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On some conditions for univalence and starlikeness in the unit disc

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RIASSUNTO – *Si stabiliscono alcuni criteri per l'univalenza e la stellazione delle funzioni analitiche sul disco unitario. Si usano metodi connessi alla relazione di subordinazione.*

ABSTRACT – *By using the method of differential subordinations, we give some criteria for univalence and starlikeness in the unit disc.*

KEY WORDS – *Starlike - Univalent - Subordination.*

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1 – Introduction and preliminaries

In the beginning we cite the following well-dnown definitions [3].

For a function f analytic in the unit disc $U = \{z\colon |z| < 1\}$, with $f'(0) \neq 0$ and $f(0) = 0$, we say that it is starlike (univalent) if and only if $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, $z \in U$.

For a function f analytic in U with $f'(0) \neq 0$ we say that it is convex (univalent) if and only if $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$, $z \in U$.

We note that f is convex if and only if zf' is starlike.

Let A denote the class of functions f analytic in U with $f(0) = f'(0) - 1 = 0$, and let $S^*(\alpha)$ denote the subclass of A consisting of starlike functions of order α, i.e. the subclass for which $\mathrm{Re}\{zf'(z)/f(z)\} > \alpha$ for some $\alpha (0 \le \alpha < 1)$ and for all $z \in U$. We write S^* instead of $S^*(0)$.

Also we need some notations about subordination.

Let f and F be analytic in U. The function f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

By using the method of differential subordination (see [1] and [2]) we give some criteria for univalence expresing by $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$, and for starlikeness in the unit disc. This paper is motivated by the previous paper of RAM and SANDER SINGH [4].

For the proofs of the coming results we use the following lemma due to MILLER and MOCANU [2].

LEMMA A. *Let q be univalent in U and let θ and ϕ be analytic in a domain in D containing $q(U)$, with $\phi(w) \ne 0$ when $w \in q(U)$. Set $Q(z) = zq'\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) *Q is starlike in U, and*

$$\text{(ii)} \qquad \mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0, \quad z \in U.$$

If p is analytic in U, with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \tag{1}$$

then $p \prec q$, and q is the best dominant of (1).

We note that the univalent function q is said to be a dominant of differential subordination (1) if $p \prec q$ for all p satisfying (1). If $\tilde{q}$ is a dominant of (1) and $\tilde{q} \prec q$ for all dominants q of (1), then $\tilde{q}$ is said to be the best dominant of (1).

2 – Results and consequences

THEOREM 1. *If $f \in A$ satisfies in U the following condition*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\gamma} \left|\frac{zf''(z)}{f'(z)}\right|^{\gamma} < \frac{1-a}{1+a}\left(\frac{3}{2}\right)^{\gamma}, \quad z \in U, \tag{2}$$

for some $0 \le \gamma \le 1$ *and* $0 \le a < 1$, *then* $f \in S^*$ *and* $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+z}{1+az}$.

Proof. First if $\gamma = 0$, then the condition (2) is equal to

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1-a}{1+a}, \quad z \in U,$$

i.e. $\dfrac{zf'(z)}{f(z)} \prec 1 + \dfrac{1-a}{1+a}z$ which implies $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+z}{1+az}$ (because the function $\dfrac{1+z}{1+az}$ maps the unit disc U onto the disc with diameter end points 0 and $\dfrac{2}{1+a}$ on the real axis).

Now, let suppose that $0 < \gamma \le 1$ and let show that in this case for a function p analytic in U with $p(0) = 1$ and $q(z) = \dfrac{1+z}{1+az}$, we have that the following implication

$$\begin{aligned}(3)\qquad &(p(z)-1)^{\frac{1}{\gamma}} + zp'(z)\frac{(p(z)-1)^{\frac{1}{\gamma}-1}}{p(z)} \prec \\ &\prec (q(z)-1)^{\frac{1}{\gamma}} + zq'(z)\frac{(q(z)-1)^{\frac{1}{\gamma}-1}}{q(z)} \equiv h(z) \implies p(z) \prec q(z),\end{aligned}$$

is true and that the function q is the best dominant.

Really, if we choose $\theta(w) = (w-1)^{\frac{1}{\gamma}}, \phi(w) = \dfrac{(w-1)^{\frac{1}{\gamma}-1}}{w}$ and $q(z) = \dfrac{1+z}{1+az}$ $(0 < \gamma \le 1, 0 \le a < 1)$ in Lemma A, then we have that the function

$$Q(z) = zq'(z)\frac{(q(z)-1)^{\frac{1}{\gamma}-1}}{q(z)} = \frac{1}{1+z}\left(\frac{(1-a)z}{1+az}\right)^{\frac{1}{\gamma}}$$

is starlike in U, because

$$z\frac{Q'(z)}{Q(z)} = \frac{1}{2}\frac{1-z}{1+z} + \frac{1}{2}\frac{1-az}{1+az} + \left(\frac{1}{\gamma} - 1\right)\frac{1}{1+az}.$$

Also, we get

$$\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}+z\frac{Q'(z)}{Q(z)}\right\}=\mathrm{Re}\left\{\frac{1}{\gamma}q(z)+z\frac{Q'(z)}{Q(z)}\right\}>0,\quad z\in U.$$

Therefore, the conditions (i) and (ii) of Lemma A are satisfied and the implication (3) follows from Lemma A.

Let consider the function h defined in (3). After some transformations we have

$$h(z)=(1-a)^{\frac{1}{\gamma}}\cdot\left(\frac{z}{1+az}\right)^{\frac{1}{\gamma}}\frac{2+z}{1+z},$$

and from there

$$|h(e^{i\varphi})|=\frac{(1-a)^{\frac{1}{\gamma}}}{2}\cdot\frac{1}{(1+2a\cos\varphi+a^2)^{1/2\gamma}}\sqrt{9+\mathrm{tg}^2\frac{\gamma}{2}}\geq$$

$$\geq\frac{(1-a)^{\frac{1}{\gamma}}}{2}\cdot\frac{1}{(1+2a+a^2)^{1/2\gamma}}\sqrt{9}=\frac{3}{2}\left(\frac{1-a}{1+a}\right)^{\frac{1}{\gamma}}.$$

That's why, the image $h(U)$ contains the disc $|w|<\frac{3}{2}(\frac{1-a}{1+a})^{\frac{1}{\gamma}}$.

Therefore, if p is analytic in U with $p(0)=1$ and if

$$\left|(p(z)-1)^{1/\gamma}+zp'(z)\frac{(p(z)-1)^{\frac{1}{\gamma}-1}}{p(z)}\right|<\frac{3}{2}\left(\frac{1-a}{1+a}\right)^{1/\gamma}$$

which is equivalent to

$$|p(z)-1|^{1-\gamma}\left|p(z)-1+\frac{zp'(z)}{p(z)}\right|^{\gamma}<\frac{1-a}{1+a}\left(\frac{3}{2}\right)^{\gamma},\tag{4}$$

then from the implication (3) we have that $p\prec q$. Finally, if we put $\frac{zf'(z)}{f(z)}$, $f\in A$, instead of p in (4), we get the statement of this theorem.

REMARK 1. For $a=0$ in Theorem 1 we obtain for $0\leq\gamma\leq1$ the result given in [4].

THEOREM 2. *Let q be a convex function in U with $q(0)=1$ and* $\mathrm{Re}\{q(z)\}>\frac{1}{2}$, $z\in U$. *If $0\leq\alpha<1$, p is analytic in U with $p(0)=1$ and* *if*

$$(5)\qquad \begin{aligned}&(1-\alpha)p^2(z)+(2\alpha-1)p(z)-\alpha+(1-\alpha)zp'(z)\prec(1-\alpha)q^2(z)+\\&\quad+(2\alpha-1)q(z)-\alpha+(1-\alpha)zq'(z)\equiv h(z),\end{aligned}$$

then $p\prec q$, and q is the best dominant of (5).

PROOF. For $\alpha=1$ it is evident. Suppose that $0\le\alpha<1$. In Lemma A we choose $\theta(w)=(1-\alpha)w^2+(2\alpha-1)w-\alpha$ and $\phi(w)=1-\alpha$.

Then the function $Q(z)=zq'(z)\phi(q(z))=(1-\alpha)zq'(z)$ is starlike because q is a convex function. Further,

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}+z\frac{Q'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{2q+\frac{2\alpha-1}{1-\alpha}+z\frac{Q'(z)}{Q(z)}\right\}>$$

$$>2\cdot\frac{1}{2}+\frac{2\alpha-1}{1-\alpha}+\operatorname{Re}\left\{z\frac{Q'(z)}{Q(z)}\right\}=\frac{\alpha}{1-\alpha}+\operatorname{Re}\left\{z\frac{Q'(z)}{Q(z)}\right\}>0,\quad z\in U.$$

Therefore, the statement of Theorem 2 easily follows from Lemma A.

If we put $p(z)=z\dfrac{f'(z)}{f(z)}$, $f\in A$, in Theorem 2, then the left hand side of (5) is equal to

$$(6)\qquad \alpha\left(\frac{zf'(z)}{f(z)}-1\right)+(1-\alpha)\frac{z^2f''(z)}{f(z)},$$

COROLLARY 1. *Let $f\in A$ and let*

$$(7)\qquad \operatorname{Re}\left\{\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+(1-\alpha)\frac{z^2f''(z)}{f(z)}\right\}>-\frac{1}{2},\quad z\in U,$$

for some $\alpha, 0\le\alpha\le1$. Then $f\in S^*(\frac{1}{2})$.

PROOF. If we take $q(z) = \dfrac{1}{1-z}$ in Theorem 2, then we have that the function h defined in (5) is equal to

$$h(z) = \frac{z}{(1-z)^2}(2-\alpha-\alpha z),$$

and from there

$$\operatorname{Re}\{h(e^{i\varphi})\} = -\frac{1}{2} - \frac{1-\alpha}{2}\operatorname{ctg}^2\frac{\varphi}{2} \le -\frac{1}{2}.$$

Now, if the relation (7) is satisfied, then the function (6) is subordinate to the function h and from Theorem 2 we get $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1}{1-z}$, i.e. $f \in S^*(\frac{1}{2})$.

Taking $\alpha = 0$ in Corollary 1, we get

COROLLARY 2. *If $f \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{z^2 f''(z)}{f(z)}\right\} > -\frac{1}{2}, \quad z \in U,$$

then $f \in S^*(\frac{1}{2})$.

COROLLARY 3. *If $f \in A$ and if*

$$\left|\left(\frac{zf'(z)}{f(z)} - 1\right) + (1-\alpha)\frac{z^2 f''(z)}{f(z)}\right| < \frac{5-\alpha}{8}, \quad z \in U, \tag{8}$$

for some $\alpha, 0 \le \alpha \le \frac{1}{2}$, then $f \in S^*(\frac{1}{2})$ and $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+\frac{1}{3}z}{1-\frac{1}{3}z}$.

PROOF. For the function $q(z) = \dfrac{1+\frac{1}{3}z}{1-\frac{1}{3}z}$ we have that $q(U)$ is the disc with diameter end points $\frac{1}{2}$ and 2 on the real axis, i.e. $\operatorname{Re}\{q(z)\} > \frac{1}{2}$,

$z \in U$. Further, for such q we have that the function h defined in (5) is equal to

$$h(z) = \frac{2z}{(3-z)^2}\Big[(1-2\alpha)z + 3(2-\alpha\Big]. \tag{9}$$

From (9) we obtain that

$$|h(e^{i\varphi})| =$$
$$= \frac{2}{10-6\cos\varphi}\sqrt{(1-2\alpha)^2 + 6(1-2\alpha)(2-\alpha)\cos\varphi + 9(2-\alpha)^2} \geq$$
$$\geq \frac{2}{16}\sqrt{[(1-2\alpha)-3(2-\alpha)]^2} = \frac{1}{8}(5-\alpha),$$

and from there we conclude that $h(U)$ contains the disc $|w| < \frac{1}{8}(5-\alpha)$.

Finally, if (8) is satisfied, then the function (6) is subordinate to the function h defined by (9) and from Theorem 2 we get

$$\frac{zf'(z)}{f(z)} \prec \frac{1+\frac{1}{3}z}{1-\frac{1}{3}z},$$

which was to be proved.

Taking $\alpha = 0$ in Corollary 3, we obtain

COROLLARY 4. *If for $f \in A$ we have*

$$\left|\frac{z^2 f''(z)}{f(z)}\right| < \frac{5}{8}, \quad z \in U,$$

then $f \in S^*(\frac{1}{2})$ and $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+\frac{1}{3}z}{1-\frac{1}{3}z}$.

THEOREM 3. *Let $f \in A$, $\alpha \geq 1$ and*

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{1}{f'(z)} \prec \alpha+(1-\alpha)\frac{1+z}{1-z}+\alpha\frac{2z}{1-z^2} \equiv h(z), \tag{10}$$

then $\mathrm{Re}\{f'(z)\} > 0$, $z \in U$.

PROOF. First, we want to show that for a function p analytic in U with $p(0) = 1$ we have that the following implication

$$\alpha + (1-\alpha)\frac{1}{p(z)} + \alpha\frac{zp'(z)}{p(z)} \prec h(z) \implies p(z) \prec \frac{1+z}{1-z}, \tag{11}$$

where h is defined in (10), is true. That fact easily follows from Lemma A by taking $\theta(w) = \alpha + (1-\alpha)\frac{1}{w}$, $\phi(w) = \frac{\alpha}{w}$ and $q(z) = \frac{1+z}{1-z} (\alpha \geq 1)$. If in (11) we put f', $f \in A$, instead of p, then we have that the condition (10) implies that $f'(z) \prec \frac{1+z}{1-z}$, i.e. $\operatorname{Re}\{f'(z)\} > 0$.

For the funciton h defined by (10) we conclude that it maps the unit disc U onto the complex plane slit along the half-lines $\operatorname{Re}\{w\} = \alpha$, $\operatorname{Im}\{w\} \geq \sqrt{2\alpha(3\alpha-2)}$ and $\operatorname{Re} w = \alpha$, $\operatorname{Im}\{w\} \leq -\sqrt{2\alpha(3\alpha-2)}$. Combining this fact with Theorem 3, we get

COROLLARY 5. *Let $f \in A$ and $\alpha > 1$. Then each of the following conditions*

$$\operatorname{Re}\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{1}{f'(z)}\right\} < \alpha, \quad z \in U, \tag{12}$$

$$\left|\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{1}{f'(z)}\right| < \sqrt{\alpha(7\alpha-4)}, \quad z \in U, \tag{13}$$

$$\arg\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{1}{f'(z)}\right\} < \operatorname{arctg}\sqrt{\frac{2(3\alpha-2)}{\alpha}}, \quad z \in U, \tag{14}$$

$$\left|\operatorname{Im}\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{1}{f'(z)}\right\}\right| < \sqrt{2\alpha(3\alpha-2)}, \quad z \in U, \tag{15}$$

imply $\operatorname{Re}\{f'(z)\} > 0$, $z \in U$.

For $\alpha = 1$ we have the corresponding conditions in the cases (13), (14) and (15).

REMARK 2 The condition (12) is weaker than that given in [4] (Th. 5), but the conditions (13), (14) and (15) are new.

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