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A note on $|\overline{N}, p_n|_k$ summability factors

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RIASSUNTO – *Si generalizzano due teoremi di* LAL *[4]*

ABSTRACT – *In this paper two theorems of* LAL *[4] on* $|\overline{N}, p_n|$ *summability methods have been generalized for* $|\overline{N}, p_n|_k$ *summability methods, where* $k \geq 1$.

KEY WORDS – *Absolute summability - Summability factors - Infinite series - Fourier series.*

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1 – Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty, \ (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.1}$$

The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}\left|t_n - t_{n-1}\right|^k < \infty . \tag{1.3}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\left|\overline{N}, p_n\right|_k$ summability is the same as $|C, 1|_k$ (resp. $\left|\overline{N}, p_n\right|$) summability. Also it should be noted that in the special case when $p_n = 1/(n+1)$ and $k = 1$, $|\overline{N}, p_n|_k$ summability is equivalent to the summability $|R, \log n, 1|$.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. The Fourier series of $f(t)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) . \tag{1.4}$$

2 – Lal's results

LAL [4] proved the following theorems for $\left|\overline{N}, p_n\right|$ summability methods.

THEOREM A. *If the sequence (s_n) is bounded and (λ_n) is a sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\left|\lambda_n\right| = 0(1) \tag{2.1}$$

$$\sum_{n=1}^{m}\left|\Delta\lambda_n\right| = 0(1) \quad as \quad m \longrightarrow \infty , \tag{2.2}$$

then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N}, p_n\right|$.

THEOREM B. *The summability $\left|\overline{N}, p_n\right|$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions* (2.1)-(2.2) *of Theorem A are satisfied.*

3 – Main results

The aim of this paper is to generalize above theorems for $\left|\overline{N},p_n\right|_k$ summability methods, where $k \geq 1$. Now, we shall prove the following theorems.

THEOREM 1. *Let $k \geq 1$. If the sequence (s_n) is bounded and the sequence (λ_n) is such that conditions* (2.1)-(2.2) *of Theorem A are satisfied with the condition* (2.1) *replaced by;*

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\left|\lambda_n\right|^k = 0(1) \quad as \quad m \longrightarrow \infty, \tag{3.1}$$

then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N},p_n\right|_k$.

THEOREM 2. *Let $k \geq 1$. The summability $\left|\overline{N},p_n\right|_k$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the condition* (2.2) *and* (3.1) *are satisfied.*

It may be remarked that if we take $k = 1$ in our theorems, then we get Theorem A and Theorem B, respectively. Also it should be noted that Theorem 2 includes as particular cases the wellknown results due to BHATT [1], MATSUMOTO [5] and MOHANTY [6].

4 – Proof of theorem 1

Let (T_n) be the sequence of the $(\overline{N},p_n)$ means of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{w=0}^{v} a_w\lambda_w = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v\,. \tag{4.1}$$

Then, for $n \geq 1$, we have that

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v\,, \quad (P_{-1} = 0)\,. \tag{4.2}$$

Using Abel's transformation we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\Big(P_{v-1}\lambda_v\Big) s_v + \frac{p_n s_n \lambda_n}{P_n} =$$

$$= - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v +$$

$$+ \frac{p_n s_n \lambda_n}{P_n} = T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say}.$$

To complete the proof of Theorem 1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \Big(P_n/p_n\Big)^{k-1} \Big|T_{n,r}\Big|^k < \infty, \quad \text{for} \quad r = 1, 2, 3. \tag{4.3}$$

Now, applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \Big(P_n/p_n\Big)^{k-1} \Big|T_{n,1}\Big|^k \le \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \Big\{ \sum_{v=1}^{n-1} p_v |s_v|^k |\lambda_v|^k \Big\} \times$$

$$\times \Big\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \Big\}^{k-1} = 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v |\lambda_v|^k =$$

$$= 0(1) \sum_{v=1}^{m} p_v |\lambda_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = 0(1) \sum_{v=1}^{m} \frac{p_v}{P_v} |\lambda_v|^k = 0(1) \quad \text{as } m \longrightarrow \infty,$$

by virtue of the hypotheses.

Again

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k \le \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \Big\{ \sum_{v=1}^{n-1} P_v \Big|\Delta\lambda_v\Big| \, |s_v| \Big\} \times$$

$$\times \Big\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \Big|\Delta\lambda_v\Big| \Big\}^{k-1} = 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Big|\Delta\lambda_v\Big| =$$

$$= 0(1) \sum_{v=1}^{m} P_v |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = 0(1) \sum_{v=1}^{m} \Big|\Delta\lambda_v\Big| = 0(1) \text{ as } m \longrightarrow \infty,$$

by virtue of the hypotheses.

Finally, we have that

$$\sum_{n=1}^{m} \left(P_n/p_n\right)^{k-1}\left|T_{n,3}\right|^k = O(1)\sum_{n=1}^{m} \frac{p_n}{P_n}|\lambda_n|^k = O(1) \quad \text{as} \quad m \longrightarrow \infty,$$

by virtue of the hypotheses. Therefore, we get that

$$\sum_{n=1}^{m} \left(P_n/p_n\right)^{k-1}\left|T_{n,r}\right|^k = O(1) \quad \text{as} \quad m \longrightarrow \infty, \quad \text{for} \quad r = 1,2,3.$$

This completes the proof of Theorem 1.

5 – Proof of theorem 2

Since the behaviour of the Fourier series for a particular value of x, as for as convergence is concerned, depends on the behaviour of the function in the immediate neighbourhood of this point only, Theorem 2 is an immediate consequence of Theorem 1.

REMARK. If we take $p_n = 1$ for all values of n in our theorems, then we get two results for $|C,1|_k$ summability methods.

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