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On the integration in convergence spaces

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RIASSUNTO – *Si definisce un integrale, rispetto ad una misura non negativa e finita, per le funzioni a valori in uno spazio pseudo-topologico di Husdorff completo. Se ne studiano alcune proprietà*

ABSTRACT – *An integral with respect to a non negative and finite measure of functions valued in a complete Husdorff pseudo-topological vector space in defined and some of their properties are stated*

KEY WORDS – *Convergence space - Pseudo-topological space - Filter - Filter basis - Measure - Simple function - Integrable function - Integral.*

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1 – Introduction

The convergence spaces (also called pseudo-topological spaces or limit spaces) have had a great utility for solving the different difficulties found in the extension of the Fréchet differentiation in normed spaces to more general topological vector spaces. Analogously to the extension process in the differentiation theory, on extending the Bochner integral to the case of functions valued in locally convex spaces, several theories have appeared, which present some handicaps relative to different questions like the completeness and the dual of the spaces L^p and the permanence of the existence of densities of measures (see [1], [2], [5], and [12]), among others.

The main object of this paper is to present an integration theory for functions valued in convergence spaces. This let us to give a global treatment for diverse known theories and it is useful for solving the difficulties mentionned before and also in the study of the problems where it appears the duality between the differential (in pseudo-topological spaces) and the integral, mainly related with the solution of differential equations in these spaces.

The integral here has been defined with respect to a non negative and finite measure but it could be extended for non necessarily finite measures following a standard way. Also a similar development could be made starting from measurable functions with countable image instead of simple functions.

2 – Integration in convergence spaces

Let X be complete Husdorff pseudo-topological vector space and (Ω, Σ, μ) a non negative and finite measure space. Let assume that for every filter $\mathcal{F}$ on X which converges to zero (we will write $\mathcal{F} \downarrow 0$) there exists a filter $\mathcal{F}'$ coarser that $\mathcal{F}$, which has a basis formed by absorbing, balanced, convex sets. The countably additive vector measures $m\colon \Sigma \longrightarrow X$, the simple functions and their integrals will be defined in a standard way.

DEFINITION 1. *A non empty subset $A \subset X$ is said to be bounded if $\mathcal{V} \bullet [A] \downarrow 0$, where $\mathcal{V}$ denotes the filter of the zero neighborhoods in* $\mathbb{R}$ *(with the usual topology) and $[A]$ is the filter generated by the filter basis $\{A\}$.*

Clearly, if A is a non empty balanced, convex, bounded subset of X, then the convergence structure defined by the gauge q_A of A, in the linear subspace X_A of X (generated by A) is such that a filter $\mathcal{F} \downarrow 0$ (in (X_A, Q_A)) if and only if it is finer than the filter $\mathcal{V} \bullet [A]_A$ (being $[A]_A$ the trace of the filter $[A]$ on X_A), and every bounded subset of the normed vector space (X_A, q_A) is a bounded subset of the pseudo-topological space X. Then we can state the following:

PROPOSITION 2. *A countably additive measure $m\colon \Sigma \longrightarrow X$ is bounded (i.e. $m(\Sigma)$ is a bounded subset of X) if there exists a balanced,* *convex, bounded subset $A \subset X$ such that $m(\Sigma) \subset X_A$ and (X_A, q_A) is a Banach space sugh that Γ_A is a total subset of the dual space of (X_A, q_A), being Γ_A the family of the restrictions to X_A of the linear and continuous mappings from the convergence space X into $\mathbb{R}$.*

PROOF. It follows from the DIEUDONNÉ-GROTHENDIECK theorem (see the corollary I.3.3, p.16, of [8]) since $x'm$ is a real value countably additive measure, and therefore bounded, for every $x' \in \Gamma_A$.

Remark that the restrictions to X_A of the linear and continuous mappings from the convergence space X into $\mathbb{R}$, are continuous linear forms on (X_A, q_A).

DEFINITION 3. *A filter $\mathcal{S}$ of simple functions is said to be uniformly convergent on $A \in \Sigma$ to a function $f: \omega \longrightarrow X$ if the filter generated by the filter-basis $\{\{f(t) - s(t): t \in A, s \in S\}: S \in \mathcal{S}\}$, is convergent to zero.*

As usual we say that a filter $\mathcal{S}$ of simple functions is *almost uniformly convergent to a function* $f: \Omega \longrightarrow X$ if for every $\varepsilon > 0$ there exists $A \in \Sigma$ such that $\mu(\Omega - A) \leq \varepsilon$ and the filter $\mathcal{S}$ is uniformly convergent to f on A.

PROPOSITION 4. *Let $\mathcal{S}$ be a filter of simple functions which is uniformly convergent on $A \in \Sigma$ to a function $f: \omega \longrightarrow X$, then the filter $I_A(\mathcal{S})$ generated by the filter-basis*

$$\left\{\left\{\int_A s d\mu: s \in S\right\}: S \in \mathcal{S}\right\},$$

is a Cauchy filter on X and therefore it converges to a vector, that we will denote by $\int_A f d\mu$. It is easily proved that the vector $\int_A f d\mu$ is independent of the filter $\mathcal{S}$ of simple functions.

PROOF. Let us suppose that $\mu(A) > 0$ and consider a filter $\mathcal{F} \downarrow 0$ having a basis $\mathcal{B}$ formed by adsorbing, balanced, convex sets such that it is finer than the filter generated by the filter-basis $\{\{f(t) - s(t): t \in A.s \in S\}: S \in \mathcal{S}\}$. Then if $B_1, B_2 \in \mathcal{B}$ there exists $S \in \mathcal{S}$ such that $\{f(t) - s(t): t \in A, s \in S\} \subset B_i$ for $i = 1, 2$, and therefore, if

$$s_i = \sum_{j=1}^{n} x_j^i \chi_{A_j} \in S \quad (i = 1, 2)$$

we have that

$$\int_A s_1 d\mu - \int_a s_2 d\mu = \sum_{j=1}^{n} (x_j^1 - x_j^2)\mu(A_j \cap A) \in \mu(A)(B_1 + B_2),$$

from where it follows immediately that $I_A(\mathcal{S})$ is a Cauchy filter.

DEFINITION 5. *A function $f : \Omega \longrightarrow X$ is said to be μ-measurable if there exists a filter $\mathcal{S}$ of simple functions and a filter (on X) $\mathcal{F} \downarrow 0$ such that $\mathcal{S}$ is almost uniformly convergent to f and for every $F \in \mathcal{F}$ there exists $S \in \mathcal{S}$ verifying that*

$$\left\{ \int_A s d\mu - \int_A f d\mu : s \in S \right\} \subset F$$

for every $A \sin \Sigma$ such that the filter $\mathcal{S}$ is uniformly convergent on A to f.

PROPOSITION 6. *With the notations of the definition 5, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint members of Σ such that $\mu(\Omega - \bigcup_{n\in\mathbb{N}} A_n) = 0$ and $\mathcal{S}$ is uniformly convergent on A_n to f, for every $n \in \mathbb{N}$. Then the filter $I(f)$ generated by the basis*

$$\left\{ \left\{ \int_{\bigcup_{k=1}^{n} A_k} d\mu : n \geq m \right\} : m \in \mathbb{N} \right\},$$

is convergent to a vector of X that will be denoted by $\int_\Omega f d\mu$ and it is independent of the filters $\mathcal{S}$ and $\mathcal{F}$ and also of the sequence $(A_n)_{n\in\mathbb{N}}$.

PROOF. In fact, we can assume that there exists a basis $\mathcal{B}$ of $\mathcal{F}$, formed by absorbing, balanced, convex sets. Then if $B \in \mathcal{B}$ there exists $S \in \mathcal{S}$ such that

$$\int_{\bigcup_{k=n}^{m} A_k} s d\mu - \int_{\bigcup_{k=n}^{m} A_k} f d\mu \in B$$

for every $s \in S$ and $m, n \in \mathbb{N}$ with $n \leq m$. If $s_0 \in S$ there exists $n_0 \in \mathbb{N}$ such that

$$\int_{\bigcup_{k=n}^{m} A_k} s_0 d\mu \in B$$

and

$$\int_{\bigcup_{k=n}^{m} A_k} f d\mu \in B + B$$

for every $m \geq n \geq n_0$, from where it follows that the filter $I(f)$ is a cauchy filter on X and so it converges to a vector $\int_\Omega f d\mu \in X$.

Let $(C_n)_{n\in\mathbb{N}} \subset \Sigma$ another pairwise disjoint sequence such that $\mu(\Omega - \bigcup_{n\in\mathbb{N}} C_n) = 0$ and the filter $\mathcal{S}$ is uniformly convergent on C_n to f for every $n \in \mathbb{N}$. Let us prove that the filter generated by the basis

$$\left\{\left\{ \int_{\substack{\cup(A_k \cap C_p) \\ k=1,\dots,m \\ p=1,\dots,n}} f d\mu \colon m, n \geq r \right\} : r \in \mathbb{N} \right\}$$

is convergent to $\int_\Omega f d\mu$. In fact, let be $B \in \mathcal{B}$ then there exists $s_0 \in S_0 \in \mathcal{S}$ and $r_0 \in \mathbb{N}$ such that if $m, n, p \geq r_0$, $u = \min(m, n)$ and $v = \max(m, n)$ then we have that

$$\int_{\bigcup_{k=1}^{m} A_k} f d\mu - \int_{\substack{\cup(A_i \cap C_j) \\ i=1,\dots,n \\ j=1,\dots,p}} f d\mu \in \int_{\bigcup_{k=1}^{m} A_k} s_0 d\mu - \int_{\substack{\cup(A_i \cap C_j) \\ i=1,\dots,n \\ j=1,\dots,p}} s_0 d\mu + B + B$$

and

$$\int_{\bigcup_{k=1}^{m} A_k} s_0 d\mu - \int_{\substack{\cup(A_i \cap C_j) \\ i=1,\dots,n \\ j=1,\dots,p}} s_0 d\mu \in \int_{\substack{\cup(A_i \cap C_j) \\ k=u,\dots,v \\ j=1,\dots,p}} s_0 d\mu + B \subset B + B,$$

from where the result follows immediately, since the independence of $\mathcal{F}$ is trivial and the independence of $\mathcal{S}$ is a consequence of the proposition 4.

DEFINITION 7. *A μ-measurable function $f\colon \Omega \longrightarrow X$ is said to be μ-integrable if there exists a filter $\mathcal{S}$ of simple functions almost uniformly convergent to f and a filter $\mathcal{F} \downarrow 0$ on X verifying that for every $F \in \mathcal{F}$ there exists $S \in \mathcal{S}$ such that*

$$\int_A s d\mu - \int_A f d\mu \in F \tag{7.1}$$

holds if $s \in S$ and $A \in \Sigma$, where $\int_A f d\mu$ is the vector $\int_\Omega f \chi_A d\mu$ whose existence has been stated in the proposition 6.

THEOREM 8. *Let $f\colon \Omega \longrightarrow X$ be a μ-integrable function and consider the set function $m_f\colon \Sigma \longrightarrow X$ defined by*

$$m_f(A) = \int_A f d\mu \qquad (A \in \Sigma),$$

then the following assertions hold:

8.1. *m_f is a countable additive measure.*

8.2. *m_f is μ-continuous (i.e. the filter on X generated by the basis $\{\{m_f(A)\colon \mu(A) \le \varepsilon\}\colon \varepsilon \in \mathbb{R}^+\}$ is convergent to zero).*

8.3. *Let $g\colon \Omega \longrightarrow X$ be another μ-integrable function and $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is μ-integrable and $m_{\alpha f+\beta g}(A) = \alpha m_f(A) + \beta m_g(A)$ for every $A \in \Sigma$.*

8.4. *Let Y be a complete Hausdorff pseudo-topological space and $T\colon X \longrightarrow Y$ a linear and continuous mapping, then the function $T \circ f$ is μ-integrable and*

$$T\Big(\int_A f d\mu\Big) = \int_A (T \circ f) d\mu$$

holds for every $A \in \Sigma$.

PROOF. 8.1. Let us consider two sequences of pairwise disjoint subsets of X, $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \Sigma$, a filter $\mathcal{S}$ of simple functions which is uniformly convergent on A_n to f, for every $n \in \mathbb{N}$, and a filter $\mathcal{F} \downarrow 0$ on X which has a basis $\mathcal{B}$ formed by absorbing, balanced, convex sets, such that $\mu(\Omega - \bigcup_{n\in\mathbb{N}} A_n) = 0$ and $\mathcal{S}$ and $\mathcal{F}$ verify the conditions of the definition 7. Then for every $B \in \mathcal{B}$ there exists $s_0 \in S_0 \in \mathcal{S}$ and $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ we have that

$$\int_{(\bigcup_{k=1}^{+\infty} B_k)\cap(\bigcup_{i=1}^{n} A_i)} f d\mu - \int_{\bigcup_{k=1}^{m} B_k} f d\mu \in B + \int_{(\bigcup_{k=1}^{+\infty} B_k)\cap(\bigcup_{i=1}^{n} A_i)} s_0 d\mu - \int_{\bigcup_{k=1}^{m} B_k} s_0 d\mu + B$$

and

$$\int_{(\bigcup_{k=1}^{+\infty} B_k)\cap(\bigcup_{i=1}^{n} A_i)} s_0 d\mu - \int_{\bigcup_{k=1}^{m} B_k} s_0 d\mu =$$

$$= \int_{(\bigcup_{k=m+1}^{+\infty} B_k)\cap(\bigcup_{i=1}^{n} A_i)} s_0 d\mu - \int_{(\bigcup_{k=1}^{m} B_k)\cap(\bigcup_{i=n+1}^{+\infty} A_i)} s_0 d\mu \in B + B,$$

from where the result follows easily.

8.2. Let be $\mathcal{S}$ and $\mathcal{F}$ two filters verifying the conditions of the definition 7 and suppose that the filter $\mathcal{F}$ has a basis $\mathcal{B}$ formed by absorbing, balanced, convex subsets of X, then if $B \in \mathcal{B}$ there exists $s_0 \in S_0 \in \mathcal{S}$ and $\varepsilon > 0$ such that

$$\int_A f d\mu = \int_A (f - s_0) d\mu + \int_A s_0 d\mu \in B + B$$

for every $A \in \Sigma$ with $\mu(A) \leq \varepsilon$ and so the result holds.

8.3. and 8.4. are trivial.

DEFINITION 9. *A countably additive measure $m \colon \Sigma \longrightarrow X$ is said to have locally small average range if there exists a filter $\mathcal{F} \downarrow 0$ on X such that for every $A \in \Sigma^+ = \{C \in \Sigma \colon \mu(C) > 0\}$ and $F \in \mathcal{F}$ there exists $A' \in \Sigma_A^+ = \{C \in \Sigma \colon C \subset A, \mu(C) > 0\}$ such that*

$$\frac{m(C)}{\mu(C)} - \frac{m(D)}{\mu(D)} \in F$$

for every $C, D \in \Sigma_{A'}^+$.

THEOREM 10. *If the space $\mathcal{L}(X)$ of the μ-integrable functions $h\colon \Omega \longrightarrow X$, edowed with the following convergence structure: a filter $\mathcal{I}$ on $\mathcal{L}(X)$ is convergent to a function $f \in \mathcal{L}(X)$ if there exists a filter $\mathcal{F} \downarrow 0$ on X such that for every $A \in \Sigma$, the filter generated by the basis $\{\{\int_A (g-f)d\mu\colon g \in I\}\colon I \in \mathcal{I}\}$ is finer than $\mathcal{F}$, is a complete pseudo-topological space then every μ-continuous countably additive measure $m\colon \Sigma \longrightarrow X$ having locally small average range, has a density in $\mathcal{L}(X)$.*

PROOF. Let us consider the filter of simple functions S_π generated by the basis[1]

$$\Big\{\Big\{\sum_{C\in\pi'} \frac{m(C)}{\mu(C)}\chi_C\colon \pi' \geq \pi\Big\}, \pi \in \Pi\Big\}$$

where Π denotes the family of all measurable and finite partitions of Ω ordered by refinement (i.e. $\pi_1 \leq \pi_2$ if every element of π_1 is modulo μ the union of elements of π_2). Let us prove first that S_π is a Cauchy filter. In fact, let $\mathcal{F} \downarrow 0$ be a filter verifying the condition of the definition 9 and which is coarser than the filter generated by the basis $\{\{m(A)\colon A \in \Sigma, \mu(A) \leq \varepsilon\}\colon \varepsilon \in \mathbb{R}^+\}$. We can assume the existence of a basis $\mathcal{B}$ of the filter $\mathcal{F}$, formed by absorbing, balanced, convex subsets of X.

For every $B \in \mathcal{B}$ and $A \in \Sigma^+$ there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint subsets of X belonging to Σ_A^+ such that $\mu(A - \bigcup_{n\in\mathbb{N}} A_n) = 0$ and

$$\frac{m(C)}{\mu(C)} - \frac{m(D)}{\mu(D)} \in B$$

for every $C, D \in \Sigma_{A_n}^+$ and $n \in \mathbb{N}$. Moreover, since the measure m is μ-continuous there exists $n_0 \in \mathbb{N}$ such that $m(C) \in B$ for every $C \in \Sigma_{A_0}$ being $A_0 = \bigcup_{n_0+1}^{+\infty} A_n$. Let be $\pi_0 = \{A_1, \ldots, A_{n_0}, A_0, \Omega - A\}$ and consider $\pi_1, \pi_2 \geq \pi_0$, then if

$$s_i = \sum_{C_i\in\pi_i} \frac{m(C_i)}{\mu(C_i)}\chi_{C_i} \quad (i = 1, 2)$$

[1] We will take $\frac{0}{0} = 0$.

we have that

$$\int_A (s_1 - s_2)d\mu \in \Big(\sum_{k=1}^{n_0} \sum_{\substack{C_1,C_2,\subset A_k \\ C_i\in\pi_i \\ i=1,2}} \Big(\frac{m(C_1)}{\mu(C_1)} - \frac{m(C_2)}{\mu(C_2)}\Big)\mu(C_1\cap C_2)\Big) +$$

$$B + B \subset \sum_{k=1}^{n_0} \mu(A_k)B + B + B + \subset \mu(\Omega)B + B + B\,,$$

from where it follows that $\mathcal{S}_\pi$ is a Cauchy filter in $\mathcal{L}(X)$, and therefore, there exists $f \in \mathcal{L}(X)$ such that $\mathcal{S}_\pi$ converges to f (in $\mathcal{L}(X)$). Then $I_A(\mathcal{S}_\pi)$ converges to $\int_A f d\mu$ and $m(A) = \int_A f d\mu$ for every $A \in \Sigma$, since $I_A(\mathcal{S}_\pi)$ is convergent to $m(A)$ and the limit structure on X is Hausdorff.

REMARK 11. In the case of being X a Banach space if we consider the limit structure on it defined by its topology, then the integral stated here contains the Bochner integral. Moreover, if X is a complete Hausdorff locally convex space and we consider the limit structure defined by its topology, then the integral defined in [13] (see also [12]) is also contained in the integration defined here, since for every μ-integrable function (following [13]) there exists a pirwise disjoint sequence $(K_n)_{n\in\mathbb{N}} \subset \Sigma$ such that $\mu(\Omega - \bigcup_{n\in\mathbb{N}} K_n) = 0$ and for every $n \in \mathbb{N}$ there exists a net $(f_i^n)_{i\in I_n}$ of simple functions (vanishing outside of A_n) which is uniformly convergent to f on A_n and the filter generated by the basis

$$\Big\{\Big\{\sum_{j=1}^{r} f_{i_j}^j : i_j \in I_j, j = 1,\dots,r, r \geq n\Big\} : n \in \mathbb{N}\Big\}$$

verifies all the conditions of the definition 7. The integrals of [7], [11] and [14] are similar to the integral of [13].

Moreover, if X is a complete Husdorff convex bornological space, then the bornological integral defined in [6] is also contained in this one if we consider on X the Mackey limit structure associated to the bornology of the space (i.e. $\mathcal{F} \downarrow 0$ on X iff there is a bounded, balanced and convex subset $B \subset X$ such that the filter on X generated by the basis formed by the zero neighborhoods in (X_B, q_B) is coarser than $\mathcal{F}$).

REFERENCES

[1] Mª E. BALLVÉ: *On the stability of the completeness of L^α spaces*, Rend. C. Mat. Palermo, Ser. II, 39(1990), 58-64.

[2] Mª E. BALLVÉ - J.L. DE MARIA: *On the dual of L^α for locally convex spaces*, Atti Sem. Mat. Fis. Univ. Modena, 38(1990), 165-169.

[3] E. BINZ: *Continuous convergence on $C(X)$*, Lect. Notes in Math., 469. Springer-Verlag, Berlin, 1975.

[4] C. BLONDIA: *Integration in locally convex spaces*, Simon Stevin, 55(1981), 81-102.

[5] C. BLONDIA: *On the Radon-Nikodym property in locally convex spaces and the completeness of L^1_E*, Rev. R. Acad. Ci. Madrid, 81(1987), 635-647.

[6] F. BOMBAL: *Medida e integración en espacios bornológicos*, Rev. R. Acad. Ci. Madrid, 75(1981), 115-137.

[7] G.Y.H. CHI: *On the Radon-Nikodym theorem in locally convex spaces*, Lect. Notes in Math., 541, Springer-Verlag, Berlin, 1976.

[8] J. DIESTEL - J. UHL: *Vector measures*, Math. Surveys, 15. Amer. Math. Soc., Providence, R.I., 1977.

[9] H.R. FISCHER: *Limesraüme*, Math. Ann., 137(1959), 169-303.

[10] A. FRÖLICHER - W. BUCHER: *Calculus in vector spaces without norm*, Lect. Notes in Math., 30. Springer-Verlag, Berlin, (1966).

[11] D. GILLIAN: *On integration and the Radon-Nikodym theorem in quasi-complete locally convex spaces*, J. Reine Angew. Math., 292(1977), 125-137.

[12] P. JIMÉNEZ GUERRA - B. RODRIGUEZ-SALINAS: *On the completeness of L^α for locally convex spaces*, Arch. Math., 52(1989), 82-91.

[13] B. RODRIGUEZ-SALINAS: *Integración de funciones con valores en un espacio localmente convexo*, Rev. R. Acad. Ci. Madrid, 73(1979), 361-387.

[14] G.E.F. THOMAS: *The Lebsgue-Nikodym theorem for vector valued Radon measures*, Mem. Amer. Math. Soc., 139, (1974).

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