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On Ruscheweyh derivatives of meromorphic functions

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RIASSUNTO – *Si denota con* Σ *la classe di funzioni* $f(z) = z^{-1} + a_0 + a_1 z + \dots$, *analitiche nel disco unitario* $\{z\colon 0 < |z| < 1\}$. *Si denota con* Σ_n *la classe di funzioni* $f \in \Sigma$ *che verificano la condizione*

$$\operatorname{Re} S_n(f(z)) = \operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{n-1}{n} \quad (|z| < 1,\ n = 1,2,\dots)$$

dove $D^n f(z) = f(z) * \frac{1}{z}(1 - (\frac{z}{z-1})^n)$ *e* $*$ *è la convoluzione di Hadamard. Una funzione* $f \in \Sigma$ *si dice che appartiene alla classe* $\Sigma_n(\alpha,\beta)$ *se* $\operatorname{Re}\{n(\alpha,\beta)S_n(f(z)) - \alpha(n+1)S_{n+1}(f(z))\} > n - \alpha - 1$, *dove* α,β *sono numeri reali e* $n = 1,2,\dots$. *In questo lavoro si dimostrerà che* $\Sigma_n(\alpha,\beta) \subset \Sigma_n$. *Infine si studia una classe di operatori integrali definiti in* Σ_n.

ABSTRACT – *Let* Σ *denote the class of functions* $f(z) = z^{-1} + a_0 + a_1 z + \dots$, *which are analytic in the annulus* $\{z\colon 0 < |z| < 1\}$. *Let* Σ_n *denote the class of functions* $f \in \Sigma$ *which satisfy the condition*

$$\operatorname{Re} S_n(f(z)) = \operatorname{Re}\frac{D^{n+1}f(z)}{D^n f(z)} > \frac{n-1}{n} \quad (|z| < 1,\ n = 1,2,\dots)$$

where $D^n f(z) = f(z) * \frac{1}{z}(1 - (\frac{z}{z-1})^n)$ *and* $*$ *is the Hadamard convolution. A function* $f \in \Sigma$ *is said to belong to the class* $\Sigma_n(\alpha,\beta)$ *if* $\operatorname{Re}\{n(\alpha,\beta)S_n(f(z)) - \alpha(n+1)S_{n+1}(f(z))\} > n - \alpha - 1$, *where* α,β *are real numbers and* $n = 1,2,\dots$. *In this paper we shall show that* $\Sigma_n(\alpha,\beta) \subset \Sigma_n$. *Finally we study a class of integral operators defined on* Σ_n.

KEY WORDS – *Univalent meromorphic functions - Convolution - Radius of convexity - Starlike functions - Convex functions.*

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1 – Introduction

Let A be the class of functions $f(z) = z + a_2 z^2 + \dots$ which are regular in the unit disc $E = \{z\colon |z| < 1\}$. Let $f(z) = z^{-1} + a_0 + a_1 z + \dots$ be regular in $E - \{0\}$. Denote this class of functions by Σ. The Hadamard product or convolution of two functions $f, g \in \Sigma$ is denoted by $f \star g$. Let

$$(1.1) \qquad D^n f(z) = f(z) \star \frac{1}{z}\left(1 - \left(\frac{z}{z-1}\right)^n\right),$$

which implies that

$$(1.2) \qquad D^n f(z) = \frac{(-1)^{n-1}}{(n-1)!} z^{n-1} f^{(n-1)}(z),$$

where $n = 1, 2, \dots$ and $z \in E$. We shall refer to $D^n f$ as the n th order Ruscheweyh derivative of a meromorphic function f.

In this paper we shall define two new classes which are Σ_n and $\Sigma_n(\alpha, \beta)$. Let Σ_n denote the class of functions $f \in \Sigma$ that satisfy

$$(1.3) \qquad \operatorname{Re} S_n\big(f(z)\big) = \operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \frac{n-1}{n}$$

for $z \in E$ and $n = 1, 2 \dots$ Note that $S_n(f(0)) = 1$ for all n. Also note that $\Sigma_1 = \Sigma^*$ and $\Sigma_2 \equiv \Sigma_K$ are class of functions that are known as the meromorphic starlike and convex functions, respectively. We denote by $\Sigma_n(\alpha, \beta)$ the set of all functions f in Σ such that

$$(1.4) \qquad \operatorname{Re} M_n(\alpha, \beta, f(z)) > n - \alpha - 1, \qquad (z \in E),$$

where

$$(1.5) \qquad M_n(\alpha, \beta, f(z)) = n(\alpha + \beta) S_n(f(z)) - \alpha(n+1) S_{n+1}(f(z)),$$

and α, β are real numbers and $n = 1, 2, \dots$. For each n the class $\Sigma_n(\alpha, \beta)$ reduces to the class of meromorphic functions: $\Sigma_n(0, 1) \equiv \Sigma_n$ and $\Sigma_n(-1, 1) \equiv \Sigma_{n+1}$.

In section 2 we shall show that $\Sigma_n(\alpha, \beta) \subset \Sigma_n$. Substituting $n = 1$ in the above relation it follows that $\Sigma_1(\alpha, \beta) \subset \Sigma^*$. This result is a generalization of the result obtained by BAJPAJ and MEHROK in [1] when $\gamma = 0$. In section 3 we study Libera integral operator on the class Σ_n. We show that this operator preserves the class Σ_n.

2 – The classes Σ_n and $\Sigma_n(\alpha, \beta)$

THEOREM 1. *$\Sigma_n(\alpha,\beta) \subset \Sigma_n$ for all $n = 1, 2, \dots, \alpha > 0$ and $0 < \beta < 1$.*

PROOF. It can easly be verifyed from (1.1) that

$$z\Big(D^n f(z)\Big)' = (n-1)D^n f(z) - nD^{n+1}f(z) \tag{2.1}$$

for all $n = 1, 2, \dots$. Let $f \in \Sigma_n(\alpha, \beta)$. Let $\omega(z)$ be a regular function in E defined by

$$S_n(f(z)) = \frac{n + (n-2)\omega(z)}{n(1+\omega(z))}. \tag{2.2}$$

Clearly $\omega(0) = 0$ and $\omega(z) \neq -1$. To complete the proof we need to show that

$$\operatorname{Re} S_n(f(z)) > \frac{n-1}{n}, \qquad (z \in E \quad \text{and} \quad n = 1, 2, \dots).$$

To this end, it is sufficient to show $|\omega(z)| < 1$, $z \in E$. Taking the logarithmic derivative of both sides of (2.2) and using (2.1) we get

$$(n+1)S_{n+1}(f(z)) = 1 + nS_n(f(z)) + \frac{2z\omega'(z)}{(1+\omega(z))(n+(n-2)\omega(z))}. \tag{2.3}$$

Substituting from (2.2) and (2.3) in (1.4) we obtain

$$\begin{aligned} M_n(\alpha,\beta,f(z)) = &-\alpha + \beta\frac{n+(n-2)\omega(z)}{1+\omega(z)} + \\ &- 2\alpha\frac{z\omega'(z)}{(1+\omega(z))(n+(n-2)\omega(z))}. \end{aligned} \tag{2.4}$$

We claim that $|\omega(z)| < 1$, $z \in E$. For otherwise by the lemma of JACK [4] there exists $z_0 \in E$ such that result to (2.4) we get

$$\operatorname{Re} M_n(\alpha,\beta,f(z_0)) - (n-\alpha-1) = -(n-1)(1-\beta) - \frac{2\alpha k(n-1)}{|n+(n-2)\omega(z_0)|^2} < 0$$

which is a contradiction to our hypothesis that $f \in \Sigma_n(\alpha,\beta)$. Hence $|\omega(z)| < 1$ and from (2.2) we conclude that $f \in \Sigma_n$.

We shall need the following lemma ([2], p. 25).

LEMMA. *If $\omega(z)$ is regular in E and satisfies the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in E$, then*

$$|z\omega'(z) - \omega(z)| \le \frac{|z|^2 - |\omega(z)|^2}{1-|z|^2}, \qquad (|z| < 1). \tag{2.5}$$

THEOREM 2. *Let $f \in \Sigma_n$, $\alpha > 0$ and $0 < \beta < 1$. Then*

$$\frac{2}{\alpha n} \operatorname{Re} M_n(\alpha,\beta,f(z)) + 2 \ge \begin{cases} P_1(r) & \text{for } R_0 \le R_1 \\ P_2(r) & \text{for } R_0 \ge R_1 \end{cases}$$

where

$$P_1(r) = \frac{(\alpha+2\beta)(n+(n-2)r)}{\alpha n(1+r)} + \frac{(n-2)(1+r)}{n+(n-2)r},$$

$$P_2(r) = \frac{4}{n}\left[\frac{(n-2)(\alpha+\beta)(n-(n-2)r^2)}{\alpha(1-r^2)}\right]^{1/2} - \frac{2(n-(n-2)r^2)}{1-r^2}$$

$$R_0^2 = \frac{\alpha(n-1)(n-(n-2)r^2)}{n^2(\alpha+\beta)(1-r^2)}$$

$$R_1 = \frac{n+(n-2)r}{n(1+r)}.$$

The result is sharp.

PROOF. Since $f \in \Sigma_n$ we can write

$$S_n(f(z)) = \frac{n + (n-2)\omega(z)}{n(1+\omega(z))}, \tag{2.6}$$

where $\omega(z)$ is regular in E, $\omega(0) = 0$ and $|\omega(z)| < 1$. As in Theorem 1 we find that from (2.6) and (1.5)

$$\begin{aligned} M_n(\alpha,\beta,f(z)) + n =& n\frac{\alpha+2\beta}{2}p(z) + \alpha\frac{n-2}{2}\frac{1}{p(z)} + \\ & -\frac{2}{n}\frac{z\omega'(z) - \omega(z)}{(1+\omega(z))^2(p(z)}, \end{aligned} \tag{2.7}$$

where $p(z) = \frac{n+(n-2)\omega(z)}{n(1+\omega(z))}$. From (2.5) and (2.7) we have

$$\begin{aligned} \frac{2}{\alpha n}\operatorname{Re} M_n(\alpha,\beta,f(z)) + 2 \geq & \operatorname{Re}\left\{\frac{\alpha+2\beta}{\alpha}p(z) + \frac{n-2}{np(z)}\right\} + \\ & -\frac{|z|^2|p(z) - \frac{n-2}{n}|^2 - |1-p(z)|^2}{(1-|z|^2)|p(z)|}. \end{aligned}$$

An application of Lemma 1, KARUNAKARAN [5], with $C = 1 + \frac{2\beta}{\alpha}$, $D = \frac{n-2}{n}$ and $B = 1$ gives immediately the inequality stated in Theorem 2.

THEOREM 3. *Let $f \in \Sigma_n$ and $n = 1, 2, \ldots$ Then $f \in \Sigma_{n+1}$ holds for $|z| < \rho(n)$, where*

$$\rho(n) = n\left(n^2 - 2n + 8 + 4\sqrt{(n^2-2n+4)}\right)^{-1/2}$$

PROOF. For $f \in \Sigma_n$ let $p(z)$ be the regular function defined in E by

$$S_n(f(z)) = \frac{n-1+p(z)}{n}. \tag{2.8}$$

Here $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in E. Logarithmic differentiation of (2.8) and from (2.1) should yield

$$(n+1)S_{n+1}(f(z)) = n + p(z) + \frac{zp'(z)}{n-1+p(z)}. \tag{2.9}$$

The conclusion of the theorem follows immediately by Corollary 1 of RUSCHEWEYH and SING [6] or Theorem 1 of YOSHIKAWA and YOSHIKAI [7].

COROLLARY. *Taking $n = 1$, it follows that if $f \in \Sigma^*$ that $f \in \Sigma_K$ for $|z| < 2 - \sqrt{3}$.*

3 – Integral operators

Let c be a real number. We define h_c by

$$h_c(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{c}{c+k+1} z^k, \qquad 0 < |z| < 1.$$

Let the operator $L\colon \Sigma \to \Sigma$ be defined by $F = L(f)$, where

$$F(z) = cz^{-c-1} \int_0^z t^c f(t)dt, \qquad c > 0.$$

Then the function F can be written in the form $f(z) \star h_c(z)$. We shall refer to L as the Libera integral operator. We first give a condition of $f \in \Sigma$ for which the function $L(f)$ belongs to Σ_n.

THEOREM 4. *$S_n(f(z))$ is given by (1.3). Let $f \in \Sigma$ and satisfies the condition*

$$\operatorname{Re} S_n(f(z)) > \frac{n-1}{n} - \frac{1}{2n(c+1)}, \tag{3.1}$$

where $c > 0$, $n = 1, 2, \dots$, then the function $F = L(f) \in \Sigma_n$ for $F \neq 0$ in $E - \{0\}$.

PROOF. Since $F(z) = f(z) \star h_c(z)$ it can be easily verified that

$$z(D^n F(z))' = cD^n f(z) - (c+1)D^n F(z). \tag{3.2}$$

Let $\omega(z)$ be a regular function in E defined by

$$S_n(F(z)) = \frac{n + (n-2)\omega(z)}{n(1+\omega(z))}. \tag{3.3}$$

Here $\omega(0) = 0$ and $\omega(z) \neq -1$ in E. Logarithmic derivative (3.3) and using (3.2) we obtain

$$\begin{aligned} cS_n(f(z)) = & \frac{c}{n}\frac{n + (n-2)\omega(z)}{1+\omega(z)} + \\ & - \frac{2z\omega'(z)}{(n + (n-2)\omega(z))(1+\omega(z))}\frac{D^{n+1}F(z)}{D^n f(z)}. \end{aligned} \tag{3.4}$$

We can write the identity (2.1) for F

$$z(D^n F(z))' = (n-1)D^n F(z) - nD^{n+1}F(z). \tag{3.5}$$

From (3.2) and (3.5), after a simple computation we get

$$c\frac{D^n f(z)}{D^n F(z)} = \frac{c + (c+2)\omega(z)}{1+\omega(z)}. \tag{3.6}$$

(3.4) in conjuktion with (3.6) gives

$$S_n(f(z)) = \frac{1 + \frac{n-2}{n}\omega(z)}{1+\omega(z)} - \frac{2}{n}\frac{z\omega'(z)}{(1+\omega(z))(c + (c+2)\omega(z))} \tag{3.7}$$

and the conclusion of the theorem follow from (3.7), as show in Theorem 1.

By $\Sigma^*(\gamma)$ we denote the class of functions $f \in \Sigma$ starlike in $E - \{0\}$ and satisfying in this region the condition

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \gamma.$$

COROLLARY. *If we put $n = c = 1$ in Theorem 4, we find that* $L\left(\Sigma^*(\frac{-1}{4})\right) \subset \Sigma^*$.

It is easy to show that if $f \in \Sigma_n$, then f satisfies the condition (3.1). Thus it follows from Theorem 4 that $L(\Sigma_n) \subset \Sigma_n$.

More precisely we state the result in:

THEOREM 5. *If $f \in \Sigma_n$ then the function $F = L(f)$ is again an element of Σ_n.*

COROLLARY. *Substituting $n = 1$ and $n = 2$ in the above theorem it follows that if $f \in \Sigma^*$ (or Σ_K), then $L(f) \in \Sigma^*$ (or Σ_K).*

THEOREM 6. *Let $F \in \Sigma_n$ and $c > 0$, $n = 1, 2, \ldots$ Let f be defined as $F = L(f)$. Then $f \in \Sigma_n$ for $0 < |z| < \sqrt{\frac{c}{c+2}}$. The result is sharp.*

PROOF. Since $F \in \Sigma_n$ we can write

$$S_n(F(z)) = \frac{1 + \frac{n-2}{n}\omega(z)}{1 + \omega(z)}, \qquad (z \in E, n = 1, 2, \ldots).$$

where $\omega(0) = 0$, $\omega(z) \neq -1$ and $|\omega(z)| < 1$ in E. As in Theorem 4 we find that

$$nS_n(f(z)) - n + 1 = \frac{1 - \omega(z)}{1 + \omega(z)} - \frac{2z\omega'(z)}{(1 + \omega(z))(c + (c + 2)\omega(z))}.$$

The conclusion of the theorem follows immediately as in Theorem 4 of GOEL-SOHI [3].

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