla}_{\overline{U}}\overline{\Phi})(\overline{U},X) =&\frac{1}{2}\Big(- (L_{\overline{U}}\bar{g})(\overline{U},\varphi X) - \bar{g}\Big((L_{\overline{U}}J)\overline{U}, X) +\\ &+\bar{g}(\overline{U}, (L_{\overline{U}}J)X)\Big)\end{aligned} \tag{3.4}$$

$$\begin{aligned}(\delta\overline{\Phi})(X) =&\delta\Phi(X) - \sum_{r+1}^{n}\Big((\overline{\nabla}_{\overline{X}_j}\overline{\Phi})(\overline{X}_j, X) + (\overline{\nabla}_{J\overline{X}_j})(J\overline{X}_j, X)\Big) +\\ &+\frac{1}{2}\eta(X)\Big(\sum_{i=1}^{r}(L_{\overline{U}}\bar{g})(X_i, X_i) + +(L_{\overline{U}}\bar{g})(\varphi X_i, \varphi X_i)\Big)\end{aligned} \tag{3.5}$$

where $\{\overline{X}_1, \dots, \overline{X}_n, \overline{U}, J\overline{X}_1, \dots, J\overline{X}_n, \xi\}$ is a local J-basis of $\overline{M}$, being $X_i = \overline{X}_i$, $i = 1, \dots, r$ and $\{X_1, \dots, X_r, \varphi X_1, \dots, \varphi X_r, \xi\}$ a local φ-basis of M.

Now, we suppose that $(\overline{M}, J, \bar{g})$ is a l.c.K. manifold with Lee form $\overline{\omega}$ and let $\overline{B}$ be the field vector given by $\overline{\omega}(\overline{X}) = \bar{g}(\overline{X}, \overline{B})$ for all $\overline{X} \in \mathfrak{X}(\overline{M})$. We recall that a vector field $\overline{X}$ on $\overline{M}$ is analytic if $L_{\overline{X}}J = 0$. Then we have

PROPOSITION 3.2. *If $\overline{U}$ is a vector field analytic and Killing orthogonal to $\overline{B}$, then M is l.c.C. of class C_5.*

PROOF. We consider a J-basis on $\overline{M}$ obtained as in proposition 3.1. Since $\overline{M}$ is a l.c.K manifold, we have,

$$(\overline{\nabla}_{\overline{X}_j}\overline{\Phi})(\overline{X}_j, X) = (\overline{\nabla}_{J\overline{X}_j}\overline{\Phi})(J\overline{X}_j, X) = -(1/2n)\overline{\delta\Phi}(X),$$

for all $X \in \mathfrak{X}(M)$ and $j = r+1, \dots, n$. Thus, from (3.5) and since $\overline{U}$ is analytic and Killing, it follows

$$\overline{\delta\Phi}(X) = (n/r)\delta\Phi(X)\,. \tag{3.6}$$

On the other hand,

$$\begin{aligned}(\overline{\nabla}_{X_i}\overline{\Phi})(J\overline{U}, JX_i) &= -\,(\overline{\nabla}_{JX_i}\overline{\Phi})(J\overline{U}, X_i) = \\ &= -\,(1/2n)\overline{\delta\Phi}(\overline{U}), \quad \text{for} \quad i = 1, \dots, r\,.\end{aligned}$$

Thus,

$$\begin{aligned}\delta\eta &= -\sum_{i=1}^{r}\Big((\nabla_{X_i}\eta)X_i + (\nabla_{\varphi X_i}\eta)\varphi X_i\Big) = \\ &= -\sum_{i=1}^{r}\Big((\nabla_{X_i}\Phi)(\xi, \varphi X_i) + \Big((\nabla_{\varphi X_i}\Phi)(\xi, \varphi^2 X_i)\Big) = \\ &= -\sum_{i=1}^{r}\Big((\overline{\nabla}_{X_i}\overline{\Phi})(J\overline{U}, JX_i) - (\overline{\nabla}_{JX_i}\overline{\Phi})(J\overline{U}, X_i)\Big) = (r/n)\overline{\delta\Phi}(\overline{U})\,,\end{aligned}$$

that is

$$\delta\eta = (r/n)\overline{\delta\Phi}(\overline{U})\,. \tag{3.7}$$

Also, from (3.4) and since $\overline{M}$ is a l.c.K. manifold, we obtain

$$0 = (\overline{\nabla}_{\overline{U}}\overline{\Phi})(\overline{U}, X) = -(1/2n)\Big(\overline{\delta\Phi}(X) + \bar{g}(\overline{U}, JX)\overline{\delta\Phi}(\xi)\Big)\,,$$

and thus

$$\delta\Phi(\varphi X) = 0\,, \quad X \in \mathfrak{X}(M)\,. \tag{3.8}$$

Moreover, since $\overline{U}$ is orthogonal to $\overline{B}$, and by (3.6), we deduce that $\delta\Phi(\xi) = 0$ and, consequently,

$$\delta\Phi = 0\,. \tag{3.9}$$

Therefore, from (3.6), (3.7) and (3.9) and since $\overline{M}$ is a l.c.K. manifold and $\overline{U}$ analytic and Killing, we obtain,

$$(\nabla_X\Phi)(Y,Z) = (\delta\eta/2r)\Big(\Phi(X,Z)\eta(Y) - \Phi(X,Y)\eta(Z)\Big)$$

whence

$$d\Phi = -2(\delta\eta/2r)\eta\wedge\Phi\,, \quad d\eta = 0 \quad \text{and} \quad N_\varphi = 0\,.$$

Moreover, using (3.7) and (3.8), $\overline{\omega}(X) = -(\delta\eta/2r)\eta(X)$, which implies that $(\delta\eta/2r)\eta$ is closed. Thus, from theorem 2.1, M is l.c.C. of class C_5.

A l.c.K. manifold is said to be strongly no Kähler l.c.K. (s.n.l.c.K.) if its Lee form $\omega \neq 0$ at every point [15].

Next, we shall study semi-invariant submanifolds of s.n.l.c.K. manifolds with respect $\overline{U}$, where $\overline{U}$ is in the direction of $J\overline{B}$. First, one has

LEMMA 3.1. *On a l.c.K. manifold we have:*

$$(L_{J\overline{B}}\overline{g})(\overline{X}, J\overline{Y}) = -\overline{g}(\overline{X}, (L_{J\overline{B}}J)\overline{Y})\,,$$

for all $\overline{X}, \overline{Y} \in \mathfrak{X}(M)$. In particular, $J\overline{B}$ is analytic iff it is Killing.

PROOF. Using $L_{J\overline{B}}\overline{\Phi} = 0$ and (3.2) it follows the assertion.

PROPOSITION 3.3. *Let $\overline{M}$ be a s.n.l.c.K. manifold and M a semi-invariant submanifold with respect $J\overline{B}/\|\overline{B}\|$. If $(L_{J\overline{B}}\overline{g})(X,Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$, then M is l.c.C. of class C_5.*

PROOF. Since $(L_{J\overline{B}}\bar{g})(X,Y) = 0$ and $J\overline{B}$ is orthogonal to M we have

$$(L_{J\overline{B}/\|\overline{B}\|}\bar{g})(X,Y) = 0\,, \tag{3.10}$$

for all $X, Y \in \mathfrak{X}(M)$.

From (3.3), (3.10) and since $\overline{M}$ is a l.c.K. manifold it follows

$$(\nabla_X\Phi)(Y,Z) = 1/2\Big(\Phi(X,Y)\omega(Z) - \Phi(X,Z)\omega(Y)\Big) \tag{3.11}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Moreover, since $\eta = -\omega/\|\omega\|$, by using (3.11) we deduce that $\delta\eta/2r = -\|\omega\|/2$, which shows that M is l.c.C. of class C_5.

Now, we suppose that $\dim \overline{M} \geq 6$ and the 1-form $\overline{\omega} \circ J$ is completely integrable. Then

PROPOSITION 3.4. *Are equivalently:*

i) The leaves of the foliation $\overline{\omega} \circ J = 0$ *are l.c.C. semi-invariant submanifolds of* $\overline{M}$.

ii) $(L_{J\overline{B}}\bar{g})(X,Y) = 0$, *for all* X, Y *orthogonals to* $J\overline{B}$.

iii) $\bar{g}\Big((L_{J\overline{B}}J)JX, Y\Big) = 0$, *for all* X, Y *orthogonals to* $J\overline{B}$.

PROOF. (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) they follows directly from lemma 3.1 and proposition 3.3.

(i) $\Longrightarrow$ (ii) is proved as follows. Let M be a generic leaf of the foliation $\omega \circ J = 0$ and X, Y, Z vector fields tangent to M. Since M is l.c.C. it is of type $C_4 \oplus C_5 \oplus C_{12}$ (see theorem 2.1 of [6]). Moreover, since $\overline{M}$ is a l.c.K. manifold, using (3.3) we have

$$\begin{aligned}(\nabla_X\Phi)(Y,Z) &= (\|\omega\|/2)\Big(\Phi(X,Z)\eta(Y) - \Phi(X,Y)\eta(Z)\Big)+ \\ &+ 1/(2\|\omega\|)\Big(\eta(Z)(L_{J\overline{B}}\bar{g})(X,Y) - \eta(Y)(L_{J\overline{B}}\bar{g})(X,Z)\Big)\end{aligned} \tag{3.12}$$

whence $(\nabla_X\Phi)(Y,Z) = 0$, for all X, Y, Z tangents to M and orthogonal to $\xi = -B/\|B\|$. Thus, M is of type $C_5 \oplus C_{12}$. Moreover, since $\dim M \geq 5$, by proposition 2.3 of [6], M is of type C_5.

On the other hand, from (3.1), (1.5) and (3.12), $\delta\eta = n\|\omega\|$. Then, using again (3.12) we have that $\eta(Z)(L_{J\overline{B}}\bar{g})(X,Y) - \eta(Y)(L_{J\overline{B}}\bar{g})(X,Z) = 0$.

This shows that (i) $\Longrightarrow$ (ii).

4 – Examples

- Let (M, J, h) be an almost Hermitian manifold, $\dim M = 2n (n \geq 2)$, and θ an arbitrary 1-form on M. In $M \times \mathbb{R}$ we consider the almost contact metric structure (φ, ξ, η, g) given by

$$\varphi(X, ad/dt) = (JX, -t\theta(JX)d/dt), \qquad \xi = (0, d/dt),$$
$$\eta(X, ad/dt) = t\theta(X) + a$$
$$g\Big((X, ad/dt), (Y, bd/dt)\Big) = h(X, Y) + ab + t^2\theta(X)\theta(Y) + \Big(\theta(X)b + \theta(Y)a\Big)t$$

where a and b are C^∞ functions on $M \times \mathbb{R}$, $X, Y \in \mathfrak{X}(M)$. Then, if M is locally conformal Kähler manifold, with Lee form ω, put $\theta = -\omega/2$, the above almost contact metric structure (φ, ξ, η, g) is l.c.C.. Moreover, if M is not globally conformal Kähler then $M \times \mathbb{R}$ is not globally conformal cosymplectic.

- A more interesting example is that of the $(2n+1)$-dimensional real Hopf manifold $\mathbb{R}H^{2n+1}$ (see [19]), which is defined as follows. We consider the transformation $\psi_\lambda \colon \mathbb{R}^{2n+1} - \{0\} \longrightarrow \mathbb{R}^{2n+1} - \{0\}$ given by

$$\bar{x}^i = \lambda x^i, \quad \lambda \in \mathbb{R}, \quad \lambda > 0, \quad \lambda \neq 1,$$

and denote by Ψ_λ the infinite cyclic group generated by Ψ_λ. Then $\mathbb{R}H^{2n+1} = (\mathbb{R}^{2n+1} - \{0\})/\Psi_\lambda$.

Using the diffeomorphism f of $\mathbb{R}^{2n+1} - \{0\}$ on $S^{2n} \times \mathbb{R}$ given by

$$(x^i) \longrightarrow (x^i/\|x\|, \ln\|x\|/\ln\lambda)$$

we obtain that $\mathbb{R}H^{2n+1}$ is diffeomorphic to $S^{2n} \times S$, which proves that $\mathbb{R}H^{2n+1}$ is a compact connected differentiable manifold.

Now, we consider in $\mathbb{R}^{2n+1} - \{0\}$ the metric

$$g = \frac{\sum_{i=1}^{2n+1} (dx^i)^2}{\sum_{i=1}^{2n+1} (x^i)^2}$$

where $(x^1, \dots, x^{2n+1})$ are the coordinates in $\mathbb{R}^{2n+1} - \{0\}$. The vector fields

$$X_i = \Big(\sum_{i=1}^{2n+1} (x^i)^2\Big)^{1/2} \partial/\partial x^i, \quad i = 1, \dots, 2n+1,$$

form an orthonormal basis for the Riemannian manifold ($\mathbb{R}^{2n+1} - \{0\}$). Let (φ, ξ, η, g) be the almost contact metric structure on $\mathbb{R}^{2n+1} - \{0\}$ given by

$$\varphi X_i = X_{n+i}, \quad \varphi X_{n+i} = -X_i \quad i = 1, \dots, n$$
$$\xi = X_{2n+1}, \quad \eta = \frac{dx^{2n+1}}{\left(\sum_{i=1}^{2n+1}(x^i)^2\right)^{1/2}}$$

The structure (φ, ξ, η, g) is g.c.C. of type $C_4 \oplus C_5 \oplus C_{12}$, where the Lee form ω is given by

$$\omega = \frac{\sum_{i=1}^{2n+1} x_i dx_i}{\sum_{i=1}^{2n+1} (x^i)^2}.$$

Then the tensors φ, ξ, η and g on $\mathbb{R}^{2n+1} - \{0\}$ all descend to $\mathbb{R}H^{2n+1}$. We denote by $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ the structure induced on $\mathbb{R}H^{2n+1}$. Thus, $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is l.c.C. of class $C_4 \oplus C_5 \oplus C_{12}$. Now, by the definition of the diffeomorphism f, one gets $\omega = -\ln \lambda f^* dt$, and consequently, by descend to $\mathbb{R}H^{2n+1}$, the structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is not globally conformal cosymplectic.

REMARK. Since $\mathbb{R}H^{2n+1} \approx S^{2n} \times S^1$ the Betti numbers of $\mathbb{R}H^{2n+1}$ are

$$b_0 = b_1 = b_{2n} = b_{2n+1} = 1, \quad b_i = 0 \quad 2 \leq i \leq 2n-1,$$

and thus, $\mathbb{R}H^{2n+1}$ can have not cosymplectic structures for $n \geq 2$ (see [3]).

- Let $G(k)$ be the connected solvable non-nilpotent Lie group of dimension 3 consisting of real matrices of the form

$$a = \begin{bmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $x, y, z \in \mathbb{R}$ and k is real number such that $e^k + e^{-k}$ is an integer different to 2. An standard calculation shows that

$$\{dx - kxdz, dy + kydz, dz\}$$

is a basis for the right invariant 1-forms on $G(k)$. Let $\Gamma(k)$ be a discret subgroup of $G(k)$ such that the quotient space $M(k) = G(k) \setminus \Gamma(k)$ compact (see [1]). Hence the 1-forms $dx - kxdz, dy + kydz, dz$ descend to 1-forms α, β, γ on $M(k)$. The manifold $M(k)$ can be not cosymplectic structures (see [10]). Indeed, this manifold can have not normal structures.

Now, define

$$\varphi = \alpha \otimes Z - \gamma \otimes X, \ \xi = Y, \ \eta = \beta \quad \text{and} \quad g = \alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma$$

where $\{X, Y, Z\}$ is the dual basis of vector fields to $\{\alpha, \beta, \gamma\}$.

Then (φ, ξ, η, g) is an almost contact metric structure on $M(k)$ l.c.C. of class C_{12} and it is not g.c.C.

- An example of l.c.A.C. manifold which is not globally is obtained as follows. Let M be the product manifold $\mathbb{R}^{2n} \times S^1$, where S^1 is the 1-dimensional sphere, and we consider the global basis of vector fields given by:

$$X_i = \partial/\partial x^i \quad i = 1, \dots 2n \quad X_{2n+1} = 2\sum_{i=1}^{n} x^i \partial/\partial x^i + T,$$

where $(x_1, \dots, x_{2n})$ are the usual coordinates in $\mathbb{R}^{2n}$ and T is the dual vector field of the canonical 1-form (lenght element) θ of S^1. Define on M a metric by

$$g = \sum_{i=1}^{n} \Big((-2x^i\theta + dx^i) \otimes (-2x^i\theta + dx^i) + dx_{n+i} \otimes dx_{n+i}\Big) + \theta \otimes \theta .$$

Then $\{X_i\}_{i=1,\dots,2n+1}$ is an orthonormal frame with respect to g. Now, we consider in M the almost contact metric structure (φ, ξ, η, g) given by

$$\varphi X_i = -X_{n+i}, \ \varphi X_{n+i} = X_i, \ i = 1, \dots, n, \ \xi = X_{2n+1}, \ \eta = \theta .$$

It is easy to check that $(M, \varphi, \xi, \eta, g)$ is l.c.A.C. with Lee form θ, and consequently it is not globally conformal. Moreover, since $N_\varphi \neq 0$, we deduce that $(M, \varphi, \xi, \eta, g)$ is not l.c.C.

Finally, we obtain an example of foliation l.c.C. by using the proposition 3.3.

- Let S_M be the Inoue surfaces, i.e. $S_M = \mathbb{H} \times \mathbb{C}/G_M$, where $\mathbb{H}$ is the upper half of the complex plane $\mathbb{C}$ and G_M is the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ constructed as follows. Let $M \in SL(3, \mathbb{Z})$ be a unimodular matrix with a real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta \neq \bar{\beta}$. We choose a real eigenvector (a_1, a_2, a_3) and an eigenvector (b_1, b_2, b_3) of M corresponding to α and β, respectively. Then the group G_M is defined as the group generated by

$$\varphi_0\colon (w,z) \longrightarrow (\alpha w, \beta z)\,,$$
$$\varphi_i\colon (w,z) \longrightarrow (w + a_i, z + b_i)\,, \quad i = 1,2,3\,.$$

The action of G_M on $\mathbb{H} \times \mathbb{C}$ is properly discontinous and has not fixed points. Hence $S_M = \mathbb{H} \times \mathbb{C}/G_M$ is a compact complex surface ($S_M = \mathbb{H} \times \mathbb{C}/G_M$ is a fiber bundle over the circle S^1 with the 3-torus T^3 as fiber). Moreover, the first Betti number of S_M is equal to 1. Therefore S_M does not admit any Kähler metrics (for further details see [8], [14]).

We consider on $\mathbb{H} \times \mathbb{C}$ the Hermitian metric g, the 1-form ω and the vector field B given by

$$g = \frac{dw \otimes d\overline{w}}{(w_2)^2} + w_2 dz \otimes d\bar{z}\,, \quad \omega = \frac{dw_2}{w_2}\,, \quad B = w_2 \frac{\partial}{\partial w_2}$$

where (w, z) are the coordinates in $\mathbb{H} \times \mathbb{C}$ and $w_2 = \mathrm{Im}(w) > 0$.

Then g, ω and B are invariant by G_M, hence they induce a metric $\bar{g}$, a 1-form $\overline{\omega}$ and a vector field $\overline{B}$ on S_M.

In [14], Tricerri has proved that $(S_M, \bar{g})$ is an locally conformal Kähler manifold with Lee form $\overline{\omega}$ and $\bar{g}(X, \overline{B}) = \overline{\omega}(X)$, for X tangent to S_M.

If we denote by $\overline{J}$ the complex structure on S_M, then it is easy to check that $\overline{w} \circ \overline{J}$ is completely integrable and, moreover, $(L_{\overline{JB}}\bar{g})(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(S_M)$ and normals to $\overline{JB}$. Thus, by using proposition 3.3, we conclude that the leaves of the foliation $\overline{w} \circ \overline{J} = 0$ are l.c.C. of class C_5 (more precisely, the leaves are Kenmotsu and consequently they are not compact). Moreover it is not difficult to prove that the leaf of the foliation through the origin is not g.c.C..

REFERENCES

[1] L. Auslander - L. Green - F. Hahn: *Flows on Homogeneous spaces*, Annals of Math. Studies 53, Princeton Univ. Press. (1963).

[2] D.E. Blair: *Contact Manifolds in Riemannian Geometry*, Lecture Notes Math., 509, Springer (1976).

[3] D.E. Blair - S.I. Goldberg: *Topology of almost contact manifolds*, Journal Differential Geometry 1(1967) 347-354.

[4] D.E. Blair - G.D. Ludden - K. Yano: *Semi-invariant immersions*, Kodai Math. Sem. Rep. 27(1976), 313-319.

[5] D. Chinea - J.C. González: *A classification of almost contact metric manifolds*, Annali di Mat. Pura ed appl. (IV) 156 (1990) 15-36.

[6] D. Chinea - J.C. Marrero: *Conformal changes of almost contact metric structures*, To appear in "Rivista di Matematica della Università di Parma".

[7] A. Gray - L.M. Hervella: *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. di Mat. pura ed appl. (IV) 123 (1980) 35-58.

[8] M. Inoue: *"On surfaces of class VII_0"*, Inventiones Math. 24(1974), 269-310.

[9] D. Janssens - L. Vanhecke: *Almost contact structures and curvature tensors*, Kodai Math. J. 4(1981), 1-27.

[10] M. León: *Compact almost contact solvmanifolds admitting neither Sasakian nor cosymplectic structures*, Riv. Mat. Univ. Parma (4) 15(1989), 105-110.

[11] P. Libermann: *Sur les structures presque complexes et autres structures infinitesimales regulieres*, Bull. Soc. Mat. France 83(1955) 195-224.

[12] P. Libermann: *Sur les automorphismes infinitèsimaux des structures symplectiques ed des structures de contact*, Colloq. Gèomètrie Diffèrentielle Globale (Bruxelles, 1958), Louvain (1959), 37-59.

[13] J.C. Marrero: *Cambios conformes en variedades casi contacto. Variedades localmente conforme almost co-Kähler*, Thesis, Universidad de La Laguna, Spain, (1990).

[14] F. Tricerri: *Some Examples of locally conformal Kähler Manifolds*, Rend. Sem. Mat. Univ. Politec. Torino, vol.40, 1(1982), 81-92.

[15] I. Vaisman: *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976) 338-351.

[16] I. Vaisman: *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. Roma 12(1979), 263-284.

[17] I. Vaisman: *Conformal changes of almost contact metric structures*, Lecture Notes in Math. 792 (1979), 435-443. Springer.

[18] I. Vaisman: *Generalized Hopf Manifolds*, Geometriae Dedicata 13(1982), 231-255.

[19] I. VAISMAN - C. REISCHER: *Local similarity manifolds*, Annali di Matematica Pura ed Applicata (IV) Vol.135 (1983) 279-292.

Lavoro pervenuto alla redazione il 27 ottobre 1989
ed accettato per la pubblicazione il 7 maggio 1992
su parere favorevole di A. Cossu e di S. Marchiafava

INDIRIZZO DELL'AUTORE:

D. Chinea - J.C. Marrero - Departamento de Matemática Fundamental - Universidad de La Laguna - La Laguna, Tenerife - Canary Island - Spain