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Free G-spaces, principal G-fibrations and maps between classifying spaces

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RIASSUNTO – *Sia G un gruppo topologico, M uno spazio ed $H(M)$ il monoide di composizione delle auto equivalenze omotopiche di M. Si dimostra che se G ha lo stesso tipo omotopico di un complesso CW, allora l'insieme delle classi di equivalenza G-omotopica di un qualunque G-spazio libero, che sia omotopicamente equivalente a M, è classificato dall'insieme delle classi omotopiche libere $[B_G, B_{H(M)}]$. Generalizzando al caso in cui G è un monoide topologico provvisto di inverso omotopico libero, si ottiene una classificazione simile per G-fibrazioni principali. Sono esposti alcuni esempi basilari.*

ABSTRACT – *Let G be a topological group, M be a given space and $H(M)$ the monoid under composition of self-homotopy equivalences of M. We show that if G has the homotopy type of CW-complex then the set of G-homotopy equivalence classes of free G-spaces that are homotopy equivalent to M is classified by the set of free homotopy classes $[B_G, B_{H(M)}]$. Generalizing to the case where G is a topological monoid with a free homotopy inverse, we obtain a similar classification result for principal G-fibrations. Some basic examples are given.*

KEY WORDS – *Classifying spaces - G-spaces - principal fibrations.*

A.M.S. CLASSIFICATION: 55R15 - 55R35

1 – Basic and background ideas

In this paper we refer to classification questions for both principal G-bundles and principal G-fibrations. In the former case we assume that G is a topological group, in the latter that it is a *topological monoid with* *a free homotopy inverse*, i.e. that there exists a map $u: G \longrightarrow G$ with the property that the map $g \longrightarrow g.u(g)$ of G into G is freely homotopic to the constant map that has value the identity of G.

(1.1) In either case there exists a classifying space B_G and a universal object $p_G: E_G \longrightarrow B_G$, i.e. a principal G-bundle or principal G-fibration, as the case may be, with the property that E_G is contractible (see [7:thm. 7.5]and [15:p.333] respectively). We most frequently refer to the fibration case; to avoid any confusion the bundle case construction [29] will be denoted by $p_G^b: E_G^b \longrightarrow B_G^b$.

In classifying principal G-bundles or principal G-fibrations the usual procedure is to fix G and the base space B. Then:

(1.2) pulling p_G^b back over maps $B \longrightarrow B_G^b$ induces principal G-bundles over B, thereby classifying principal G-bundles over B up to G-isomorphism by means of the set $[B, B_G^b]$ (see [29] and [20:section 12 of ch.4]). Pulling p_G back over maps $B \longrightarrow B_G$ induces principal G-fibrations over B, thereby classifying principal G-fibrations over B up to G-fibre homotopy equivalence ($=G-FHE$, see [15:section 5] and also [35] and [26:cor. 9.4]).

An alternative view, and the one adopted here, is to fix G and the homotopy types of the G-spaces under consideration (= the total spaces of the principal bundles or fibrations involved), but at the same time allow B to vary. Recalling that a G-space X is *free* if the projecton $X \longrightarrow X/G$ can be given the structure of a principal G-bundle, we will now make some elementary observations concerning this second view of the classification question.

If G is a topological group and M is any given space then:

(1.3) there exists at least one free G-space homotopy equivalent to M, and

(1.4) in general there will be more than one homotopy type of free G-spaces homotopy equivalent to M.

The former statement can be verified by considering the principal G-bundle $p_G^b \times 1_M: E_G^b \times M \longrightarrow B_G^b \times M$.

In the latter case we may consider any example where G is not contractible and M is the underlying space of G. Then the free G-space $E_G^b \times G$ (with the usual action on E_G^b and the trivial action on G) and G (with the usual action) have the same homotopy type, yet cannot have the same G-homotopy type. For if there were a G-homotopy equivalence from $E_G^b \times G$ to G it would induce a homotopy equivalence between the quotient spaces $E_G^b \times G/G$ and G/G, i.e. $B_G \times G$ and a singleton space.

Remarks similar to (1.3) and (1.4) apply to principal G-fibrations.

One paper in which this second approach has been followed is [9].

Let V and W be simplicial sets on which a given simplicial group G acts. A simplicial map $V \longrightarrow W$ will be a called a *G-weak homotopy equivalence* if it is compatible with the action of G and is a weak homotopy equivalence.

If M is a given simplicial set then part (i) of theorem 1.2 of [9] classifies those G-weak homotopy equivalence classes of simplicial sets on which G acts that are also weak homotopy equivalent to M. The object that is used to model this set is a set of simplicial homotopy classes between simplicial classifying complexes, i.e. $[\overline{W}_G, \overline{W}_{sgaM}]$, where $sgaM$ denotes the simplicial group of automorphisms of M. Further it is stated that each equivalence class of simplicial sets contains simplicial sets which satisfy the extension condition and on which the G-action is free.

The reader can find basic definitions for the above simplicial concepts in [25]. The result [9:thm.1.2(i)] is derived by restricting an equivalence of simplicial homotopy theories [9:section 2], i.e. between G-equivariant homotopy theory and that of fibrations over the corresponding classifying complex $\overline{W}_G$. The proof of this last result is rooted in the theory of model categories of [11] and [32].

A somewhat analagous equivalence of homotopy theories is discussed in [17] and [18]. In these cases the equivariant theory used is that of G-spaces and G_∞-maps, the latter concept being as developed in [16]. The related theory is that of the associated Borel fibrations over classifying spaces for G.

A further equivalence of homotopy theories is described in [4], this time between theories for fibrations with additional structure and for Dold fibrations over the appropriate classifying spaces. Principal bundles and principal fibrations are two of the examples discussed.

Let $\mathbf{W}$ denote the class of all space having the homotopy types of CW-complexes. The classification results for free G-spaces and principal G-fibrations that could be derived from this last equivalence of homotopy theories, i.e. our theorems 4.8 and 3.6 respectively, refer to G-spaces that have the homotopy type of a given space M and require $G \in \mathbf{W}$. The classification is then by means of the sets $[B_G^b, B_{H(M)}]$ and $[B_G, B_{H(M)}]$

respectively.

These results of this paper, and their proofs, differ from thm.1.2(i) of [9] in several ways.

(i) We work in a topological rather than a simplicial context, giving separate results for the principal bundle and principal fibration cases.

(ii) No use in made of weak homotopy equivalences or G-weak homotopy equivalences (as in [9]); we choose to give an expanded argument (using definition 3.1 and proposition 3.2) that allows us to work with appropriate types of homotopy equivalence.

(iii) We do not deduce our results as consequences of an equivalence of homotopy theories, as this would present them as being dependent on the relatively complex fibred mapping space machinery of [4]. Instead we prefer to proceed in a simpler more direct fashion. Actually the proof of the main theorem (2.1) of [4] is not complete as given; it quotes and depends on a direct generalization of proposition 3.2 of this paper (numbered 1.9 in [4]).

In recent years there has been much interest in the topic of maps out of classifying spaces, a notable result being MILLER'S proof [28] of the Sullivan Conjecture [37: p. 5.118]. In particular there have been numerous papers concerning maps between the classifying spaces of topological groups [1],[2],[12],[13],[14],[19],[21],[22],[23],[24],[27],[30],[31],[43],[44] and [45]. We have thus clarified a link between a natural extension of this last topic and that of the classification of principal bundles and fibrations.

We present our main line of argument (sections 2 and 3) in terms of principal G-fibrations. The bundle version is given in section 4. Finally some basic examples are discussed in section 5.

Our work is in the context of the category of *compactly generated spaces (=cg-spaces)*, i.e. spaces having the weak topology relative to all incoming maps from compact Hausdorff spaces [40:example (ii) of section 5]. Known results can, of course, be *cg-ified* by *cg-ifying* the spaces involved (i.e. by giving them this weak topology).

If M is a space then $H(M)$ will denote the space of self-homotopy equivalences of M, equipped with the (cg-ification of the) compact-open topology. It follows from the exponential law [40:thm. 3.6] that $H(M)$, with the binary operation of composition, is a topological monoid.

2 – Principal G-fibrations

Let us assume, throughout this and the following section, that G is a topological monoid with a free homotopy inverse.

DEFINITION 2.1. *From [15:p.329]. A principal G-fibration consists of a pair (q, a), q being a map $Y \longrightarrow B$ and $a: Y \times G \longrightarrow Y$ being a right action of G on Y such that:*

(i) $q \circ a = q \circ \pi$, where $\pi: Y \times G \longrightarrow Y$ denotes the projection, and

(ii) there exists a numerable cover $\mathbf{U}$ of B such that for each $U \in \mathbf{U}$ $q|q^{-1}(U): q^{-1}(U) \longrightarrow U$ is $G-FHE$ to the projection $G \times U \longrightarrow U$. In the case of this projection, G is assumed to act on $G \times U$ from the right in the obvious fashion.

DEFINITIONS 2.2. *If $q: Y \longrightarrow B$ is a principal G-fibration and $f: A \longrightarrow B$ is a map then f^*Y will denote the fibred product or pullback space with underlying set $\{(y, a) \in Y \times A | q(y) = f(a)\}$; then the projection $f^*q: f^*Y \longrightarrow A$ is an induced principal G-fibration. The projection $f^*Y \longrightarrow Y$ will be denoted by q^*f.*

LEMMA 2.3. *If $G \in \mathbf{W}$ then $B_G \in \mathbf{W}$.*

The following proof is a rephrased version of some comments that were included in a 1981 letter from M. Fuchs to the author.

PROOF. Let B_G^{DL} denote the Dold-Lashof classifying space for G [8:p.293]. Then B_G has the homotopy type of B_G^{DL} [15:p.335] and B_G^{DL} is constructed out of G by taking products (in the compactly generated topology), mapping cylinders, mapping cones and direct limits. Further the category of spaces having the homotopy type of a CW-complex is closed under these operations so $B_G^{DL} \in \mathbf{W}$ and hence $B_G \in \mathbf{W}$.

DEFINITION 2.4. *If $p: X \longrightarrow A$ and $q: Y \longrightarrow B$ are principal G-fibrations then a G-pairwise map $\langle f, g \rangle$ from p to q consists of a G-map $f: X \longrightarrow Y$ and a map $g: A \longrightarrow B$ such that $q \circ f = g \circ p$. We notice that if $A = B$ and $g = 1_B$ then such an f is just a G-map over B.*

The $\langle f, g\rangle$ notation should be distinguished from the following more or less standard notation that is also used later: if $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Z$ are G-maps then $(f,g)\colon X \longrightarrow Y \times Z$ denotes the G-map $(f,g)(x) = (f(x), g(x))$, where $x \in X$.

In the following definitions it is assumed that I carries the trivial G-structure.

DEFINITION 2.5. *Let us assume that Y and B are G-spaces and that $q\colon Y \longrightarrow B$ is a G-map. If for all choices of a G-space W, a G-map $h\colon W \longrightarrow Y$ and a G-homotopy $H\colon W \times I \longrightarrow B$ such that $qh(w) = H(w,0)$ for all $w \in W$, there exists a G-homotopy $K\colon W \times I \longrightarrow Y$ with $K(w,0) = h(w)$ for all $w \in W$ and such that $q \circ K = H$, then q will be said to satisfy the G-covering homotopy property $(G - CHP)$.*

If this property holds in the cases of all homotopies H that are stationary on $[0, \frac{1}{2}]$, i.e. such that $H(w,t) = H(w,0)$, for all $w \in \mathbf{W}$ and $t \in [0, \frac{1}{2}]$, then q will be said to satisfy the G-weak covering homotopy property $(G - WCHP)$.

THEOREM 2.6. *If $q\colon Y \longrightarrow B$ is a principal G-fibration then it satisfies the $G - WCHP$.*

PROOF. Viewing B as a G-space under the trivial action we see that q is a G-map. It follows from [7:prop.5.2 and thm.5.12] that if $p\colon X \longrightarrow A$ is locally fibre homotopy trivial, relative to a numerable cover of A, then p satisfies the $WCHP$ [7:p.238]. Applying the same argument to the category of G-spaces and noticing that the spaces and maps that occur in the proofs of the G-analogues of the above results are G-spaces and G-maps respectively, we obtain the required result.

REMARK 2.7. A G-covering homotopy theorem for principal G-bundles is proved in [7:thm.7.8]. This depends on knowing that if $p\colon X \longrightarrow A$ and $q\colon Y \longrightarrow B$ are principal G-bundles and $f\colon A \longrightarrow B$ a map then a functional bundle (p, q, f) is a Hurewicz fibration. There is a corresponding proof of a version of our 2.6 using a functional fibration (pf^*q), in the (pq) notation of [3:section 1] and [5:def.7.2], using [7:thm. 5.12] to prove that this satisfies the $WCHP$. However that argument requires B to be weak Hausdorff, which then complicates the use of such a version of

2.6 in the proof of our proposition 3.2 and lemma 3.4 and hence in that for theorem 3.6, and thereby seems to force an extra assumption in that last result (e.g. $M \in \mathbf{W}$).

3 – Classifying principal G-fibrations

DEFINITION 3.1. *If $\langle f_0, g_0 \rangle$ and $\langle f_1, g_1 \rangle$ are G-pairwise maps from $p\colon X \longrightarrow A$ to $q\colon Y \longrightarrow B$ then a G-pairwise homotopy from $\langle f_0, g_0 \rangle$ to $\langle f_1, g_1 \rangle$ is a G-pairwise map from $p \times 1_I\colon X \times I \longrightarrow A \times I$ to q that restricts to $\langle f_0, g_0 \rangle$ and $\langle f_1, g_1 \rangle$ in the expected fashion. The concept of G-pairwise homotopy equivalence ($G - PHE$) can now be defined in the obvious manner. The reader will notice that taking $A = B$ and considering only the identity map from B to B, we retrieve the concepts of G-homotopy over B and $G - FHE$.*

PROPOSITION 3.2. *If $q\colon Y \longrightarrow B$ is a principal G-fibration and $f\colon A \longrightarrow B$ is a homotopy equivalence then the principal G-fibration $f^*q\colon f^*Y \longrightarrow A$ is $G - PHE$ to q.*

Our proof requires a preliminary definition and lemma.

DEFINITION 3.3. *If $p\colon X \longrightarrow A \times I$ and $q\colon Y \longrightarrow A \times I$ are principal G-fibrations and $f_i\colon p^{-1}(A \times \{i\}) \longrightarrow q^{-1}(A \times \{i\})$ are G-maps over $A \times \{i\}$, for i = both 0 and 1, then a G-map $F\colon X \longrightarrow Y$ over $A \times I$ that restricts to f_0 and f_1 will be called a G-translation of f_0 into f_1, from X to Y, and denoted by $f_0 \sim f_1$.*

It will be convenient, in figure 3, to represent such a deformation in the following fashion.

$$
\begin{array}{ccc}
p^{-1}(A \times \{0\}) & \xrightarrow[f_0]{\qquad\qquad} & q^{-1}(A \times \{0\}) \\
X & F & Y \\
p^{-1}(A \times \{1\}) & \xrightarrow[f_1]{\qquad\qquad} & q^{-1}(A \times \{1\})
\end{array}
$$

Fig. 1

LEMMA 3.4. *Let $q\colon Y \longrightarrow B$ be a principal G-fibration.*

(i) If $L\colon A \times I \longrightarrow B$ is a homotopy from u to v then there are G-maps $\alpha\colon u^*Y \longrightarrow u^*Y$, $\beta\colon v^*Y \longrightarrow u^*Y$ and $\gamma\colon v^*Y \longrightarrow v^*Y$, all over A, and G-translations:

(a) $\Phi_1\colon 1_{u*Y} \sim \alpha$, from $(u^*Y) \times I$ to $(u^*Y) \times I$,

(b) $\Phi_2\colon \alpha \sim \beta$, from L^*Y to $(u^*Y) \times I$,

(c) $\Phi_3\colon \beta \sim \gamma$, from $(v^*Y) \times I$ to L^*Y, and

(d) $\Phi_4\colon \gamma \sim 1_{v^*Y}$, from $(v^*Y) \times I$ to $(v^*Y) \times I$.

(ii) Given a second homotopy $M\colon A \times I \longrightarrow B$, from u to v, such that $L \simeq M$ relative to their common end points u and v, then there are G-maps $\lambda\colon u^*Y \longrightarrow u^*Y$ and $\mu\colon v^*Y \longrightarrow v^*Y$, both over A, and G-translations:

(a) $\Psi_1\colon 1_{u^*Y} \sim \lambda$, from $(u^*Y) \times I$ to $(u^*Y) \times I$,

(b) $\Psi_2\colon \lambda \sim \mu$, from L^*Y to M^*Y, and

(c) $\Psi_3\colon \mu \sim 1_{v^*Y}$, from $(v^*Y) \times I$ to $(v^*Y) \times I$.

PROOF. (i) We define $\hat{L}\colon A \times I \times I \longrightarrow B$ by $\hat{L}(a,s,t) = L(a,s)$ for $t \leq \frac{1}{2}$, $= L(a, s+1-2t)$ for $\frac{1}{2} \leq t \leq \frac{1}{2}(s+1)$, and $L(a,0)$ $t \geq \frac{1}{2}(s+1)$, with $a \in A$ and $s,t \in I$. Identifying $(L^*Y) \times \{0\}$ with L^*Y we have $q \circ (q^*L) = \hat{L} \circ ((L^*q) \times 1_I)|(L^*Y) \times \{0\})$ and the $G-WCHP$ allows us to obtain $L'\colon (L^*Y) \times I \longrightarrow Y$ such that $\langle L', \hat{L}\rangle$ is a G-pairwise map from $(L^*q) \times 1_I$ to q extending $\langle q^*L, L\rangle$. Pulling q back over $\hat{L}, L'$ determines a G-map $L''\colon L^*Y \times I \longrightarrow (\hat{L})^*Y$ over $A \times I \times I$ by $L''(y,a,s,t) = (L'(y,a,s,t),a,s,t)$, where $(y,a,s) \in L^*Y(\subset Y \times A \times I)$ and $t \in I$; restricting L'' to the subspaces of $(L^*Y) \times I$ that are the inverse images under $(L^*q) \times I$ of the subspaces $A \times \{(0,1)\}$, $A \times \{(1,1)\}$, $A \times \{(1,\frac{1}{2})\}$, $A \times \{0\} \times I$ and $A \times I \times \{1\}$ yields the maps α, β and γ and the translations Φ_1 and Φ_2, respectively. The restrictions of L'' over $A \times \{1\} \times [\frac{1}{2}, 1]$ and $A \times \{1\} \times [0, \frac{1}{2}]$, modified using the identifications, $[\frac{1}{2}, 1] \longrightarrow [0,1]$ by $t \longrightarrow 2t-1$ and $[0,\frac{1}{2}] \longrightarrow [0,1]$ by $t \longrightarrow 2t$, yield the G-translations Φ_3 and Φ_4 respectively.

(ii) This follows by a similar argument, using the relative homotopy $L \simeq M$ in place of $\hat{L}$.

In proving the above proposition we use a homotopy inverse $g\colon B \longrightarrow A$ of f; the reader may wish to refer to the pullback diagram of figure 2.

$$\begin{array}{ccccccc}
f^*g^*f^*Y & \xrightarrow[(g^*f^*q)^*f]{} & g^*f^*Y & \xrightarrow[(f^*q)^*g]{} & f^*Y & \xrightarrow[q^*f]{} & Y \\
\Big\downarrow{\scriptstyle f^*g^*f^*q} & & \Big\downarrow{\scriptstyle g^*f^*q} & & \Big\downarrow{\scriptstyle f^*q} & & \Big\downarrow{\scriptstyle q} \\
A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & A & \xrightarrow[f]{} & B
\end{array}$$

Fig. 2

The canonical identifications $(1_B)^*Y = Y$, $(1_B)^*q = q$, $q^*(1_B) = 1_Y$, $(f \circ g)^*Y = g^*f^*Y$, $(f \circ g)^*q = g^*f^*q$, $q^*(f \circ g) = (q^*f) \circ ((f^*q)^*q)$ and $(f^*q)^*(g \circ f) = ((f^*q)^*g) \circ ((g^*f^*q)^*f)$ will be made in the course of the proof.

PROOF OF 3.2. Let $H\colon f \circ g \simeq 1_B$ and $K\colon g \circ f \simeq 1_A$ be homotopies chosen in such a way that there is a homotopy $W\colon f \circ K \simeq H \circ (f \times 1_I)$ relative to their end points $f \circ g \circ f$ and f (see [41]).

Defining $\beta = \beta(H)\colon Y \longrightarrow g^*f^*Y$ as in 3.4 (i) and $\omega = ((f^*q)^*g) \circ \beta(H)\colon Y \longrightarrow f^*Y$ we will show that the G-pairwise map $\langle \omega, g\rangle\colon q \longrightarrow f^*q$ is a G-pairwise homotopy inverse to $\langle q^*f, f\rangle\colon f^*q \longrightarrow q$.

First we notice that:

$$\begin{aligned}
&\langle q^*f, f\rangle \circ \langle \omega, g\rangle = \langle (q^*f) \circ (((f^*q)^*g) \circ \beta(H)), f \circ g\rangle \\
&= \langle ((q^*f) \circ ((f^*q)^*g)) \circ \beta(H), f \circ g\rangle = \langle (q^*(f \circ g) \circ \beta(H), f \circ g\rangle \\
&\simeq \gamma \text{ (using the } G\text{-homotopy } (q^*H) \circ \Phi_3(H)) \\
&\simeq 1_Y \text{ (via the } G\text{-homotopy } \Phi_4(H)).
\end{aligned}$$

The proof that $\langle \omega, g\rangle \circ \langle q^*f, f\rangle \simeq \langle 1_{f^*Y}, 1_A\rangle$ requires a more complicated G-pairwise homotopy, as illustrated in figure 3. The idea behind this method of illustration is to allow the reader to picture the way the various G-translation "pieces" fit together in a jigsaw puzzle type fashion. The conventions to be followed in interpreting this diagram are as follows:

(i) when maps (including translations) are adjacent, one to the left of the other, they should be undestood as being composed, e.g. the eighth line of figure 3 refers to the homotopy

$$((f^*q)^*K, \pi_I) \circ \Phi_3(f \circ K)\colon (f^*Y) \times I \longrightarrow (f^*Y) \times I,$$

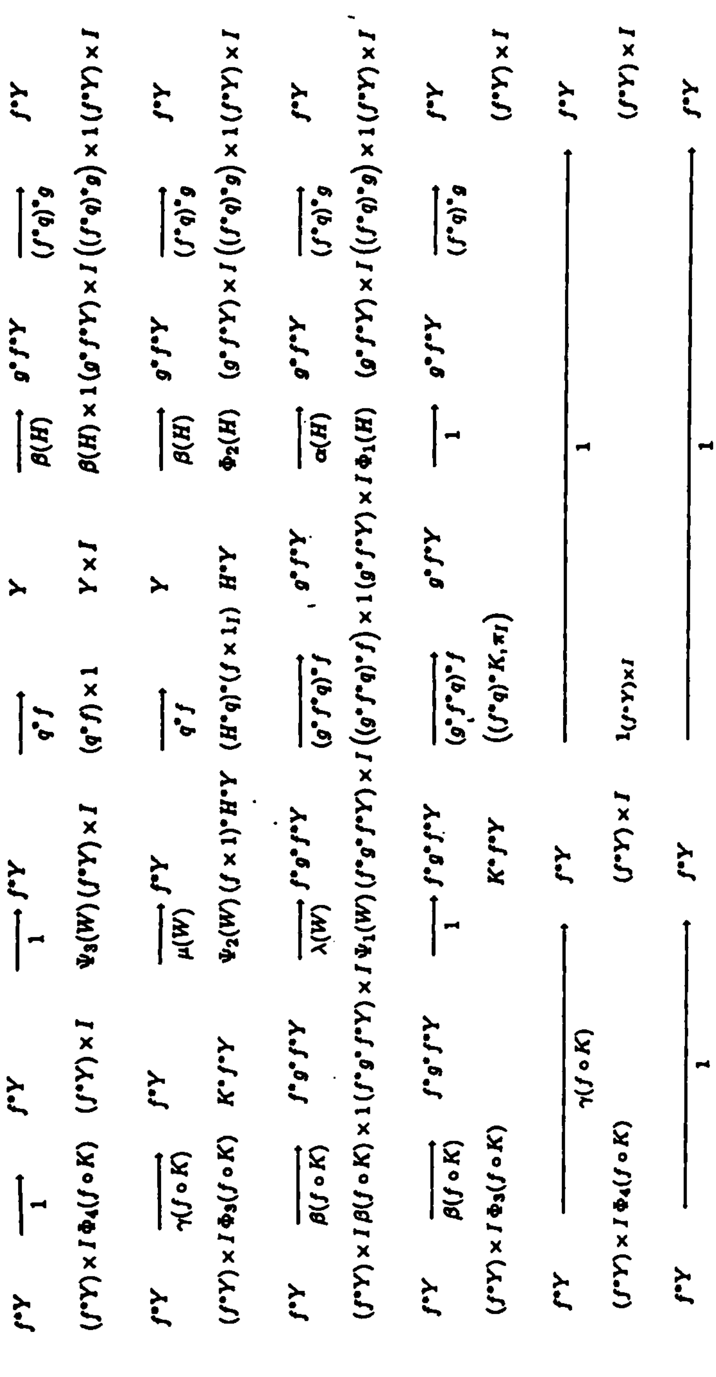

Fig. 3

where $\pi_I: K^*f^*Y \longrightarrow I$ is the projection and so $((f^*q)^*K, \pi_I): K^*f^*Y \longrightarrow (f^*Y) \times I$;

(ii) when homotopies or translations are adjacent, one above the other, it should be understood that they are added, i.e. that the "end" of one homotopy or translation is to be considered as being attached to the "beginning" of the other; and

(iii) the compound homotopy illustrated, composed with the projection into f^*Y, determines the required homotopy

$$f^*Y \times I \longrightarrow f^*Y.$$

(3.5) If M is a given space then there exists a Hurewicz fibrations $p_\infty: E_\infty \longrightarrow B_{H(M)}$, with fibres homotopy equivalent to M, that is universal in the following sense.

If $B \in \mathbf{W}$ then pulling p_∞ back over maps $B \longrightarrow B_{H(M)}$ determines a bijection:

$$[B, B_{H(M)}] \longrightarrow FHE(M: B),$$

where $FHE(M: B)$ denotes the set of fibre homotopy equivalence (= FHE) classes of Hurewicz fibrations over B with fibres homotopy equivalent to M (see [26:cor,9.5] or [33.thm.2]). We mention that in the literature $B_{H(M)}$ is frequently denoted by B_∞.

THEOREM 3.6. *Let $G \in \mathbf{W}$ be a topological monoid with a free homotopy inverse, M be a given space and $PHE(G: M)$ denote the set of $G-PHE$ classes of principal G-fibrations $q: Y \longrightarrow B$, where B ranges over the class of all (of course cg-) spaces, but Y is required to be homotopy equivalent to M.*

(i) Then there are bijections Q_1 and Q_2:

$$[B_G, B_{H(M)}] \xrightarrow[Q_1]{} FHE(M: B_G) \xrightarrow[Q_2]{} PHE(G: M),$$

where Q_1 is as described in 3.5 and Q_2 defined as follows. Let $p: X \longrightarrow B_G$ denote a Hurewicz fibration with fibres homotopy equivalent to M and $[p]$ denote the FHE class of p. Then we define $Q_2([p])$ to be the $G-PHE$ class of the principal G-fibration $p^*(p_G)$.

(ii) Combining these steps the bijection

$$Q(= Q_2 \circ Q_1) : [B_G, B_{H(M)}] \longrightarrow PHE(G\colon M)$$

is defined by the following procedure. Any map $f\colon B_G \longrightarrow B_{H(M)}$ enables us to construct a double pullback diagram as illustrated in figure 4.

$$\begin{array}{ccccc} p^*(E_G) & \xrightarrow[p^*(p_G)]{} & f^*(E_\infty) & \xrightarrow[p^*_\infty(f)]{} & E_\infty \\ \Big\downarrow{\scriptstyle (p_G)^*p} & & \Big\downarrow{\scriptstyle p = f^*(p_\infty)} & & \Big\downarrow{\scriptstyle p_\infty} \\ E_G & \xrightarrow[p_G]{} & B_G & \xrightarrow[f]{} & B_{H(M)} \end{array}$$

Fig. 4

Thus f induces the Hurewicz fibration $p = f^*(p_\infty)$ and this, in turn, induces the principal G-fibration $p^*(p_G)\colon p^*(E_G) \longrightarrow f^*(E_\infty)$. Then Q is the rule that takes the homotopy class of f to the $G - PHE$ class of $p^*(p_G)$.

PROOF. We just have to verify that Q_2 is a bijection. Taking $p\colon X \longrightarrow B_G$ to be a Hurewicz fibration with fibres of the homotopy type of M, then the fibration $(p_G)^*p\colon p^*(E_G) \longrightarrow E_G$ has fibres homotopy equivalent to M and a contractible base space (1.1), so $p^*(E_G)$ has the homotopy type of M [39:thm.1.3].

Further if $q\colon Y \longrightarrow B_G$ is a Hurewicz fibration and there is an FHE $h\colon X \longrightarrow Y$ then $p = q \circ h$ and $p^*(p_G) = (q \circ h)^*(p_G) = h^*q^*(p_G)$ which is $G - PHE$ (3.2) to $q^*(p_G)$. Hence we have shown that Q_2 is well defined.

Given a principal G-fibration $r\colon Z \longrightarrow B$ with Z homotopy equivalent to M, let $g\colon B \longrightarrow B_G$ be a map such that r is $G - FHE$ to $g^*(p_G)$ (see 1.2). Factoring g as the composite of a homotopy equivalence $B \longrightarrow \overline{B}$ and a fibration $\bar{g}\colon \overline{B} \longrightarrow B_G$ [34:2.8.9] it follows from 3.2 that r is $G - PHE$ to $\bar{g}^*(p_G)$. Hence we know that M, Z and $\bar{g}^*(E_G)$ all have the same homotopy type. Now the fibres of $\bar{g}$ have the homotopy types of the fibres of $(p_G)^*\bar{g}$ and, since $(p_G)^*\bar{g}\colon \bar{g}^*(E_G) \longrightarrow E_G$ is a fibration with contractible base space (1.1), the homotopy type of $\bar{g}^*(E_G)$ and of M. Hence $Q_2([\bar{g}]) = [r] \in PHE(G\colon M)$ and so Q_2 is surjective.

Let us assume that we are given Hurewicz fibrations $p\colon X \to B_G$ and $q\colon Y \longrightarrow B_G$; further we will suppose that there is a $G-PHE$ $\langle u, v\rangle$ from

$p^*(p_G)$ to $q^*(p_G)$. Using the universal property of pullbacks we see that u factors through $v^*q^*(E_G)$, giving the commutative diagram of figure 5.

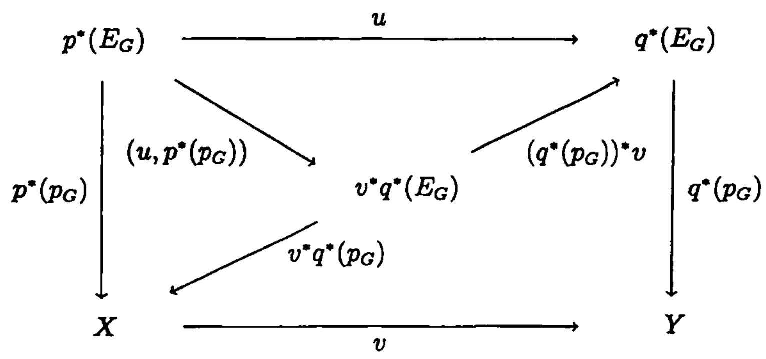

Fig. 5

Now u and $(q^*(p_G))^*v$ are G-homotopy equivalences, by the data and 3.2 respectively, hence $(u, p^*(p_G))\colon p^*(E_G) \longrightarrow v^*q^*(E_G)$ is a G-homotopy equivalence. It follows from the G-version of [7:thm.6.1] that this last map is a $G - FHE$ from $p^*(p_G)$ to $v^*q^*(p_G) = (q \circ v)^*(p_G)$. Hence $p \simeq q \circ v$ (see 1.2). Now q has the covering homotopy property so there is a map $w\colon X \longrightarrow Y$ that is homotopic to the homotopy equivalence v with $p = q \circ w$; hence w is a homotopy equivalence and, by [7:thm.6.1], a FHE between p and q. Hence Q_2 is injective.

4 – Classifying principal G-bundles

An argument similar to that of section 2 and 3 produces a classification theorem for (of course numerable) principal G-bundles and hence for free G-spaces. The modifications to the principal fibration argument are as described below.

(4.1) We assume that G is a topological group.

(4.2) The universal principal bundle $p_G^b\colon E_G^b \longrightarrow B_G^b$ will be used in place of $p_G\colon E_G \longrightarrow B_G$ (see (1.1) and (1.2)).

LEMMA 4.3. *If $G \in \mathbf{W}$ then $B_G^b \in \mathbf{W}$.*

PROOF. It follows from [6:thm.3.2 and example 3] that $p_G^b \colon E_G^b \longrightarrow B_G^b$ is universal amongst principal G-fibrations (as well as amongst principal G-bundles) in the sense of 1.2.

Thus p_G and p_G^b both posses the same universal property relative to principal G-fibrations. Let $f \colon B_G \longrightarrow B_G^b$ and $g \colon B_G^b \longrightarrow B_G$ be representatives of the homotopy classes of maps that induce p_G and p_G^b to within $G-FHE$, from p_G^b and p_G respectively. Then $f \circ g$ and $g \circ f$ induce p_G^b and p_G from p_G^b and p_G respectively, again to within $G-FHE$. It follows from 1.2 that $f \circ g$ and $g \circ f$ are homotopic to the appropriate identity maps. Hence B_G^b is homotopy equivalent to B_G and, by 2.3, $B_G^b \in \mathbf{W}$.

(4.4) If M is a given space then $Fr(G \colon M)$ will denote the set of all G-homotopy types of free G-spaces that have the homotopy type of M.

(4.5) The $G-CHP$ theorem that is proved for principal G-bundles in [7], i.e. theorem 7.8, can take the place of our theorem 2.6.

(4.6) The various G-pairwise concepts simplify in the bundle case. If $p \colon X \longrightarrow A$ and $q \colon Y \longrightarrow B$ are principal G-bundles and $f \colon X \longrightarrow Y$ is a G-map then there is a function $g \colon A \longrightarrow B$ determined by the rule $g(a) = qf(x)$, where $a \in A$ and $x \in \{q^{-1}(a)\}$. Now q is an identification so it follows that g is continuous and we have the following result.

LEMMA 4.7. *If X and Y are free G-spaces then there are bijective correspondences between:*

(i) the set of G-maps $f \colon X \longrightarrow Y$ and the set of G-pairwise maps $\langle f, g \rangle$, from $p \colon X \longrightarrow X/G$, $p(x) = xG$ where $x \in X$, to $q \colon Y \longrightarrow Y/G$, $q(y) = yG$, $y \in Y$; and also between:

(ii) the set of G-homotopies $F \colon X \times I \longrightarrow Y$ and the set of G-pairwise homotopies $\langle F, H \rangle$ from $p \times 1_I$ to q.

Further (iii) X and Y are G-homotopy equivalent if and only if p and q are G-pairwise homotopy equivalent.

(iv) $Fr(G \colon M) = Pws(G \colon M)$.

Repeating the argument of section 3, but for principal G-bundles and with 4.1...4.7 incorporated, we obtain the following result.

THEOREM 4.8. *Let G be a topological group in $\mathbf{W}$ and M be a given space. Then there are bijections:*

$$[B_G^b, B_{H(M)}] \xrightarrow[R_1]{} FHE(M \colon B_G^b) \xrightarrow[R_2]{} Fr(G \colon M).$$

In particular the bijection

$$R = R_2 \circ R_1 \colon [B_G^b, B_{H(M)}] \longrightarrow Fr(G\colon M)$$

is defined, for any choice of a map $f\colon B_G^b \longrightarrow B_{H(M)}$, by taking $R([f])$ to be the G-homotopy class of $p^*(E_G^b)$ (in the notation of figure 4).

5 – Examples

(5.1) Let G in $\mathbf{W}$ be a path-connected topological monoid with a free homotopy inverse and M be the Eilenberg-McLane space $K(\pi, m)$, for some positive integer m and Abelian group π. The path-connectivity of G ensures that B_G is simply connected (see 1.1), so it is standard that $FHE(K(\pi, m)\colon B_G)$ is in bijective correspondence with the set of orbits of $H^{m+1}(B_G, \pi)$ under the obvious left action of the group of automorphisms $\operatorname{aut}\pi$. This follows, for example, from the equivalence of conditions (a) and (d) on p.335 of [3], and from [7;thm.6.3]. Then 3.6 implies that:

$$PHE(G\colon K(\pi, m)) = H^{m+1}(B_G, \pi)/\operatorname{aut}\pi\,.$$

Of course if G is a path connected topological group in $\mathbf{W}$ then by (4.8):

$$Fr(G\colon K(\pi, m)) = H^{m+1}(B_G, \pi)/\operatorname{aut}\pi\,.$$

(5.2) If A is a path-connected pointed space then PA will denote the *space of Moore paths* in A, i.e. maps of the interval $[0, e_f]$ into A with $f(0) = *$ and e_f a non-negative real number. The *space of Moore loops in* A, denoted by ΩA, consists of maps $f\colon [0, e_f] \longrightarrow A$ which start and end at $*$, i.e. $f(0) = f(e_f) = *$. Clearly ΩA is a grouplike topological monoid under the operation of attaching loops. The path fibration $q_A\colon PA \longrightarrow A$ that evaluates at the end of paths, i.e. $q_A(f) = f(e_f)$ where $f \in PA$, is a *universal principal* ΩA-fibration, the distinguished fibre being ΩA. Thus $B_{\Omega A} = A$. In particular if A is $K(\pi, m)$ then $\Omega K(\pi, m) = K(\pi, m-1)$.

(5.3) Considering now the special cases of 5.1 with $G = M = K(\mathbf{Z}_p, m)$ and with p a prime, then by 5.1 we have:

$$\begin{aligned} PHE(K(\mathbf{Z}_p, m)\colon K(\mathbf{Z}_p, m)) &= H^{m+1}(K(\mathbf{Z}_p, m+1), \mathbf{Z}_p)/\operatorname{aut}\mathbf{Z}_p = \\ &= \operatorname{Hom}(\mathbf{Z}_p, \mathbf{Z}_p)/\operatorname{aut}\mathbf{Z}_p\,, \end{aligned}$$

a set with cardinality 2.

The following are examples of such $K(\mathbf{Z}_p, m)$-fibrations:

(a) $K(\mathbf{Z}_p, m) \longrightarrow *$, and

(b) $q_m \times 1\colon PK(\mathbf{Z}_p, m+1) \times K(\mathbf{Z}_p, m) \longrightarrow K(\mathbf{Z}_p, m+1) \times K(\mathbf{Z}_p, m)$, where $q_m\colon PK(\mathbf{Z}_p, m+1) \longrightarrow K(\mathbf{Z}_p, m+1)$ denotes the universal path fibration over $K(\mathbf{Z}_p, m+1)$ and 1 is the identity on $K(\mathbf{Z}_p, m)$. Now the base spaces $*$ and $K(\mathbf{Z}_p, m+1) \times K(\mathbf{Z}_p, m)$ have different homotopy types so these two principal fibrations do not have the same $K(\mathbf{Z}_p, m)$-pairwise homotopy type. Thus we have exhibited members of each of the $K(\mathbf{Z}_p, m) - PHE$ classes of such principal $K(\mathbf{Z}_p, m)$-fibrations.

(5.4) It is well known that the isomorphism classes of those fibre bundles over S^n that have a given fibre and given group G may be classified be means of the orbit set of $\pi_{n-1}(G)$ that is obtained from an action $\pi_0(G) \times \pi_{n-1}(G) \longrightarrow \pi_{n-1}(G)$ (see [36:thm.18.5] or [20:thm.7.8.2]). We will show that there is a similar result classifying Hurewicz fibrations.

If X and Y are based spaces then $[X, Y]_*$ will denote the set of based homotopy classes of based maps from X to Y. If X has a non-degenerate base point there is an action $\pi_1(Y) \times [X, Y]_* \longrightarrow [X, Y]_*$ [42, III 1.10] and if Y is also path-connected then $[X, Y]$ is in bijective correspondence with the orbit set $[X, Y]_*/\pi_1(Y)$ [42, III 1.11]. It follows from 3.5 that $FHE(M\colon S^n)$ is in bijective correspondence with the orbit set $\pi_n(B_{H(M)})/\pi_1(B_{H(M)})$. Recalling that if $n > 0$ it is a consequence of 1.1 that $\pi_n(B_{H(M)}) = \pi_{n-1}(H(M))$; we see that there is a bijection:

$$FHE(M\colon S^n) = \pi_n(H(M))/\pi_0(H(M))$$

We are now in a position, via 3.6 to classify principal ΩS^n-fibrations $(n > 0)$ with total spaces homotopy equivalent to M. We have:

$$PHE(\Omega S^n\colon M) = \pi_{n-1}(H(M))/\pi_0(H(M)).$$

(5.5) Let G be ΩS^n and $M = K(\pi, m)$, with m and n both > 0: we are therefore considering the intersection of examples 5.1 and 5.4. In the latter case we have $B_G = S^n$, in the former that the path-component of $H(K(\pi, m))$ that contains the identity is a $K(\pi, m)$ [38:p.31]. We recall that $H^{m+1}(S^n, \pi) = 0$ if $m + 1 \neq n$ but is π if $m + 1 = n$.

Hence (a) if $m + 1 \neq n$ then $PHE(\Omega S^n\colon K(\pi, m)) = 0$, i.e. all principal ΩS^n-fibrations with total spaces of the homotopy type of $K(\pi, m)$

are $\Omega S^n - PHE$ to the principal ΩS^n-fibration:

$$q_n \times 1 \colon PS^n \times K(\pi, m) \longrightarrow S^n \times K(\pi, m) .$$

(b) If $m+1=n$ then $PHE(\Omega S^n \colon K(\pi,m))$ is in bijective correspondence with the orbit set $\pi/\operatorname{aut}\pi$ (determined by the evaluation action $(\operatorname{aut}\pi) \times \pi \longrightarrow \pi$).

REFERENCES

[1] J.F. ADAMS - Z. MAHMUD: *Maps between classifying spaces*, Invent. Math. 35(1976), 1-41.

[2] J.F. ADAMS: *Maps between classifying spaces*, Enseign. Math. (2) 24(1978), 79-85.

[3] P.I. BOOTH: *A unified treatment of some basic problems in homotopy theory*, Bull. Amer. Math. Soc. 79(1973), 331-336.

[4] P.I. BOOTH: *Equivalent homotopy theories and groups of self-equivalences*, In *Groups of Self-homotopy Equivalences*, Lecture Notes in Math. vol. 1425 (Springer-Verlag, 1990), pp. 1-16.

[5] P.I. BOOTH - R. BROWN: *Spaces of partial maps, fibred mapping spaces and the compact-open topology*, Gen. Top. and its Applics. 8(1978), 181-195.

[6] P.I. BOOTH - P.R. HEATH - R. PICCININI: *Characterizing universal fibrations*, In *Algebraic topology, Vancouver 1977*, Lecture Notes in Math. vol. 673 (Springer-Verlag, 1978), pp. 168-184.

[7] A. DOLD: *Partitions of unity in the theory of fibrations*, Ann. of Math. 78(1963), 223-255.

[8] A. DOLD - R. LASHOF: *Principal quasifibrations and fibre homotopy equivalence of bundles*, Illinois J. of Math. 3(1959), 285-305.

[9] R. DROR - W.G. DWYER - D.M. KAN: *Equivariant maps which are self-homotopy equivalences*, Proc. Amer. Math. Soc. 80(1980), 670-672.

[10] W.G. DWYER - D.M. KAN: *Reducing equivariant homotopy theory to the theory of fibrations*, Contemp. Math. 37(1985), 35-49.

[11] W.G. DWYER - D.M. KAN: *Function complexes in homotopical algebra*, Topology 19(1980), 427-440.

[12] W.G. DWYER - G. MISLIN: *On the homotopy type of the components of map$_*$ (BS^3, BS^3)*, In *Algebraic topology, Barcelona 1986*, Lecture Notes in Math. vol. 1298, (Springer-Verlag, 1987), pp. 82-89.

[13] W.G. DWYER - C.W. WILKERSON: *Maps of BZ/pZ to BG*, In *Algebraic topology-rational homotopy, Louvian-la-Neuve 1986*, Lecture Notes in Math. vol. 1318 (Springer-Verlag, 1988), pp. 92-98.

[14] W.G. DWYER - A. ZABRODSKY: *Maps between classifying spaces*, In *Algebraic topology, Barcelona 1986*, Lecture Notes in Math. vol. 1298 (Springer-Verlag 1987), pp. 106-119.

[15] M. FUCHS: *A modified Dold-Lashof construction that does classify H-principal fibrations*, Math. Ann. 192 (1971), 328-340.

[16] M. FUCHS: *Homotopy equivalences in equivariant topology*, Proc. Amer. Math. Soc. 58(1976), 347-352.

[17] M. FUCHS: *Equivariant maps up to homotopy and Borel spaces*, Publ. Mat. Univ. Autonoma Barcelona 28(1981), 79-102.

[18] M. FUCHS: *Borel fibrations and G-spaces*, Manuscripta Math. 58(1987), 377-380.

[19] J.R. HUBBOCK: *Homotopy homomorphisms of Lie groups*, In *New developments in topology (Proc. Sympos. algebraic topology, Oxford 1972)*, London Math. Soc. Lecture notes series (11), (Cambridge University Press, 1974), pp. 33-41.

[20] D. HUSEMOLLER: *Fibre Bundles*, McGraw-Hill, 1966.

[21] N. IWASE: *On the splitting of mapping spaces between classifying spaces*, I. Publ. Res. Inst. Math. Sci. 23(1987), 445-453.

[22] S. JACKOWSKI - J.E. MCCLURE - B. OLIVER: *Self-maps of classifying spaces of compact simple Lie groups*, Bull. Amer. Math. 22(1990), 65-72.

[23] G.R. MAMEDOV: *Mappings of classifying spaces I*, Uspekhi Math. Nauk 41(1986) 3(249), 191-192.

[24] G.R. MAMEDOV: *Mappings of classifying spaces II*, Izv. Akad. Nauk. Azerbaidzhan SSR Ser. Fiz. - Tekhn. Mat. Nauk 7(1986) 5, (1987) 7-10.

[25] J.P. MAY: *Simplicial objects in algebraic topology*, Van Nostrand Math. Studies (Van Nostrand, 1967).

[26] J.P. MAY: *Classifying spaces and fibrations*, Memoirs of the Amer. Math. Soc. 155, 1975.

[27] H. MILLER: *Massey-Peterson towers and maps from classifying spaces*, In *Algebraic topology, Aarhus 1982*, Lecture Notes in Math. vol. 1051 (Springer-Verlag, 1984), pp. 401-417.

[28] H. MILLER: *The Sullivan Conjecture on maps from classifying spaces*, Ann. of Math. 120(1984), 39-87.

[29] J. MILNOR: *Construction of universal bundles II*, Ann. of Math. (3) 63(1956), 430-436.

[30] D. NOTBOHM: *Abbildungen zwichen klassifizierenden raumen I*, Math. Gottingensis, Schriftenr. Sonderforchungsbereichs Geom. Anal. 10, 29S. (1988).

[31] D. NOTBOHM: *Abbildungen zwischen klassifizierenden raumen II*, Math. Gottingensis, Schriftenr. Sonderforchungsbereichs Geom. Anal. 10, 8S. (1988).

[32] D.G. QUILLEN: *Homotopical algebra*, Lecture Notes in Math. vol. 43 (Springer-Verlag, 1967).

[33] R. SCHON: *The Brownian classification of fiber spaces*, Arch. Math. 39(1982), 359-365.

[34] E. SPANIER: *Algebraic topology*, McGraw-Hill, 1966.

[35] J.D. STASHEFF: *H-spaces and classifying spaces: foundations and recent developments (part II)*, Proceedings of Symposia in Pure Math., vol. XXII, Algebraic topology, (Amer. Math. Soc. 1971), pp. 254-263.

[36] N. STEENROD: *The topology of fibre bundles*, Princeton University Press, 1951.

[37] D. SULLIVAN: *Geometric Topology*, Part I: *Localization, Periodicity and Galois Symmetry*, (M.I.T. Press, 1970).

[38] R. THOM: *L'homologie des espaces fonctionnels*, In *Colloque de Topologie Algebrique, Louvain 1956* (Masson and Cie, Paris, 1957) pp. 29-39.

[39] K. VARADARAJAN: *On fibrations and category*, Math. Zeit. 88(1965), 267-273.

[40] R.M. VOGT: *Convenient categories of topological spaces for homotopy theory*, Archiv. der Math. 22(1971), 545-555.

[41] R.M. VOGT: *A note on homotopy equivalences*, Proc. Amer. Math. Soc. (2) 32(1972), 627-629.

[42] G.W. WHITEHEAD: *Elements of homotopy theory*, Graduate text in mathematics 61 (Springer-Verlag, 1978).

[43] C.M. WILKERSON: *Self-maps of classifying spaces*, In *Localization in group theory and homotopy theory, Battelle Research Center 1974*, Lecture Notes in Math. Vol. 418 (Springer-Verlag, 1974), pp. 150-157.

[44] Z. WOJTKOWIAK: *A remark on maps between classifying spaces of compact Lie groups*, Canad. Bull. Math. 31(1988), 452-458.

[45] A. ZABRODSKY: *Maps between classifying spaces*, In *Algebraic topology and algebraic K-theory, Princeton 1983*, Ann. of Math. Stud. vol. 113 (Princeton Univ. Press, Princeton, N.J., 1987), pp. 228-246.

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