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# Variational inequality for a viscous drum vibrating in the presence of an obstacle

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RIASSUNTO – *Si prova l'esistenza della soluzione debole, globale nel tempo, della disequazione variazionale per un tamburo viscoso che vibra in presenza di un ostacolo. Sono prese in esame condizioni al contorno omogenee del tipo di Neumann e del tipo di Dirichlet.*

ABSTRACT – *The existence of global in time weak solution of the variational inequality for a viscous drum vibrating in the presence of an obstacle is proved. Both homogeneous Neumann and homogeneous Dirichlet boundary conditions are considered. Some regularity results are obtained.*

KEY WORDS – *Vibrating string equation - Variational inequality - Compact imbedding - Penalty method.*

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## 1 – Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with the lipschitz-like continuous boundary $\partial\Omega$. Let $T > 0$, $I = (0,T)$, $t \in (0,T\rangle$ and $Q_t = \Omega \times (0,t)$. The model describing vibrations with a unilateral constraint $u(x,t) \geq 0$ is governed in the points from $Q_T$, where $u(x,t) > 0$, by the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \,. \tag{1.1}$$

In the sequel, we will denote $\dot{u} = \dfrac{\partial u}{\partial t}$, $\ddot{u} = \dfrac{\partial^2 u}{\partial t^2}$.

We will study the classical mixed boundary value problem

$$u(x,0) = u_0(x)\,,\ x \in \Omega \tag{1.2}$$

$$\dot{u}(x,0) = u_1(x)\,,\ x \in \Omega \tag{1.3}$$

and Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1.4}$$

The problem (1.1)–(1.4) can be formally written in the form

$$\ddot{u} - \Delta u = f \tag{1.5}$$

with (1.2)–(1.4) and with

$$u \geq 0\,,\ f \geq 0\,,\ uf = 0 \text{ in } Q_T\,, \tag{1.6}$$

where $f$ has the meaning of a reaction force.

From the mid–seventies, the problems of the vibrating string equation with unilateral constraints in one space dimension were intensively studied by a group of Italian and French mathematicians, namely L. Amerio, C.Citrini, C. Marchionna, H. Cabannes, A. Bachelot–Motet. A survey of their results concerning mostly the classical solution can be found in the Proceedings [2]. The global existence of the weak solution is proved by A. ARTJUSHIN [1]. In several space dimensions , the existence of the weak solution remains open.

The aim of this paper is to solve the problem for a "viscous drum" , i.e. to solve the equation in the form

$$\ddot{u} - \Delta u - \Delta \dot{u} = f. \tag{1.7}$$

Let us note that this formulation includes the elastic impact law.

## 2 – Definition of the problem and penalty method

If $k$ is a positive integer, $1 \le p \le \infty$, we denote by $W^{k,p}(\Omega)$ the usual Sobolev spaces of $L^p(\Omega)$ functions with $L^p(\Omega)$ derivatives up to the order $k$. As usually, we denote by $D(\Omega)$ the set of infinitely differentiable functions with the compact support in $\Omega$. By $W_0^{k,p}(\Omega)$, $p < \infty$, we denote the closure of $D(\Omega)$ in $W^{k,p}(\Omega)$.

If $E$ is a Banach space, then $L^p(I; E)$, $1 \le p < \infty$, denotes the usual Bochner space. $B(I; E)$ is a space of bounded functions from $I$ into $E$ with the sup-norm.

Let us suppose $g \in L^2(\Omega)$, $u_0 \in W^{1,2}(\Omega)$ and $u_1 \in W^{1,2}(\Omega)$[1]. We say that

$$
\begin{aligned}
&u \in B(I;\, W^{1,2}(\Omega))\,,\\
&\dot u \in B(I;\, L^2(\Omega)) \cap L^2(I;\, W^{1,2}(\Omega))\,,\\
&u \ge 0 \text{ in } Q_T \text{ almost everywhere}
\end{aligned}
\tag{2.1}
$$

is a generalized solution to the problem

$$\ddot u - \Delta u - \Delta \dot u - g = f \text{ in } Q_T\,, \tag{2.2}$$

$$u \ge 0 \text{ in } Q_T,\, f \ge 0 \text{ in } Q_T,\, fu = 0 \text{ in } Q_T\,, \tag{2.3}$$

$$\frac{\partial \dot u}{\partial n} + \frac{\partial u}{\partial n} \ge 0,\ u \ge 0,\ \Big(\frac{\partial \dot u}{\partial n} + \frac{\partial u}{\partial n}\Big)u = 0 \text{ on } \partial\Omega \times I\,, \tag{2.4}$$

$$u(\cdot, 0) = u_0 \ge 0\,,\ (\dot u(\cdot, 0) - u_1) \ge 0\,,\ (\dot u(\cdot, 0) - u_1)u_0 = 0\,, \tag{2.5}$$

if for every $v \in K$, we have

$$
\begin{aligned}
&\iint_{Q_T} (\nabla u \nabla (v-u) + \nabla \dot u \nabla (v-u) - g(v-u))\, dx\, dt -\\
&- \iint_{Q_T} \dot u(\dot v - \dot u)\, dx\, dt + \int_{\Omega_T} \dot u(v-u)\, dx - \int_{\Omega} u_1(v-u_0)\, dx \ge 0\,.
\end{aligned}
\tag{2.6}
$$

[1] The last condition can be slightly weakened.

Here

(2.7) $K = \{v \in L^2(I;\, W^{1,2}(\Omega)), \dot v \in L^2(I;\, L^2(\Omega)); v \geq 0$ a.e. in $Q_T\}$.

Of course $u(0,\cdot) = u_0$ in the sense of traces. If we suppose the solution to the problem (2.1), (2.6), is smooth enough, we get in a formal way (2.2)–(2.5).

We consider a convex penalty function $F : \mathbb{R}^1 \longmapsto \mathbb{R}^1$ such that

(2.8) $$F(\xi) = 0 \Longleftrightarrow \xi \geq 0 \text{ in } \Omega .$$

For simplicity, we suppose that

(2.9) $$-c\xi \leq F'(\xi) \leq 0 ,$$

where $c$ is some positive constant.
We can take for example

(2.10) $$F(\xi) = (\xi^-)^2 .$$

Let $u_\varepsilon$ be a solution to the penalty problem (in a formal setting)

(2.11) $$\begin{aligned} &\ddot u_\varepsilon - \Delta u_\varepsilon - \Delta \dot u_\varepsilon - g + \frac{1}{\varepsilon} F'(u_\varepsilon) = 0 \text{ in } Q_T ,\\ &\frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega \times I ,\\ &u_\varepsilon(\cdot,0) = u_0 ,\ \dot u_\varepsilon(\cdot,0) = u_1 . \end{aligned}$$

By the standard Galerkin method we get a unique solution $u_\varepsilon$ to (2.11) (see for example [4] or the appendix) such that

(2.12) $$\begin{aligned} &u_\varepsilon \in L^\infty(I;\, W^{1,2}(\Omega)) ,\\ &\dot u_\varepsilon \in L^\infty(I;\, L^2(\Omega)) \cap L^2(I;\, W^{1,2}(\Omega)) ,\\ &\ddot u_\varepsilon \in L^2(I;\, L^2(\Omega)) \end{aligned}$$

and for almost all $t \in (0,T)$, we have

(2.13) $$\int_\Omega \left( \ddot u_\varepsilon v + \nabla u_\varepsilon \nabla v + \nabla \dot u_\varepsilon \nabla v - gv + \frac{1}{\varepsilon} F'(u_\varepsilon) v \right) dx = 0$$

for every $v \in W^{1,2}(\Omega)$.

## 3 – A priori estimates and the limit process for $\varepsilon \to 0$

LEMMA 3.1. *There exists a positive constant $c$, independent of $\varepsilon$, such that for any $t \in \langle 0, T \rangle$, we have*

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t} |\dot{u}_\varepsilon|^2\,dx + \frac{1}{2}\int_{\Omega_t} |\nabla u_\varepsilon|^2\,dx + \iint_{Q_t} |\nabla \dot{u}_\varepsilon|^2\,d\tau\,dx + \\ + \frac{1}{\varepsilon}\int_{\Omega_t} F(u_\varepsilon)\,dx \quad \leq c,\end{aligned} \tag{3.1}$$

*where $\Omega_t = \{(x,t);\ x \in \Omega\}$.*

PROOF. We take $v = \dot{u}_\varepsilon$ as the test function in (2.13). □

We come actually to an easy, but fundamental estimate for the next.

LEMMA 3.2.

$$\frac{1}{\varepsilon}\iint_{Q_t} |F'(u_\varepsilon)|\,dx\,dt \leq c, \tag{3.2}$$

*where $c$ does not depend of $\varepsilon$.*

PROOF. By assumptions we have $-\frac{1}{\varepsilon}F'(\xi) \geq 0$. From (2.13) with $v = 1$, we get

$$\begin{aligned}-\iint_{Q_t} \frac{1}{\varepsilon}F'(u_\varepsilon)dx\,dt = -\iint_{Q_T} g\,dx\,dt + \int_{\Omega_T} \dot{u}_\varepsilon\,dx - \\ - \int_{\Omega} u_1\,dx \leq c.\end{aligned} \tag{3.3}$$

□

Let us choose actually a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \rightharpoonup u$ in $L^2(I;\ W^{1,2}(\Omega))$ and $\dot{u}_{\varepsilon_n} \rightharpoonup \dot{u}$ in $L^2(I;\ W^{1,2}(\Omega))$.

Let $2k > N$. We have, by the imbedding theorem, see for example [7]

$$W^{k,2} \hookrightarrow\hookrightarrow C(\bar{\Omega}). \tag{3.4}$$

It follows from (3.1), (3.2) and (3.4) that

$$\ddot{u}_{\varepsilon_n} \text{ are bounded in } L^1\left(I; (W^{k,2}(\Omega))^*\right) \tag{3.5}$$

and because

$$\dot{u}_{\varepsilon_n} \text{ are bounded in } L^2(I; W^{1,2}(\Omega)), \tag{3.6}$$

we can use the following generalization of Aubin's Theorem (see for example the survey paper [11]):

THEOREM 3.1. *Let $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$ be Banach spaces, the first reflexive and separable. Let $1 < p < \infty$, $1 \le q < \infty$. Then the space*

$$W \equiv \{v; v \in L^p(I; B_0), \dot{v} \in L^q(I; B_1)\} \hookrightarrow\hookrightarrow L^p(I; B).$$

Let us remark that the theorem holds also, if $B_1$ is a Hausdorff locally convex space, see [10].

PROOF. Let $v_n$ be a bounded sequence from $W$. Because the space $L^p(I; B_0)$ is reflexive ( see for example [4] ), we can suppose $v_n \rightharpoonup v$ in $L^p(I; B_0)$ and without loss of the generality $v = 0$. First, by the well known lemma, see [8], $\forall \varepsilon > 0\, \exists \eta = \eta(\varepsilon)$ such that

$$\|u\|_B \le \varepsilon \|u\|_{B_0} + \eta(\varepsilon)\|u\|_{B_1}, \tag{3.7}$$

so it is enough to prove that

$$\int_0^T \|v_n(t)\|_{B_1}^p \, dt \to 0. \tag{3.8}$$

Let us look at $\int_0^{\frac{T}{2}} \|v_n(t)\|_{B_1}^p$. For $0 < s \le \frac{T}{2}$ and every $t \in (0, \frac{T}{2})$, we have

$$(3.9)\qquad v_n(t) = \frac{1}{s}\int_0^s v_n(t+\tau)\,d\tau + \int_0^s \left(\frac{\tau}{s} - 1\right)\dot v_n(t+\tau)\,d\tau\,.$$

Let us look first at the second member in (3.9). We have

$$(3.10)\qquad \begin{aligned} &\int_0^{\frac{T}{2}} \Big\| \int_0^s \left(\frac{\tau}{s} - 1\right)\dot v_n(t+\tau)\,d\tau \Big\|_{B_1}^p dt \le \\ &\le \int_0^{\frac{T}{2}} \Big[\int_0^s \left(1 - \frac{\tau}{s}\right)\|\dot v_n(t+\tau)\|_{B_1}\,d\tau\Big]^p dt = \\ &= \int_0^{\frac{T}{2}} dt\Big(\int_t^{t+s} \left(1 + \frac{t-\sigma}{s}\right)\|\dot v_n(\sigma)\|_{B_1}\,d\sigma\Big)^p . \end{aligned}$$

If we put

$$(3.11)\qquad \psi_s(\lambda) = \begin{cases} 1 + \dfrac{\lambda}{s} & \text{for } -s < \lambda < 0\,, \\ 0 & \text{elsewhere}\,, \end{cases}$$

then the last integral can be rewritten as

$$(3.12)\qquad \int_0^{\frac{T}{2}} \Big(\int_{-\infty}^{\infty} \psi_s(t-\sigma)\|\dot v_n(\sigma)\|_{B_1}\,d\sigma\Big)^p .$$

Let us put $\dot v_n(\sigma) = 0$ for $\sigma \notin (0,T)$. We have

$$(3.13)\qquad \begin{aligned} &\left[\int_{-\infty}^{\infty} \psi_s(t-\sigma)\|\dot v_n(\sigma)\|_{B_1}^{\frac{1}{p}+\frac{1}{p'}}\,d\sigma\right]^p \le \\ &\le \left(\int_{-\infty}^{\infty} \|\dot v_n(\sigma)\|_{B_1}\,d\sigma\right)^{p-1} \int_{-\infty}^{\infty} \psi_s^p(t-\sigma)\|\dot v_n(\sigma)\|\,d\sigma\,; \end{aligned}$$

hence

$$
(3.14)\quad
\begin{aligned}
&\left(\int_0^{\frac{T}{2}} dt \int_{-\infty}^{\infty} \psi_s(t-\sigma)\|\dot v_n(\sigma)\|_{B_1}\, d\sigma\right)^p \le \\
&\le \left(\int_{-\infty}^{\infty} \|\dot v_n(\sigma)\|_{B_1}\, d\sigma\right)^{p-1} \int_0^{\frac{T}{2}} dt \int_{-\infty}^{\infty} \psi_s^p(t-\sigma)\|\dot v_n(\sigma)\|_{B_1}\, d\sigma \le \\
&\le \left(\int_{-\infty}^{\infty} \|\dot v_n(\sigma)\|_{B_1}\, d\sigma\right)^{p-1} \int_{-\infty}^{\infty} \|\dot v_n(\sigma)\|_{B_1}\, d\sigma \int_{-\infty}^{\infty} \psi_s^p(t-\sigma)\, dt \le \\
&\le s\left(\int_{-\infty}^{\infty} \|\dot v_n(\sigma)\|_{B_1}\, d\sigma\right)^p .
\end{aligned}
$$

So let $\varepsilon > 0$; because $\int_{-\infty}^{\infty} \|\dot v_n(\sigma)\|_{B_1}\, d\sigma \le A$, we can choose $s < \frac{\varepsilon}{2A^p}$. For $s$ fixed $\lim_{n\to\infty} \frac{1}{s}\int_0^s v_n(t+\tau)\, d\tau \rightharpoonup 0$ in $B_0$, hence $\lim_{n\to\infty} \frac{1}{s}\int_0^s v_n(t+\tau)\, d\tau \to 0$ in $B_1$ because of the imbedding $B_0 \hookrightarrow\hookrightarrow B$. On the other hand, $\|\int_0^s v_n(t+\tau)\, d\tau\|_{B_1} \le \int_0^T \|v_n(\tau)\|_{B_1}\, d\tau$, so by Lebesgue's Dominated Convergence Theorem we get

$$
(3.15)\qquad \int_0^{\frac{T}{2}} \|\int_0^s v_n(t+\tau)\, d\tau\|_{B_1}^p\, dt \to 0 . \qquad \square
$$

THEOREM 3.2. *Let $g \in L^2(Q_T)$, $u_0$ and $u_1$ in $W^{1,2}(\Omega)$. Then there exists a solution $u$ satisfying* (2.1)–(2.5) *in the sense of* (2.6).

PROOF. We consider $\{u_{\varepsilon_n}\}$. We have as mentioned

$$
(3.16)\qquad
\begin{aligned}
u_{\varepsilon_n} &\rightharpoonup u \text{ in } L^2(I; W^{1,2}(\Omega)) ,\\
\dot u_{\varepsilon_n} &\rightharpoonup \dot u \text{ in } L^2(I; W^{1,2}(\Omega)) ,
\end{aligned}
$$

and the condition (3.5) holds. So with $p = 2$, $q = 1$, $B_0 = W^{1,2}(\Omega)$, $B = L^2(\Omega)$ and $B_1 = (W^{k,2}(\Omega))^*$, we can use Theorem 3.1 to get

$$
(3.17)\qquad \dot u_{\varepsilon_n} \to \dot u \text{ in } L^2(I; L^2(\Omega)) .
$$

Let us emphasize that this strong convergence, based on the occurence of the viscosity, is essential for the following limit process. From this we also have

$$u_{\varepsilon_n}(t) \to u(t) \text{ in } L^2(\Omega) \text{ for } t \in (0,T) . \tag{3.18}$$

Let us integrate (2.13) in the time $t$ with the test function $v - u_{\varepsilon_n}$, where $v \in K$. We get

$$-\frac{1}{\varepsilon_n}\int_{\Omega} F'(u_{\varepsilon_n})(v - u_{\varepsilon_n})\, dx \geq 0 ; \tag{3.19}$$

hence

$$\begin{aligned} &\iint_{Q_T} \nabla u_{\varepsilon_n} \nabla (v - u_{\varepsilon_n}) + \nabla \dot{u}_{\varepsilon_n} \nabla (v - u_{\varepsilon_n}) - g(v - u_{\varepsilon_n})\, dx\, dt - \\ &- \iint_{Q_T} \dot{u}_{\varepsilon_n}(\dot{v} - \dot{u}_{\varepsilon_n})\, dx\, dt + \int_{\Omega_T} \dot{u}_{\varepsilon_n}(v - u_{\varepsilon_n})\, dx - \\ &- \int_{\Omega} u_1(v(0,\cdot) - u_0(\cdot))\, dx \geq 0 . \end{aligned} \tag{3.20}$$

Further,

$$\limsup_{n\to\infty} - \iint_{Q_T} \nabla u_{\varepsilon_n} \nabla u_{\varepsilon_n}\, dx\, dt \leq - \iint_{Q_T} |\nabla u|^2\, dx\, dt \tag{3.21}$$

and

$$\begin{aligned} &\limsup_{n\to\infty} - \iint_{Q_T} \nabla \dot{u}_{\varepsilon_n} \nabla u_{\varepsilon_n}\, dx\, dt = \\ &= \limsup_{n\to\infty} -\frac{1}{2}\int_{\Omega_T} |\nabla u_{\varepsilon_n}|^2\, dx\, dt + \frac{1}{2}\int_{\Omega} |\nabla u_0|^2\, dx \leq \\ &\leq -\frac{1}{2}\int_{\Omega_T} |\nabla u|^2\, dx + \frac{1}{2}\int_{\Omega} |\nabla u_0|^2\, dx = - \iint_{Q_T} \nabla \dot{u} \nabla u\, dx\, dt ; \end{aligned} \tag{3.22}$$

so the limit in (3.20) because of (3.17), (3.18), (3.21), (3.22) can be accomplished. We have from (3.1)

$$\iint_{Q_T} F(u_{\varepsilon_n})\,dxd\tau \le c\varepsilon_n\,; \tag{3.23}$$

hence

$$\iint_{Q_T} F(u)\,dx\,d\tau \le \liminf_{n\to\infty} \iint_{Q_T} F(u_{\varepsilon_n})\,dx\,d\tau = 0\,; \tag{3.24}$$

so $u \ge 0$ a.e. in $Q_T$. □

REMARK 3.1. Analogously as before, we are able to prove the existence of the weak solution of the Dirichlet problem, where we consider

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{3.25}$$

instead of the Neumann boundary condition (1.4). Of course, now we use the Sobolev spaces $W_0^{1,2}$, $W_0^{k,2}$, respectively and we get, for $\omega \in \mathcal{D}^+(\Omega)$,

$$\frac{1}{\varepsilon} \iint_{Q_t} |F'(u_\varepsilon)|\,\omega\,dx\,dt \le c \tag{3.26}$$

instead of the estimate (3.2). So, $\ddot{u}_{\varepsilon_n}$ are bounded in $L^1(I;\,(W^{k,2}_{loc})^*)$. Again , we can use Aubin's Theorem to get the strong convergence of $\{\dot{u}_{\varepsilon_n}\}$.

REMARK 3.2 We can obtain the crucial strong convergence of $\{\dot{u}_{\varepsilon_n}\}$ in $L^2(Q_T)$ by an alternative approach. Via the technique of the local straightening of the boundary and a sufficiently smooth partition of the unity (denoted by $\mathcal{R}$) on $\Omega$ (see for example [3]) and with the help of an appropriate extension technique (see [9]), we can extend $u_{\varepsilon_n}$ from $\Omega \times (0,T)$ to $\mathbb{R}^N \times (0,T)$ such that all the a priori estimates (3.1) and (3.5) remain valid independently of $\varepsilon_n$, maybe with a different constant. The extension technique applied to the time variable and the imbedding (3.4) give us finally that the sequence $\{\ddot{u}_{\varepsilon_n}\}$ is bounded in $H^{-\frac{1}{2}-\eta}\left(\mathbb{R}^1;\, H^{-\frac{N}{2}-\eta}(\mathbb{R}^N)\right)$ for every $\eta > 0$. Here $H^\alpha(\mathbb{R}^k)$ denotes $W^{\alpha,2}(\mathbb{R}^k)$ and for the definition of fractional order derivatives see [9].

Denoting by $\varphi_n$ the Fourier transform of $\dot{u}_{\varepsilon_n}$, we can find a constant $c_1$ such that for arbitrarily small $\eta > 0$

(3.27)
$$\int_{\mathbb{R}^{N+1}} \left( \frac{\tau^2}{1+\tau^{1+\eta}} \frac{1}{(1+|\xi|^2)^{\frac{N}{2}+\eta}} + (1+|\xi|^2) \right) |\varphi_n(\xi,\tau)|^2 d\xi\, d\tau \le c_1 \,, n \in \mathbb{N}\,.$$

An appropriate use of the Hölder inequality yields easily for $\alpha \in (0, \frac{1}{N+2})$ and $n \in \mathbb{N}$

$$\int_{\mathbb{R}^{N+1}} \tau^{2\alpha} |\varphi_n(\xi,\tau)|^2 d\xi\, d\tau < c_1\,. \tag{3.28}$$

The strong convergence is then a direct consequence of the compact imbedding theorem (see [9]).

The above described approach, slightly modified, can be suitable for the Dirichlet problem, too. For the sake of simplicity, we restrict ourselves to its homogeneous case. By the above mentioned method, we obtain the strong convergence for $\rho \dot{u}_{\varepsilon_n}$, where $\rho$ is an arbitrary $C^1$–smooth function with the support in $\Omega$. As (3.2) implies that for a given $\vartheta > 0$ there is $\Omega_\vartheta \subset \Omega$ such that $\overline{\Omega \setminus \Omega_\vartheta} \subset \Omega$ and $\int_0^T \int_{\Omega_\vartheta} \dot{u}_{\varepsilon_n}^2 \,dx\,dt < \vartheta$, $n \in \mathbb{N}$, we can arrive easily at the desired strong convergence of $\dot{u}_{\varepsilon_n}$.

The localization technique gives us also some better spatial regularity of $u$, naturally for more regular $u_0$. Let $\Omega$ is of class $C^{1,1}$and let $u_0 \in W^{2,2}(\Omega)$. We consider the variational inequality (3.15) on $(0,t)$, $t \le T$. We take a partition of unity $\mathcal{R}$ on $\Omega$ and suppose that every $\rho \in \mathcal{R}$ is $C^{1,1}$–smooth function. For $v$, we put $u+\rho\,(u_{-h}-u)$, where the index $h$ denotes a shift in argument in a direction $h$. If $\operatorname{supp}\rho \subset \Omega$, we take an arbitrary small $h$. If $\operatorname{supp}\rho \cap \partial\Omega \neq \emptyset$, we suppose that $\partial\Omega$ is locally straightened and we take only the vectors of this part of $\partial\Omega$ (for technical details see [3], [5]). Then we shift the whole inequality in the direction $h$ (satisfying the above described conditions) and put $v_{-h} = u_{-h} + \rho(u - u_{-h})$.

We summarize both inequalities and divide the sum by $|h|^{N+2\beta}$, $\beta \in (0, \frac{1}{2})$ in the first case or by $|h|^{N-1+2\beta}$ in the second case. We integrate the result in $h$ over $\mathbb{R}^N$, $\mathbb{R}^{N-1}$, respectively. Putting $U = \rho u$, we can see that, for the proof of regularity, we must estimate the following most important terms

$$
(3.29)\qquad \begin{aligned} &-\int_{\mathbf{R}^m} |h|^{-m-2\beta} \int_\Omega (\dot U_{-h} - \dot U)(U_{-h} - U)(x,t)\,dx\,dh+ \\ &+\int_{\mathbf{R}^m} |h|^{-m-2\beta} \iint_{Q_t} (\dot U_{-h} - \dot U)^2(x,t)dx\,dt\,dh\,, m = N \text{ or } N-1 \end{aligned}
$$

on the right hand side of the inequality. The second term, however, can be easily estimated for $\beta \in (0,1)$ by means of (3.1). For the first term, we use the Hölder inequality and (3.1) yields the result directly for $\beta \in (0, \frac{1}{2})$. But on the left hand side, we have estimated

$$
(3.30)\qquad \sup_{t\in I} \int_{\mathbf{R}^m} |h|^{-m-2\beta} \int_\Omega |\nabla (U_{-h} - U)|^2(x,t)\,dx\,dh
$$

which gives us $U = \rho u \in C^0(I;\, H^{\frac{3}{2}-\varepsilon}(\Omega))$, $\varepsilon > 0$ for $\rho$ with the support in $\Omega$ (for $\rho$ with the support crossing the boundary, we obtain the tangential regularity only). Due to this result we can proceed with an iterative procedure which gives us (together with a suitable renormation technique - see [5]) that $\rho u \in C^0(I;\, H^{2-\varepsilon}(\Omega))$, $\varepsilon > 0$ arbitrarily small, inside $\Omega$ (along $\partial\Omega$, such a regularity is proved in the tangential directions only). Multiplying (2.11) by $\Delta u_\varepsilon$, integrating it in $x$ and $t$ over $Q_T$, using the Green formula to the appropriate terms and (3.1) we derive finally

$$
(3.31)\qquad \int_{Q_T} \nabla u_\varepsilon \nabla u_\varepsilon + \Delta \dot u_\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} \nabla F'(u_\varepsilon) \nabla u_\varepsilon\,dx\,dt \le c \quad \forall \varepsilon > 0\,.
$$

For suitable $F$ (e.g. $F\colon y \mapsto (y^-)^2$) (3.31) easily yields that $\{\Delta u_\varepsilon;\ \varepsilon > 0\}$ is bounded in $L^2(Q_T)$ and in $L^\infty(I;\, L^2(\Omega))$, where $L^\infty$ can be changed for the space $B$ of bounded weakly continuous functions. The limit procedure implies that $\Delta u \in L^2(Q_T) \cap B(I;\, L^2(\Omega))$, hence the "normal regularity" of $u$ along the boundary must be of the same degree as that one in the tangential direction. Using (3.1), the above derived results and the imbedding theorem, we prove the following proposition:

THEOREM 3.3. *Let $\partial\Omega$ be of the class $C^{1,1}$ and $u_0 \in H^2(\Omega)$. Under suppositions of Theorem 3.2 the solution $u$ of* (2.6) *(or of its modification* *corresponding to* (3.25)) *belongs to* $C^0(I;\ H^{2-\eta}(\Omega))$ *for every* $\eta > 0$ *and* $\Delta u \in B(I;\ L^2(\Omega))$. *Therefore for* $N = 2$ $u$ *is* $(1-\eta)$*-Hölder continuous in the space variables and* $(\frac{1}{2}-\eta)$*-Hölder continuous in time on* $\bar{Q}_T$*, the closure of* $Q_T$*, for each* $\eta > 0$. *For* $N = 3$ *the Hölder exponent for the space variables and time on* $\bar{Q}_T$ *is* $\frac{1}{2}-\eta$, $\frac{1}{4}-\eta$, *respectively, for* $\eta > 0$ *arbitrary.*

## 4 – Appendix

LEMMA a.1. *Let* $g \in L^2(\Omega)$, $u_0 \in W^{1,2}(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Then for each* $\varepsilon > 0$ *there exists a unique weak solution* $u \equiv u_\varepsilon$ *of the problem* (2.11) *such that*

$$u \in L^\infty(I;\ W^{1,2}(\Omega)), \tag{a.2}$$

$$\dot{u} \in L^\infty(I;\ L^2(\Omega)) \cap L^2(I;\ W^{1,2}(\Omega)), \tag{a.3}$$

$$\ddot{u} \in L^2(I;\ L^2(\Omega)) \tag{a.4}$$

*and for almost every* $t \in (0,T)$, *the weak formulation*

$$\int_\Omega \left(\ddot{u}v + \nabla u \nabla v + \nabla \dot{u} \nabla v - gv + \frac{1}{\varepsilon}F'(u)v\right) dx = 0 \tag{a.5}$$

*is satisfied for every* $v \in W^{1,2}(\Omega)$.

REMARK A.1 It follows from (a.2)–(a.4) that

$$u \in C^{\frac{1}{2}}(I;\ W^{1,2}(\Omega)) \quad \text{and}$$
$$\dot{u} \in C^{\frac{1}{2}}(I;\ L^2(\Omega))\ .$$

So, the initial conditions have a good sense.

SKETCH OF THE PROOF *(i)* Unicity. Let $u_1$, $u_2$ be two solutions satisfying (a.2)–(a.5) and let $u = u_2 - u_1$. Then $u(0) = 0$, $\dot{u}(0) = 0$ and for every $v \in W^{1,2}(\Omega)$,

$$\int_\Omega (\ddot{u}v + \nabla u \nabla v + \nabla \dot{u} \nabla v)\, dx = \frac{1}{\varepsilon}\int_\Omega (F'(u_1) - F'(u_2))\, v\, dx\,. \tag{a.6}$$

Taking $v = \dot{u}(t)$ as a test function in (a.6), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\dot{u}|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla \dot{u}|^2\,dx = \tag{a.7}$$
$$= \frac{1}{\varepsilon}\int_\Omega (F'(u_1) - F'(u_2))\,\dot{u}\,dx\,.$$

Because of (2.9)

$$|\,F'(\xi_1) - F'(\xi_2)\,| \le c\max(|\xi_1|, |\xi_2|)\ . \tag{a.8}$$

After integrating (a.7) in time from 0 to $t$, we have

$$\int_{\Omega_t} |\dot{u}|^2\,dx \le \frac{cM}{\varepsilon}\iint_{Q_t} |\dot{u}|^2\,dx\,d\tau\ , \tag{a.9}$$

where $M = \iint_{Q_t} (\max(|u_1|, |u_2|))^2\,dx$. Using the Gronwall lemma, we obtain

$$\int_{\Omega_t} |\dot{u}|^2\,dx = 0 \qquad \text{for almost every } t \in (0,T). \tag{a.10}$$

*(ii)* The existence is solved by the Galerkin method. Let $\{w_j\}_{j=1}^{\infty}$ is an orthogonal dense set in $W^{1,2}(\Omega)$. Put $u^m(t,x) = \sum_{i=1}^m c_i^m(t)w_i(x)$. As usual, the system $(j = 1, 2, \ldots, m)$ of ordinary differential equations

(a.11)
$$\int_\Omega \left(\ddot{u}_m w_j + \nabla u_m \nabla w_j + \nabla \dot{u}_m \nabla w_j - g w_j + \frac{1}{\varepsilon}F'(u_m)w_j\right)dx = 0\,,$$

(a.12)
$$u_m(0) = u_{0m} \qquad \dot{u}_m = u_{1m}\,,$$

has a solution $u_m$ defined on $(0,T)$. Note that $u_{0m}$ ($u_{1m}$ respectively) is the projection in $W^{1,2}(\Omega)$ onto the space spanned by $\{w_1, w_2, \ldots, w_m\}$.

Multiplying the $j$–th equation in (a.11) first by $\dot{c}_j^m(t)$ and then by $\ddot{c}_j^m(t)$, summing for $j = 1, 2, ..., m$ and integrating over $(0, t)$, we get the following a priori estimates:

$$\text{(a.13)} \qquad \int\limits_{\Omega_t} |\dot{u}_m|^2\,dx + \int\limits_{\Omega_t} |\nabla u_m|^2\,dx + \iint\limits_{Q_t} |\nabla \dot{u}_m|^2\,d\tau\,dx \le \; const.,$$

$$\text{(a.14)} \qquad \iint\limits_{Q_t} |\ddot{u}_m|^2\,dx\,d\tau + \int\limits_{\Omega_t} |\nabla \dot{u}_m|^2\,dx \le \; const..$$

So, there exists a subsequence, still denoted $u_m$, and $u$, such that

$$\text{(a.15)} \qquad \begin{aligned} u_m &\rightharpoonup u \quad \text{*–weakly in } L^\infty(I;\, W^{1,2}(\Omega)),\\ \dot{u}_m &\rightharpoonup \dot{u} \quad \text{*–weakly in } L^\infty(I;\, W^{1,2}(\Omega)),\\ \ddot{u}_m &\rightharpoonup \ddot{u} \quad \text{weakly in } L^2(I;\, L^2(\Omega)). \end{aligned}$$

Then passing to the limit in (a.11), we see that $u$ will be a solution of (a.2)–(a.6), if

$$\text{(a.16)} \qquad \int\limits_{\Omega_t} F'(u_m)\,w_j\,dx \to \int\limits_{\Omega_t} F'(u)\,w_j\,dx.$$

However, this follows from Lebesgue's Dominated Convergence Theorem, because $|F'(u_m)| \le c\,|u_m|$. □

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