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Some results of integral geometry for density of linear subspaces of $\mathbb{C}^n$

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RIASSUNTO – *Usando tecniche di Geometria Integrale, si analizzano le densità di certi sottospazi lineari di* $\mathbb{C}^n$.

ABSTRACT – *Using techniques of Integral Geometry we analyse the densities of some linear subspaces of the complex space* $\mathbb{C}^n$.

KEY WORDS – *Complex Integral Geometry - Holomorphic density - Totally real linear subspaces.*

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– Introduction

It is not very abundant the literature about the integral geometry of complex spaces. A classical article is that of L.A. SANTALÓ, [1] in which he computes, among other things, the density and the volume of the unitary group and the complex grassmannian. In [3], Shifrin finds the proof of the kinematic formula for algebraic submanifolds of the complex projective space $\mathbb{C}P^n$. In this paper, using the same technics of integral geometry that were used to determine the densities and properties of the integral geometry of subspaces of $\mathbb{R}^n$, we generalize some of these

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properties to linear subspaces of $\mathbb{C}^n$. Here we work always with a real basis over which the group of unitary motions acts in a natural way.

In §1 we recall some well known results; in particular, for its useful and repeated applications, the theorem that allows to define a density on a homogeneous space ([2] p.166). In §2 and §3 we study some properties of the densities for linear spaces in $\mathbb{C}^n$. We consider, in particular, the case of the subspaces with a fixed degree of holomorphy, both with its natural density and its "holomorphic" density. These two types of densities seem to be useful for other works in "Complex Integral Geometry". Finally in §4 we generalize to holomophic spaces a formula by Blaschke, which could be useful to obtain Crofton-type formulas in complex integral geometry.

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1 – Some general results

Let $\mathbb{C}^n$ be the standard n-dimensional complex vector space, with its usual topology. If J denotes the standard complex structure, we say that a basis $\{e_1, \dots, e_{2n}\}$ of the underlying $2n$-dimensional real vector space is J-basis provided that $e_{i+n} = e_{i^*} = Je_i$, for all $i, 1 \le i \le n$.

Definition 1.

i) A subspace π of $\mathbb{C}^n$ of real dimension $2r$ is said to be holomorphic if $J\pi = \pi$.

We can write $\pi = \bigwedge_{i=1}^{r} (e_i \wedge e_{i^*})$, via the standard identification of subspaces and multivectors.

ii) A subspace π' of $\mathbb{C}^n$ of real dimension t is said to be antiholomorphic or totally real provided that $\langle J\pi', \pi' \rangle = 0$ and π' not contain any holomorphic subspace.

In this case, we can write

$$\pi' = e_1 \wedge \dots \wedge e_t \quad \text{with} \quad \langle e_1 \wedge \dots \wedge e_t, e_{1^*} \wedge \dots \wedge e_{t^*} \rangle = 0$$

We will express every subspace L_{2r+t} as $L_{2r+t} = \pi \wedge \pi'$, where π is a $2r$-dimensional holomorphic subspace and π' is a t-dimensional totally real subspace. We will say that the subspace L_{2r+t} has a degree of holomorphy r.

The unitary group $U(n)$ acts on $\mathbb{C}^n$ in a natural way. Therefore we can consider the unitary motions

$$x' = ax + b, a \in U(n), b \in \mathbb{C}^n$$

Following the general method of [2], and considering the real representation of $U(n)$, the group of unitary motions can be represented by the matrices of the form

$$g = \left\{ \begin{pmatrix} A & B & b_1 \\ -B & A & b_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

The structure equations with respect to a moving complex frame (P, e_i, e_{i^*}) are given by, [3]

$$\begin{pmatrix} dw_{ij} & dw_{ij^*} \\ -dw_{ij^*} & dw_{i^*j^*} \end{pmatrix} = -\begin{pmatrix} w_{ik} & w_{ik^*} \\ -w_{ik^*} & w_{i^*k^*} \end{pmatrix} \wedge \begin{pmatrix} w_{kj} & w_{kj^*} \\ -w_{kj^*} & w_{k^*j^*} \end{pmatrix} \tag{1.1}$$

and

$$\begin{pmatrix} dw_i \\ dw_{i^*} \end{pmatrix} = -\begin{pmatrix} w_{ik} & w_{ik^*} \\ -w_{ik^*} & w_{i^*k^*} \end{pmatrix} \wedge \begin{pmatrix} w_k \\ w_{k^*} \end{pmatrix} \tag{1.2}$$

whence

$$\begin{aligned} dw_{ij} &= -w_{ik} \wedge w_{kj} - w_{ik^*} \wedge w_{k^*j} \\ dw_{ij^*} &= -w_{ik} \wedge w_{kj^*} - w_{ik^*} \wedge w_{k^*j^*} \\ dw_i &= -w_{ik^*} \wedge w_k - w_{ik^*} \wedge w_{k^*} \\ dw_{i^*} &= w_{ik^*} \wedge w_k - w_{ik} \wedge w_{k^*} \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} w_{ij} &= (de_i \cdot e_j) = -(e_i \cdot de_j) = w_{i^*j^*} \\ w_{ij^*} &= -(e_i \cdot de_{j^*}) = (de_i \cdot e_{j^*}) = -w_{i^*j} \\ w_i &= dP \cdot e_i \quad ; \quad w_{i^*} = dP \cdot e_{i^*} \quad , \quad P \in \mathbb{C}^n \end{aligned} \tag{1.4}$$

We remark that equations (1.2) could be equally expressed in terms of the moving frames (P, ε_i), a complex frame of $\mathbb{C}^n$.

A calculation of the volume of the unitary group can be found in along with the proof that its density is given by

$$d(U(n)) = \bigwedge_{1\leq j<k\leq n} (w_{jk} \wedge w_{jk^*}) \wedge \Big(\bigwedge_i w_{ii^*}\Big) \tag{1.5}$$

Once the integration is accomplished the volume is,

$$\int_{U(n)} dU(n) = \prod_{h=1}^{n} \frac{(2\pi)^h}{(h-1)!} \tag{1.6}$$

Therefore, the volume element of the unitary group of motions is given by

$$d(MU(n)) = \Big(\bigwedge_{j<k} (w_{jk} \wedge w_{jk^*})\Big) \wedge \Big(\bigwedge_j (w_j \wedge w_{j^*})\Big) \wedge \Big(\bigwedge_i w_{ii^*}\Big) \tag{1.7}$$

Let G be a Lie group of dimension n and let H be a closed subgroup of G of dimension $n-m$. Then, G/H is a differentiable manifold of dimension m.

DEFINITION 2. *A density on G/H is a G-invariant m-form on G/H.*

The integral manifolds of the distribution $\mathfrak{h}$, ($\mathfrak{h}$ being the Lie algebra of H), are given by the completely integrable Pfaffian system

$$w_1 = 0, \dots, w_m = 0 \tag{1.8}$$

We know that $d(G/H) = w_1 \wedge \dots \wedge w_m$ is invariant under G and, up to a constant factor, it is the unique m-form with this property.

PROPOSITION 1, [2]. *A necessary and sufficient condition for the m-form $d(G/H)$ to be a density for G/H is that its exterior differential vanishes; that is,*

$$d(d(G/H)) = 0 \tag{1.9}$$

This is a very useful condition, because it is very manageable.

2 – Some properties of densities in linear subspaces of $\mathbb{C}^n$

2.1 – Densities in holomorphic linear subspaces of $\mathbb{C}^n$

Let $L_r^{\mathbb{C}} \equiv L_r^{\mathbb{C}(n)}$ be a holomorphic subspace through the point P with a basis $\{e_a, e_{a^*}\}$. We denote by $H_r^{\mathbb{C}}$ the closed subgroup of unitary motions $MU(n)$ that leave a fixed subspace $L_r^{o\mathbb{C}}$ invariant. Evidently, a bijective correspondence exists between the set of the holomorphic r-subspaces and the elements of the homogeneous spaces $MU(n)/_{H_r^{\mathbb{C}}}$. If, $L_r^{\mathbb{C}}$ is generated by a moving frame (p, e_a, e_{a^*}), from (1.4) and (1.8), we have

$$w_\alpha = w_{\alpha^*} = w_{a\alpha} = w_{a\alpha^*} = 0 \qquad 1 \leq a \leq r < \alpha \leq n.$$

This way we have

Proposition 2.

$$d(MU(n)/H_r^{\mathbb{C}}) = dL_r^{\mathbb{C}} = \Big(\bigwedge_\alpha (w_\alpha \wedge w_{\alpha^*})\Big) \wedge \Big(\bigwedge_{a\alpha} (w_{a\alpha} \wedge w_{a\alpha^*})\Big) \tag{2.1.1}$$
$$1 \leq a \leq r < \alpha \leq n.$$

Using the structure equations (1.3), *it follows immediately that* (2.1.1) *is a density.*

Now we can give the density of linear holomorphic spaces that contain a fixed holomorphic space; that is, $L_d^{\mathbb{C}}$ is fixed and $L_r^{\mathbb{C}}$ variable with $L_r^{\mathbb{C}} \supset L_d^{\mathbb{C}}$. With the same hypothesis, one has the following

Proposition 3.

$$dL_{r[d]}^{\mathbb{C}} = \bigwedge_{a\alpha} (w_{a\alpha} \wedge w_{a\alpha^*}) = dL_{(r-d)[0]}^{\mathbb{C}(n-d)} \tag{2.1.2}$$

where $d + 1 \leq a, \ldots \leq r < \alpha, \ldots \leq n$.

The proof that (2.1.2) is a density is immediate from (1.3) and that of Proposition 1. Therefore it is now possible to obtain the densities for linear holomorphic spaces through the origin; these are the densities of the complex grassmannians $G^{\mathbb{C}}_{r,n-r}$.

PROPOSITION 4.

$$dL^{\mathbb{C}}_{r[0]} = \bigwedge_{a\alpha} (w_{a\alpha} \wedge w_{a\alpha^*}) \tag{2.1.3}$$

The volume of the complex grassmannian has been obtained in [1] and it is given by

$$v(G^{\mathbb{C}}_{r,n-r}) = \frac{v(U(n))}{v(U(r)) \times v(U(n-r))} \tag{2.1.4}$$

where $v(U(i)), i = r, n-r, n$ are given in (1.6).

COROLLARY 1.

$$\int_{G^{\mathbb{C}}_{r,n-r}} dL^{\mathbb{C}}_{r[d]} = v\left(G^{\mathbb{C}}_{r-d,n-r}\right) \tag{2.1.5}$$

PROPOSITION 5. *With the same notations we have*

$$dL^{\mathbb{C}(r)}_{i} \wedge dL^{\mathbb{C}}_{r} = dL^{\mathbb{C}}_{r[i]} \wedge dL^{\mathbb{C}}_{i} \tag{2.1.6}$$

To prove that (2.1.6) is satisfied it is necessary to write the expression of the factors using (2.1.2) and (2.1.3)

COROLLARY 2.

$$dL^{\mathbb{C}}_{i[0]} \wedge dL^{\mathbb{C}}_{r[0]} = dL^{\mathbb{C}}_{r[i]} \wedge dL^{\mathbb{C}}_{i[0]} \tag{2.1.7}$$

2.2 – Densities for totally real linear subspaces $L_t \subset \mathbb{C}^n$.

Now we consider a totally real subspace $L_r \subset \mathbb{C}^n$. Let $(P, e_a), 1 \leq a \leq r$, be a moving frame. Using the same methods that in 2.1, we have

PROPOSITION 6. *Let L_r be a totally real r-subspace of $\mathbb{C}^n$. In this case, the density is given by*

$$d(L_r) = \Big(\bigwedge_{a\leq b}(w_{ab^*})\Big) \wedge \Big(\bigwedge_{a\alpha}(w_{a\alpha} \wedge w_{a\alpha^*})\Big) \wedge \Big(\bigwedge_{a} w_{a^*}\Big) \wedge \Big((\bigwedge_{\alpha}(w_{\alpha} \wedge w_{\alpha^*})\Big) \tag{2.2.1}$$
$$i \leq a, b, \ldots \leq r$$

The proof that $d(d(L_r)) = 0$ follows immediately from the properties of the exterior differential and from (1.3) and (1.4)

COROLLARY 3. *Let $L_{r[0]}$ be a totally real r-space through the origin; then, the density is given by*

$$d(L_{r[0]}) = \Big(\bigwedge_{ab}(w_{ab^*})\Big) \wedge \Big(\bigwedge_{a\alpha}(w_{a\alpha} \wedge w_{a\alpha^*})\Big) \tag{2.2.2}$$

Now, as a generalization of (2.1.1) and (2.1.7), we can give the density for the subspaces L_{2r+t}. In fact, we have the following

PROPOSITION 7. *The density of the $(2r+t)$-subspaces is given by*

$$d(L_{2r+t}) = \Big(\bigwedge_{au}(w_{au} \wedge w_{au^*})\Big) \wedge \Big(\bigwedge_{a\alpha}(w_{a\alpha} \wedge w_{a\alpha^*})\Big) \wedge \Big(\bigwedge_{u\leq v}(w_{uv^*})\Big) \wedge$$
$$\wedge \Big(\bigwedge_{u\alpha}(w_{u\alpha} \wedge w_{u\alpha^*})\Big) \wedge \Big(\bigwedge_{u} w_{u^*}\Big) \wedge \Big(\bigwedge_{\alpha}(w_{\alpha} \wedge w_{\alpha^*})\Big) \tag{2.2.3}$$
$$\textit{with} \quad 1 \leq a \leq r < u \leq r + t < \alpha \leq n$$

Once again Proposition 1 follows from the properties of the exterior differential (1.3) and (1.4). Evidently the density of the totally real subspaces coincide with that of the $(2(n-r)+r)$-subspaces $L_{2(n-r)+r}$, since the complement of L_r in $\mathbb{C}^n$ is $L_{2(n-r)+r}$.

If in the subspace L_{2r+t} we consider the holomorphic subspace $L_r^{\mathbb{C}}$ L_{2r+t}, then it is possible to define another density $d^h L_{2r+t}$ for that subspace, which we call holomorphic density.

PROPOSITION 8. *The holomorphic density $d^h L_{2r+t}$ is given by*

$$\begin{aligned} d^h(L_{2r+t}) = &\Big(\bigwedge_{au}(w_{au}\wedge w_{au^*})\Big)\wedge\Big(\bigwedge_{a\alpha}(w_{a\alpha}\wedge w_{a\alpha^*})\Big)\wedge\Big(\bigwedge_{u\le v}(w_{uv^*})\Big)\wedge \\ &\wedge\Big(\bigwedge_{u\alpha}(w_{u\alpha}\wedge w_{u\alpha^*})\Big)\wedge\Big(\bigwedge_{u}(w_{u^*}\wedge w_u)\Big)\wedge\Big(\bigwedge_{\alpha}(w_\alpha\wedge w_{\alpha^*})\Big) \end{aligned} \tag{2.2.4}$$

The proof that (2.2.4) is a density follows by differentiation in (2.2.4) and using (1.3)

COROLLARY 4. *Let L_r be a totally real r-space contained in $\mathbb{C}^n$. Then its holomorphic density is given by*

$$\begin{aligned} d^h(L_r) = &\Big(\bigwedge_{a\le b}(w_{ab^*})\Big)\wedge\Big(\bigwedge_{a\alpha}(w_{a\alpha}\wedge w_{a\alpha^*})\Big)\wedge \\ &\wedge\Big(\bigwedge_{a}(w_{a^*})\Big)\wedge\Big(\bigwedge_{\alpha}(w_\alpha\wedge w_{\alpha^*})\Big) \end{aligned} \tag{2.2.5}$$

2.3 – Densities for linear spaces that leave a holomorphic subspace fixed

This problem could be established in a more general form; that is, to define the dinsities of the $(2r+t)$-spaces that leave a $(2r'+t')$-space fixed. But, if a $(2r'+t')$-space is left fixed, having in mind that J is an automorphism, the $2(r'+t')$-space will be also fixed, and if $t\neq 0$ the $(2r+t)$-space would have a degree of holomorphy greater than r. So, we may consider only motions of type $L_{2r[2r']}$ and $L_{2r+t[2r]}$.

So, we have only the cases $L_{2r[2r']}$ which was studied in Proposition 3 and $L_{2r+t[2r']}$ studied in the next proposition

PROPOSITION 9.

$$d(L_{2r+t[2r']}) = \Big(\bigwedge_{b,\alpha}(w_{b\alpha} \wedge w_{b\alpha^*})\Big) \wedge \Big(\bigwedge_{b,u}(w_{bu} \wedge w_{bu^*})\Big) \wedge \wedge\Big(\bigwedge_{u\leq v}(w_{uv^*})\Big) \wedge \Big(\bigwedge_{u,\alpha}(w_{u\alpha} \wedge w_{u\alpha^*})\Big) \tag{2.3.3}$$

where $r' + 1 \leq b \leq r$.

The proof is analogous of those of the previous propositions.

COROLLARY 5. *The deinsity of the* $(2r+t)$*-spaces through the origin is given by*

$$d(L_{2r+t[0]}) = \Big(\bigwedge_{b,\alpha}(w_{b\alpha} \wedge w_{b\alpha^*})\Big) \wedge \Big(\bigwedge_{b,u}(w_{bu} \wedge w_{bu^*})\Big) \wedge \Big(\bigwedge_{u\leq v}(w_{uv^*})\Big) \wedge \wedge\Big(\bigwedge_{u,\alpha}(w_{u\alpha} \wedge w_{u\alpha^*})\Big) \tag{2.3.4}$$

where $1 \leq b \leq r$.

The set of the $L_{2r+t[0]}$ is a differentiable manifold which can be identified with the mixed grassmannian.

$$G^{2n}_{2r+t,2(n-r-t)+t} = \frac{U(n)}{U(r) \times U(n-r-t) \times 0(t)} \tag{2.3.5}$$

To determine the volume of this compact and oriented manifold, we can apply a standard method that is used to calculate the volume of the real grassmannian manifold or equivalently we can consider $U(n)$ as the total space of a fibre bundle whose fibre has type $U(r) \times U(n-r-t) \times 0(t) \times 0(t)$. So

$$\int_{G^{2n}_{2r+t,2(n-r-t)+t}} dL_{2r+t[0]} = v\Big(G^{2n}_{2r+t,2(n-r-t)+t}\Big) = \frac{v(U(n))}{v(U(r)) \times v(U(n-r-t)) \times v(0(t))} \tag{2.3.6}$$

2.4 – Some relations between densities for spaces in $\mathbb{C}^n$

If $L_{2r+t} \subset L_\ell^{\mathbb{C}}$, we have

PROPOSITION 10.

$$dL_{2r+t}^{2\ell} \wedge dL_\ell^{\mathbb{C}} = dL_{2r+t} \wedge dL_{\ell[r+t]}^{\mathbb{C}} \tag{2.4.1}$$

PROOF. The result follows easily using (2.1.1), (2.1.2) and (2.1.9).

COROLLARY 6. *With the same notations, we have*

$$dL_{2r+t[0]}^{2\ell} \wedge dL_{\ell[0]}^{\mathbb{C}} = dL_{2r+t[0]} \wedge dL_{\ell[r+t]}^{\mathbb{C}} \tag{2.4.2}$$

PROPOSITION 11.

$$dL_{2\ell+1}^{2(n-1)} \wedge dL_{2\ell-1}^{2\ell} = dL_{2\ell-1}^{2(n-1)} \wedge dL_{\ell+1[2(\ell-1)]}^{2(n-1)} \tag{2.4.3}$$

The proof follows from (2.1.9) and (2.3.3)

COROLLARY 7. *With the same hypothesis,*

$$dL_{2\ell+1[0]}^{2(n-1)} \wedge dL_{2\ell-1[0]}^{2\ell} = dL_{2\ell-1[0]}^{2(n-1)} \wedge dL_{\ell+1[2(\ell-1)]}^{2(n-1)} \tag{2.4.4}$$

If one considers holomorphic densities of spaces with a fixed degree of holomorphy, we have the following

PROPOSITION 12.

$$dL_{2r+t}^{h(2\ell)} \wedge dL_\ell^{\mathbb{C}} = dL_{2r+t}^{h} \wedge dL_{\ell[r+t]}^{\mathbb{C}} \tag{2.4.5}$$

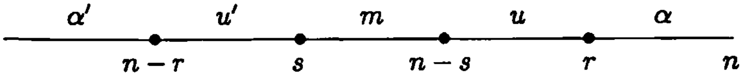
The proof follows immediately from (2.1.1) (2.1.2) and (2.2.4).

Now let $L^{\mathbb{C}}_{r[0]}$ and $L^{\mathbb{C}}_{s[0]}$ be two holomorphic orthogonal subspaces through the origin. We suppose that $r + s \geq n$. Then

PROPOSITION 13. *With the same hypothesis,*

$$dL^{\mathbb{C}}_{r[0]} \wedge dL^{\mathbb{C}}_{s[n-r]} = dL^{\mathbb{C}}_{s[0]} \wedge dL_{r[n-s]} \,. \tag{2.4.6}$$

PROOF. Without loss of generality we can assume that $r > s, s < n - s$. Then $r \geq n - s, s \geq n - r$, and $n - s > n - r$. That is, the indices vary in the following diagram



Using (2.1.1) and (2.1.2), the result follows easily. The proofs of the remaining cases are similar. Next, we give a generalization to the case of holomorphic spaces of a well known formula of densities for the intersection of subspaces, ([2] p. 206).

PROPOSITION 14. *Also with the same notations, we have*

$$dL^{\mathbb{C}}_{r} \wedge dL^{\mathbb{C}}_{s} = \Delta^{r+s-n+1} dL^{\mathbb{C}}_{r[r+s-n]} \wedge dL^{\mathbb{C}}_{s[r+s-n]} \wedge dL^{\mathbb{C}}_{r+s-n}$$

where Δ is a determinant whose components are inner products of vectors of a complex adapted basis.

The proof follows from (2.1.1), (2.1.2) and from the construction of the adapted complex bases.

3 – Another form of density for r-spaces in $\mathbb{C}^n$

a) Holomorphic spaces.

(2.1.1) gives $dL_r^{\mathbb{C}}$ in terms of a point P and $2r$ vectors of a J-basis contained in $L_r^{\mathbb{C}}$. It is often useful to introduce the elements of the holomorphic space $L_{n-r}^{\mathbb{C}}$ orthogonal to $L_r^{\mathbb{C}}$ through a fixed point 0.

With this purpose, let $dL_{n-r[0]}^{\mathbb{C}}$ be the $(n-r)$-holomorphic space orthogonal to $L_r^{\mathbb{C}}$ through 0 and let P be the intersection point. Let p be the distance from 0 to P. We choose orthonormal frames (P, e_i, e_i^*) so that $(P, e_1, \dots, e_r, e_{1^*}, \dots, e_{r^*})$ are in $L_r^{\mathbb{C}}$ and e_{r+1} is in the direction of OP. The l-forms $w_{r+h} = dP.e_{r+h}, w_{r+h}^* = dP.e_{r+h}^*$ represent the element of an arc $L_{n-r[0]}^{\mathbb{C}}$ in P in the directions e_{r+h}, e_{r+h}^*.

Therefore

$$w_{r+1} \wedge \dots \wedge w_n \wedge w_{r+1}^* \wedge \dots \wedge w_{n^*} = d\sigma_{n-r}^{\mathbb{C}} \tag{3.1}$$

represents the volume element of $L_{n-r[0]}^{\mathbb{C}}$.

From (2.1.3) and (3.1) it follows that

$$L_r^{\mathbb{C}} = d\sigma_{n-r}^{\mathbb{C}} \wedge dL_{n-r[0]}^{\mathbb{C}} \tag{3.2}$$

Note that (3.2) is the corresponding formula in the holomorphic case to that given by SANTALO ([2] p. 204).

b) Case of the L_{2r+t} spaces with degree of holomorphy r.

(2.1.9) gives L_{2r+t} in terms of $2r+t$ orthonormal vectors that form a basis of L_{2r+t} and the motions of the point P in $2(n-r-t)+t$ directions. Now we introduce the elements of the $(2(n-r-t)+t)$-space orthogonal to L_{2r+t} and containing the origin 0. With this object, let $L_{2(n-r-t)+t[0]}$ be the $L_{2(n-r-t)+t[0]}$-space orthogonal to L_{2r+t} through the origin and let $P \in L_{2r+t} \cap L_{2(n-r-t)+t[0]}$. Let p be the distance OP. We choose an orthonormal frame (P, e_i, e_{i^*}) so that $(P, e_1, \dots, e_r, e_{1^*}, \dots, e_{r^*}, e_{r+1}, \dots, e_{r+t})$ are in L_{2r+t} and $e_{r+1}^*, \dots, e_{r+t}^*$ are in the normal direction. The l-forms $w_{r+h} = dP.e_{r+h}, e_{r+h}^* = dP.e_{r+h}^*$, are the arc elements of $L_{2(n-r-t)+t}$ in P in the direction of e_{r+h} and e_{r+h}^*, respectively. Therefore $w_{r+1} \wedge \dots \wedge w_{r+t} \wedge w_{r+t+1^*} \wedge \dots \wedge w_n \wedge w_{r+t+1^*} \wedge \dots \wedge w_{n^*}$ is the volume element of $d\sigma_{2(n-r-t)+t}$ in P. Therefore

$$dL_{2r+t} = d\sigma_{2(n-r-t)+t} \wedge dL_{2r+t[0]} = d\sigma_{2(n-r-t)+t} \wedge dL_{2(n-r-t)+t[0]} \tag{3.3}$$

c) (2.2.4) gives dL_{2r+t} in terms of $2r+t$ unitary vectors that form a basis of L_{2r+t} and the motion of the point P in $2(n-r-t)$ directions. Now, proceeding analogously to the case b), we have also

$$d^h L_{2r+t} = d\sigma_{2(n-r)} \wedge dL_{2(n-r-t)+t[0]} \tag{3.4}$$

4 – A generalization of a Blaschke's formula to linear holomorphic subspaces

Let $P, P_1, \ldots, P_{2r}$, be $2r+1$ points of $L_r^{\mathbb{C}}$. An adapted moving frame to $L_r^{\mathbb{C}}$ is given by (p, e_a, e_{a^*}). Then

$$P_a - P = \sum_{j=1}^{r} \lambda_{aj} e_j + \lambda_{aj^*} e_{j^*} \tag{4.1}$$

By differentiation in (4.1) we have

$$dP_a - dP = \sum_{j=1}^{r} (d\lambda_{aj} e_j + \lambda_{aj} de_j + d\lambda_{aj^*} e_{j^*} + \lambda_{aj^*} de_{j^*}) \tag{4.2}$$

Multiplying (4.2) by e_α, e_{α^*} we have

$$\begin{aligned} (dP_a, e_\alpha) - (dP.e_\alpha) &= \sum_{j=1}^{r} (\lambda_{aj} w_{j\alpha} - \lambda_{aj^*} w_{j\alpha^*}) \\ (dP_a.e_{\alpha^*}) - (dP.e_{\alpha^*}) &= \sum_{j=1}^{r} (\lambda_{aj} w_{j\alpha^*} + \lambda_{aj^*} w_{j\alpha^*}) \end{aligned} \tag{4.3}$$

The exterior multiplication of the equations in (4.3) and by $w_\alpha \wedge w_{\alpha^*}$ gives

$$\begin{aligned} &(dP.e_\alpha) - (dP.e_{\alpha^*}) \wedge \Big(\bigwedge_a (dP_a.e_\alpha) \wedge (dP_a.e_{\alpha^*}) \Big) = \\ &= \Delta w_\alpha \wedge w_{\alpha^*} \wedge \Big(\bigwedge_j (w_{j\alpha} \wedge w^*_{j\alpha}) \Big) \end{aligned} \tag{4.4}$$

where Δ is a determinant that depends only on $\lambda_{aj}, \lambda_{aj^*}$.

By exterior multiplication of (4.4) by α and using (2.1.1), we have

$$(4.5)\quad \Big(\bigwedge_{\alpha}(dP.e_\alpha)\wedge(dP.e_{\alpha^*})\Big)\wedge\Big(\bigwedge_{a,\alpha}(dP_a.e_\alpha)\wedge(dP_a.e_{\alpha^*})\Big)=\Delta^{n-r}dL_r^{\mathbb{C}}$$

The volume element of $\mathbb{C}^n$ in P_a can be written

$$dP_a(\mathbb{C}^n)=\bigwedge_h(dP_a.e_h)\wedge(dP_a.e_{h^*})$$

and the volume element $L_r^{\mathbb{C}}$ in P_a is

$$dP_a(L_r^{\mathbb{C}})=\bigwedge_j(dP_a.e_j)\wedge(dP_a.e_{j^*})$$

Therefore the exterior multiplication of (4.5) by

$$\Big[\bigwedge_h(dP.e_h)\wedge(dP.e_{h^*})\Big]\Big[\bigwedge_{i\le j}(dP_i.e_j)\wedge(dP_i.e_{j^*})\Big]\quad (h,i,j=1,\dots,r)$$

yields

$$(4.6)\quad \begin{aligned} &dP(\mathbb{C}^n)\wedge dP_1(\mathbb{C}^n)\wedge\ldots\wedge dP_{2r}(\mathbb{C}^n)=\\ &=\Delta^{n-r}dP(L_r^{\mathbb{C}})\wedge dP_1(L_r^{\mathbb{C}})\wedge\ldots\wedge dP_{2r}(L_r^{\mathbb{C}})\wedge dL_r^{\mathbb{C}}\end{aligned}$$

which is a generalization of a well known formula by BLASCHKE ([2] p.201) to the case of holomorphic linear subspaces.

PARTICULAR CASES

a) $r=1$. Let P and P_1 be two points of the complex line $\mathbb{C}$ with coordinates (a,a_*) and (a_1,a_{1^*}) respectively with respect to a basis (e_1,e_{1^*}) of $\mathbb{C}$. Then

$$dP(\mathbb{C})=da\wedge da_*\quad;\quad dP_1(\mathbb{C})=da_1\wedge da_{1^*}$$
$$S=|(a_1-a)+i(a_{1^*}-a_*)|$$

Therefore

$$(4.7)\quad \begin{aligned} &dP(\mathbb{C}^n)\wedge dP_1(\mathbb{C}^n)=\\ &=\big[(a_1-a)^2+(a_{1^*}-a_*)^2\big]^{\frac{n-1}{2}}da\wedge da_*\wedge da_1\wedge da_{1^*}\wedge dL_1^{\mathbb{C}}\end{aligned}$$

We think that (4.7) could be a very useful expression to obtain Crofton-type formulas in $\mathbb{C}^n$, as those found by SHIFRIN [3].

b) $r = n - 1$. If $P, P_1, \ldots, P_{n-1}$ are independent points of the hyperplane $L^{\mathbb{C}}_{n-1}$, we have

$$\begin{aligned} &dP(\mathbb{C}^n) \wedge dP_1(\mathbb{C}^n) \wedge \ldots \wedge dP_{n-1}(\mathbb{C}^n) = \\ &=(n-1)! \Delta dP(L^{\mathbb{C}}_{n-1}) \wedge \ldots \wedge dP_{n-1}(L^{\mathbb{C}}_{n-1}) \wedge dL^{\mathbb{C}}_{n-1} \end{aligned} \tag{4.8}$$

It seems that it is not possible to generalize (4.8) to spaces that contain a non-zero totally real linear subspace.

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