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Stability of solutions of Neumann problems with singular potential

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RIASSUNTO – *Si studia la stabilità variazionale della soluzione di un problema di Neumann rispetto a perturbazioni del termine di potenziale che, nel modello considerato, può essere altamente singolare.*

ABSTRACT – *We study the variational stability of the solution of a Neumann problem with respect to perturbations of the potential term which, in the model considered, can be highly singular.*

KEY WORDS - *Neumann problem - Variational stability - Singular potentials.*

A.M.S. CLASSIFICATION: 35J50

– Introduction

Let us consider the following Neumann problem

$$\text{(N)}\qquad \begin{cases} -\Delta u + b \cdot Du + \mu u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

where μ is a non negative Borel measure on Ω, a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and outward normal ν, playing the role of a possibly singular potential, b and f are, respectively, a given vector field and function on Ω. The existence of a weak solution to (N) has been proved in [11] under some assumptions on the measure μ (see §1 for the precise statement).

In this paper we are interested in the behaviour of the solution of (N) under suitable perturbations of the measure μ. The motivation for such a study relies on the fact that a unified formulation for a large class of boundary value problems can be given in the form (N) for special choice of μ. Let us mention here, as an example the limit problems in the homogeneization of composite media with holes (see [4], [5]). Other significant examples can be found in [7], [8].

1 – Preliminary facts and lemmas

Following [8], [9], let us denote by $\mathcal{M}_0$ the set of all nonnegative Borel measures on Ω which are absolutely continuous with respect to the capacity (i.e. such that $\mu(E) = 0$ if $\text{cap}(E) = 0$), by $L^2(\Omega, \mu)$ the space of measurable functions whose square is summable with respect to μ on Ω and by V the Hilbert space $H^1(\Omega) \cap L^2(\Omega, \mu)$.

A weak solution of (N) is a function $u \in V$ such that

$$\int_\Omega Du \cdot D\phi dx + \int_\Omega b \cdot Du\phi dx + \int_\Omega u\phi d\mu = \int_\Omega f\phi dx \tag{1.1}$$

for every $\phi \in V$. It has been proved in [11] that if $b \in L^\infty(\Omega)$, $\mu \in \mathcal{M}_0$, $\mu(\Omega) \neq 0$, then problem (N) has a unique weak solution for any right hand side $f \in L^2(\Omega)$.

We shall always assume in the sequel that $f \in L^2(\Omega)$ and $b \in L^\infty(\Omega)$.

The resolvent operator $R^\Omega_\mu \colon L^2(\Omega) \longrightarrow V$, is well defined by $R^\Omega_\mu(f) = u$, the unique weak solution of the following problem:

$$\text{(D)} \qquad \begin{cases} -\Delta u + \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

(See [8])

According to [8] (see also [2]) a sequence $\{\mu_h\} \in \mathcal{M}_0$ is said to γ-converge to $\mu \in \mathcal{M}_0$ if $R^\Omega_{\mu_h}(f) \longrightarrow R^\Omega_\mu(F)$ strongly in $L^2(\Omega)$, for every $f \in L^2(\Omega)$ and every open bounded $\Omega \subset \mathbb{R}^N$.

This definition is related to that of Γ-convergence as introduced in [10] (see also [6]). Let us consider now the perturbed problems:

$$(\mathrm{N}_h) \qquad \begin{cases} -\Delta u_h + b\cdot Du_h + \mu_h u_h = f & \text{in } \Omega \\ \dfrac{\partial u_h}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

A basic fact which will be useful later in expressed in the following:

LEMMA 1. *Let u_h be a weak solution of $(\mathrm{N})_h$. If $\|u_h\|_{L^2(\Omega)} \le C$ then $\|Du_h\|_{L^2(\Omega)} \le C'$, for constant C' independent on h.*

PROOF. The choice $\phi = u_h$ in the weak formulation of $(N)_h$ and the nonnegativity of μ_h yield

$$\|Du_h\|^2_{L^2(\Omega)} + \int_\Omega (b\cdot Du_h)u_h dx \le \|f\|_{L^2(\Omega)}\|u_h\|_{L^2(\Omega)}\,.$$

This gives

$$\|Du_h\|^2_{L^2(\Omega)} \le \|u_h\|_{L^2(\Omega)}\Big(\|f\|_{L^2(\Omega)} + \|b\|_{L^2(\Omega)}\|Du_h\|_{L^2(\Omega)}\Big)\,;$$

by the Young inequality the assertion follows. □

Let us associate to any Borel subset E of Ω its μ-capacity $\mathrm{cap}_\mu(E,\Omega)$ by setting:

$$\mathrm{cap}_\mu(E,\Omega) = \min\Big\{\int_\Omega |Du|^2dx + \int_E |u|^2 d\mu \colon u-1 \in H^1_0(\Omega)\Big\}\,;$$

it is proved in [11] that if $\mu_h \xrightarrow{\gamma} \mu$, then

$$\lim_{h\to\infty} \mathrm{cap}_{\mu_h}(E,\Omega) = \mathrm{cap}_\mu(E,\Omega) \tag{1.2}$$

for every $E \in R_\mu$ with $E \subset\subset \Omega$, where R_μ is a special subclass of the Borel family (see [9]).

A basic tool for the sequel is the following Poincaré's type inequality (see [11] for the proof):

LEMMA 2. *Let $\mu \in \mathcal{M}_0$ with $\mu(\Omega) > 0$. Then there exists a constant $K = K(\Omega)$ independent of μ and a domain Ω', with $\Omega \subset\subset \Omega'$ such that*

$$\int_\Omega u^2(x)\,dx \le \frac{K(\Omega)}{\mathrm{cap}_\mu(\Omega,\Omega')}\left\{\int_\Omega |Du|^2 dx + \int_\Omega |u|^2 d\mu\right\}$$

for any $u \in H^1(\Omega)$.

As a consequence of this we have:

LEMMA 3. *Let us assume $\Omega \in R_\mu$, $\mu_k \in \mathcal{M}_0$ where $\mu_k \xrightarrow{\gamma} \mu$, $\mu_k(\Omega) \neq 0$ and $g_k \in L^2(\Omega)$, $g_k \longrightarrow g$ weakly in $L^2(\Omega)$.*

Then the sequence $\{u_k\}$ of the weak solutions of

$$(1.3_k)\qquad \begin{cases} -\Delta u_k + u_k\mu_k = g_k & \text{in } \Omega \\ \dfrac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

converges strongly in $L^2(\Omega)$ to a weak solution u of the problem:

$$(1.3)\qquad \begin{cases} -\Delta u + u\mu = g & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

PROOF. Indeed in [9] it has been proved that the functionals

$$F_k(u) = \begin{cases} \displaystyle\int_\Omega u^2 d\mu_k + \int_\Omega |Du|^2 dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{elsewhere in } L^2(\Omega) \end{cases}$$

are Γ-convergent in $L^2(\Omega)$ to

$$F(u) = \begin{cases} \displaystyle\int_\Omega u^2 d\mu + \int_\Omega |Du|^2 dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{elsewhere in } L^2(\Omega) \end{cases}$$

If we call $G_k = -2\int_\Omega g_k u\,dx$, applying the definition of Γ-convergence it is easy to prove that $F_k(u)+G_k(u)$ are Γ-converging in $L^2(\Omega)$ to $F(u)+G(u)$

and that for every $\lambda \in \mathbb{R}$ there exists a compact set $K_\lambda \subset L^2(\Omega)$ such that:

$$\left\{v \in L^2(\Omega)\colon F_k(v) + G_k(v) \leq \lambda\right\} \subset K_\lambda \quad \text{for every} \quad h \in \mathbb{N}.$$

Indeed, for any $v \in H^1(\Omega)$ such that $F_k(v) + G_k(v) \leq \lambda$, applying the preceeding lemma, and (1.2), we obtain

$$\begin{aligned}
&-2\|v\|_{L^2(\Omega)}\|g_k\|_{L^2(\Omega)} + \frac{\operatorname{cap}_\mu(\Omega,\Omega')}{K(\Omega)}\int_\Omega v^2dx \leq \\
&\leq -2\|v\|_{L^2(\Omega)}\|g_k\|_{L^2(\Omega)} + \frac{\operatorname{cap}_{\mu_k}(\Omega,\Omega')}{K'(\Omega)}\int_\Omega v^2dx \leq \\
&\leq -2\|v\|_{L^2(\Omega)}\|g_k\|_{L^2(\Omega)} + \int_\Omega v^2d\mu_k + \int_\Omega |Dv|^2dx \leq \\
&\leq \int_\Omega v^2d\mu_k + \int_\Omega |Dv|^2dx - 2\int_\Omega g_kvdx \leq \lambda .
\end{aligned}$$

And if we call $\|v\|_{L^2(\Omega)} = y$, we obtain $-K_1y + k_2y^2 \leq \lambda$ and then $\|v\|_{L^2(\Omega)} \leq K$. On the other hand, we have $\int_\Omega |Dv|^2dx \leq \lambda + 2\int_\Omega g_kvdx \leq 2\|v\|_{L^2(\Omega)}\|g_k\|_{L^2(\Omega)}$ and then, $\|v\|_{H^1(\Omega)} \leq K$. By the Rellich theorem we have the statement of the lemma because the weak solutions of the problem $(1.3)_k$ are the minimum points of $(F_k + G_k)(u)$ and this implies the convergence in $L^2(\Omega)$ of the weak solutions u_k of $(1.3)_k$ to the weak solution u of (1.3). □

2 – Stability results: the case $\mu(\Omega) \neq 0$

We are now in position to state a stability result for problem (N) under the main assumption that $\mu(\Omega) \neq 0$; we want to recall that this hypothesis implies that the problem (N) has a unique solution and this will be the basic tool for the proof.

THEOREM 2.1. *Let $\{\mu_h\}, \mu$ be measures in $\mathcal{M}_0$ and Ω a domain in R_μ. Let us assume that*

(A1) $\mu_h(\Omega) \neq 0$ *for every* $h \in N$ *and* $\mu(\Omega) \neq 0$.
(A2) $\{\mu_h\}$ *γ-converges to* μ, *as* $h \longrightarrow +\infty$.

Then, the weak solutions u_h *of* $(N)_h$ *converge weakly in* $H^1(\Omega)$ *as* $h \longrightarrow +\infty$ *to the weak solution* u *of (1).*

PROOF. Let us show first that

$$\|u_h\|_{L^2(\Omega)} \leq C. \tag{2.1}$$

for some constant C independent of h.

It this where not the case, then

$$\|u_{h_k}\|_{L^2(\Omega)} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow +\infty$$

for a subsequence $\{u_{h_k}\}$ of $\{u_h\}$.

Observe now that $v_k = u_{h_k}/\|u_{h_k}\|_{L^2(\Omega)}$ is a weak solution of

$$\begin{cases} -\Delta v_k + b \cdot Dv_k + \mu_{h_k} v_k = \dfrac{f}{\|u_{h_k}\|_{L^2(\Omega)}} & \text{in } \Omega \\ \dfrac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

and that

$$\|v_k\|_{L^2(\Omega)} = 1. \tag{2.3}$$

The Lemma 1 implies that

$$\|v_k\|_{H^1(\Omega)} \leq C.$$

Let v be a weak subsequential limit in $H^1(\Omega)$ of v_k.

By Lemma 3, applied with $g_k = f/\|f_{h_k}\|_{L^2(\Omega)} - b \cdot Dv_k$, we can pass to the limit in (2.2) to show that v is a solution of

$$\begin{cases} -\Delta v + b \cdot Dv + \mu v = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by uniqueness, $v = 0$, contradicting (2.3).

Therefore (2.1) is proved and lemma 1 again yields

$$\left\|u_{h_k}\right\|_{H^1(\Omega)} \leq C'.$$

At this point it is enough to apply Lemma 3 with $g_k = f - b \cdot Du_{h_k}$, to complete the proof. □

REMARK. The same result holds if we allow L^∞-perturbations of b and weak L^2-perturbations of f.

3 – Stability results: the case $\mu(\Omega) = 0$

In this section we shall establish results on the convergence of the sequence of solutions of the problems $(N)_h$ when μ_h γ-converge to $\mu = 0$ as $h \longrightarrow +\infty$.

We shall confine ourselves to measures $\mu_h \in \mathcal{M}_0$ having density with respect to the Lebesgue measure L. We shall therefore assume that

$$\text{(A3)} \qquad \mu_h = q_h L$$

with q_h a non negative Borel function on Ω. In the general case, the basic L^2 estimate in Theorems 3.1 and 3.2 below is an open question.

Let us recall that the Neumann problem

$$\text{(N}_0\text{)} \qquad \begin{cases} -\Delta u + b \cdot Du = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

has a weak solutions u if and only if the right hand side f satisfies

$$\text{(A4)} \qquad \int_\Omega fm\, dx = 0$$

where m is the strictly positive solution of the adjoint problem:

$$\begin{cases} -\Delta m - \operatorname{div}(bm) = 0 & \text{in } \Omega \\ \dfrac{\partial m}{\partial \nu} + b \cdot \nu = 0 & \text{on } \partial\Omega . \end{cases}$$

(see [1] and [3]).

THEOREM 3.1. *Let us assume* (A3), (A4) *and*
(A5) $q_h \in L^\infty(\Omega)$, $0 \le q_h \le K$, $\int_\Omega q_h(x)dx > 0$,
(A6) $\int_\Omega q_h dx \longrightarrow 0$, *as* $h \longrightarrow +\infty$,
(A7) $f \in L^s(\Omega)$, *where* $s > N$, $b \in L^\infty(\Omega)$,
then there exists a subsequence $\{u_{h_k}\}$ *of solutions of problems* $(\mathrm{N})_h$ *converging strongly in* H^1 *and uniformly to a solution of* $(N)_0$.

PROOF. Let us prove first that

$$\|u_h\|_{L^2(\Omega)} \le C, \tag{3.1}$$

for some constant C independent of h.

Let us assume by contradiction that

$$\left\|u_{h_k}\right\|_{L^2(\Omega)} \longrightarrow \infty$$

for some subsequence $\{u_{h_k}\}$ of $\{u_k\}$.

Then arguing as in the proof of the Theorem 2.1 one finds that $v_k = \dfrac{u_{h_k}}{\|u_{h_k}\|_{L^2(\Omega)}}$ satisfies

$$\begin{cases} -\Delta v_k + b\cdot Dv_k + v_k q_{h_k} v_k = \dfrac{f}{\|u_{h_k}\|_{L^2(\Omega)}} & \text{in } \Omega \\ \dfrac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.2}$$

Hence by Lemma 1,

$$\|v_k\|_{H^1(\Omega)} \le K\,. \tag{3.3}$$

If we set

$$g_k = \frac{f}{\|u_{h_k}\|_{L^2(\Omega}} - b\cdot Dv_k + v_k(1 - q_{h_k})\,,$$

then (3.2) reads as

$$\begin{cases} -\Delta v_k + v_k = g_k & \text{in } \Omega \\ \dfrac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.4}$$

By assumption (A7), a boot strap argument shows that $\|g_k\|_{L^s(\Omega)} \leq K$, so that $\{v_k\}$ has a limit, say v, in $C(\overline{\Omega})$. Taking (A6) into account we can pass to the limit in (3.4) to show that v is a weak solution of

$$\text{(3.5)} \qquad \begin{cases} -\Delta v + b \cdot Dv = 0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega . \end{cases}$$

By the maximum principle v is therefore a constant, say $v = C$. On the other hand, the choice $\phi = m$ in the weak formulation of (3.2) and condition (A4) give

$$\int_\Omega v_k q_{h_k} m \, dx = 0 \quad \text{for any} \quad k .$$

Since $m(x) > 0$ for any $x \in \Omega$ and $\int_\Omega q_h(x) dx > 0$ it follows that $\int_\Omega q_h m dx > 0$. Hence,

$$|C| = \left| \frac{\int_\Omega (v_k - C) q_{h_k} m \, dx}{\int_\Omega q_{h_k} m \, dx} \right| \leq \|(v_k - C)\|_{L^\infty(\Omega)} \longrightarrow 0 ,$$

which shows that $v = C = 0$.

We have then proved that $v_k \longrightarrow 0$ converges strongly in $C(\overline{\Omega})$ to 0; this is contradictory with $\|v_k\|_{L^2(\Omega)} = 1$ and therefore (3.1) is proved.

By Lemma 1, u_h is uniformly bounded in $H^1(\Omega)$. At this point a passage to the limit in $(\mathrm{N})_h$ using Lemma 3 completes the proof. □

REMARK. Theorems 3.1 covers the simple case $q_h = \frac{1}{h}$. The asymptotic behaviour of (N) in this case has been investigated for a large class of elliptic operators in [3].

The following result holds under a different convergence assumption on q_h.

THEOREM 3.2. *Let $N \geq 3$. Assume that hypotheses* (A3) *and* (A4) *are satisfied and that:*

(A8) $q_h = c_h \psi_h$, *with* $c_h \in \mathbb{R}$, $\psi_h \in L^p(\Omega)$, $p > 2^* = \frac{2N}{N-2}$

(A9) $\psi_h \geq 0$, $\psi_h \longrightarrow \psi$ *weakly in* $L^p(\Omega)$, $\int_\Omega \psi_h dx > 0$, $\int_\Omega \psi dx > 0$, (A10) $c_h > 0$ *and* $c_h \longrightarrow 0$.

Then the sequence of solutions of $(N)_h$ *converges weakly in* H^1 *some* u *and* u *is the solution of* $(N)_0$ *such that*

$$\int_\Omega u\psi m\,dx = 0\,. \tag{3.6}$$

PROOF. Let us prove first that

$$\|u_h\|_{L^2(\Omega)} \le C\,. \tag{3.7}$$

for some constant C independent of h.

Let us assume by contradiction that

$$\left\|u_{h_k}\right\|_{L^2(\Omega)} \longrightarrow \infty$$

for some subsequence $\{u_{h_k}\}$ of $\{u_h\}$ and set

$$v_k = \frac{u_{h_k}}{\|u_{h_k}\|_{L^2(\Omega)}}\,.$$

As in the proof of the Theorem 3.1 one obtains

$$\|v_k\|_{H^1(\Omega)} \le K\,.$$

Then there exists $v \in H^1(\Omega)$ such that

$$\begin{aligned} &v_{k_j} \longrightarrow v \quad \text{weakly in} \quad H^1\,,\\ &v_{k_j} \longrightarrow v \text{ strongly in } L^s \text{ where } s \in [1, 2^*), \end{aligned}$$

at least for a subsequence $\{v_{k_j}\}$ of $\{v_k\}$.

Observe also that, if we call $v_j = v_{k_j}$, $\psi_j = \psi_{k_j}$, $u_j = u_{h_{k_j}}$, $q_j = q_{k_j}$, then v_j satisfies (3.4) with $g_j = f/\|u_j\|_{L^2(\Omega)} - b \cdot Dv_j + v_j(1 - q_j)$.

Since by (A8), (A9)

$$\int_\Omega v_j \psi_j \phi \, dx \longrightarrow \int_\Omega v\psi\phi \, dx \quad \text{for any} \quad \phi \in V,$$

a passage to the limit in (3.4) shows that v is a weak solution of (3.5).

Hence v is a constant, say $v = C$. Taking (A4) into account we obtain:

$$\int_\Omega v_j \psi_j m \, dx = 0 \quad \text{for any} \quad j$$

and therefore

$$\int_\Omega v\psi m \, dx = 0 .$$

On the other hand

$$\int_\Omega v\psi m \, dx = C \int_\Omega \psi m \, dx .$$

By (A9), and the strict positivity of m we have $\int_\Omega \psi m dx > 0$ and therefore $v = C = 0$. This contradicts $\|v_k\|_{L^2(\Omega)} = 1$ and (3.7) is proved. By Lemma 1, u_h is uniformly bounded in $H^1(\Omega)$. We can now pass to the limit in $(\mathrm{N})_h$, taking into account the fact that $\{u_{h_k}\}$ converges strongly in L^s for any $s \in [1, 2^*)$, the limit function u satisfies $(\mathrm{N})_0$ and

$$\int_\Omega u\psi m \, dx = 0 . \tag{3.8}$$

It is easy to check that if u_1 and u_2 are solutions of $(\mathrm{N})_0$ satisfying (3.8) then $u_1 - u_2$ is a solution of (3.5) and therefore $u_1 - u_2 = C$. Hence by (3.8).

$$C \int_\Omega \psi m \, dx = 0$$

which implies $C = 0$. This proves that the whole sequence $\{u_h\}$ converges weakly in H^1 to u. □

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