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Some classes related to the components of a finite group

M.O. VALERO OLTRA

RIASSUNTO – *Vengono studiate alcune classi di gruppi finiti, costruite mediante una classe di gruppi quasisemplici oppure di Fitting, allo scopo di analizzare le relazioni esistenti tra di esse.*

ABSTRACT – *In this paper we investigate certain classes of finite groups which are defined by a class of quasisimple groups or by a Fitting class, in order to analyze the relations between them.*

KEY WORDS – *Quasisimple - Semisimple - $\mathcal{N}$-constrained - Saturated.*

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1 – Introduction. Notation

All groups considered are finite. A non-trivial group G is said to be quasisimple if it is perfect and $G/Z(G)$ is simple. A quasisimple subnormal subgroup of a group G is called a component of G. A non-trivial group G is said to be semisimple if it is a central product of quasisimple groups. By convention the trivial group is semisimple. We denote the semisimple radical of G by $E(G)$. The basic properties of the components of a group may be found in ([1], Chapter X, §13).

We denote the class of nilpotent groups by $\mathcal{N}$ and the class of all groups by $\mathcal{E}$. A group G is $\mathcal{N}$-constrained if $C_G(F(G)) \leq F(G)$ ([4]) and this is equivalent to $E(G) = 1$([5]).

Given a class of groups $\mathcal{X}$, the class map E_Z is such that

$$E_Z\mathcal{X} = (G \in \mathcal{E}| \quad \text{there exists} \quad Z \leq Z(G)|G/Z \in \mathcal{X}).$$

In this paper we investigate certain classes of groups in relation to the components of a group.

In [3] M.J. IRANZO and F. PÉREZ MONASOR introduce the class $\mathcal{K}_{\mathcal{X}}$, where $\mathcal{X}$ is a class of quasisimple groups, and in [2] the class $\mathcal{Z}_{\mathcal{X}}$, where $\mathcal{X}$ is a Fitting class. We define the classes $\mathcal{Z}_{\mathcal{X}}$, $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}'_{\mathcal{X}}$ for every class of groups $\mathcal{X}$. Our main aim is the analysis of the relations existing between these classes.

First, we will define the class $\mathcal{K}_{\mathcal{X}}$ for every class of groups $\mathcal{X}$.

DEFINITION 1. *If $\mathcal{X}$ is a class of groups, let*

$$\mathcal{K}_{\mathcal{X}} = (G \in \mathcal{E}|Q \in \mathcal{X} \quad \textit{for every component} \quad Q \textit{ of } G).$$

Obviously $\mathcal{K}_{\mathcal{X}}$ is a Fitting class for every class $\mathcal{X}$. Moreover, if $\mathcal{X}$ is E_Z-closed, then $\mathcal{K}_{\mathcal{X}}$ is extensible, though the converse is false.

PROPOSITION 2. *Let $\mathcal{X}$ be a class of groups and let G be a group. If there exist components of G that are not $\mathcal{X}$-groups we denote by $E_{\mathcal{X}}(G)$ the product of the non $\mathcal{X}$-group components of G; if these components do not exist we say that $E_{\mathcal{X}}(G) = 1$. Then $G_{\mathcal{K}_{\mathcal{X}}} = C_G(E_{\mathcal{X}}(G))$.*

PROOF. If $E_{\mathcal{X}}(G) = 1$ it is obvious.

Suppose that $E_{\mathcal{X}}(G) \neq 1$ and set $M = C_G(E_{\mathcal{X}}(G))$; we have $E_{\mathcal{X}}(G) \trianglelefteq G$ and in consequence $M \trianglelefteq G$; moreover, it is obvious that $M \in \mathcal{K}_{\mathcal{X}}$.

Now let $N \trianglelefteq G$ such that $N \in \mathcal{K}_{\mathcal{X}}$. If L is a component of N, then $L \leq M$. Therefore $E(N) \leq M$. Moreover, $[[E_{\mathcal{X}}(G), N], E_{\mathcal{X}}(G)] \leq [[E(G), N], E_{\mathcal{X}}(G)] \leq [E(N), E_{\mathcal{X}}(G)] = 1$. Therefore by the three-subgroup Lemma we have $[E_{\mathcal{X}}(G), N] = 1$ and so $N \leq M$.

Next we will define the class $\mathcal{Z}_{\mathcal{X}}$ for every class of groups $\mathcal{X}$.

DEFINITION 3. *If $\mathcal{X}$ is a class of groups, let*

$$\mathcal{Z}_{\mathcal{X}} = (G \in \mathcal{E}| \quad \textit{there exists a maximal normal } \mathcal{X}-\textit{subgroup } M \textit{ of } G|C_G(M) \leq M)$$

THEOREM 4. *Let $\mathcal{X}$ be a class of groups closed for central products and let $\mathcal{N} \subseteq \mathcal{X}$. If $G \in \mathcal{K}_{\mathcal{X}}$, then $C_G(M) \leq M$ for every maximal normal $\mathcal{X}$-subgroup M of G. In particular, $\mathcal{K}_{\mathcal{X}} \subseteq \mathcal{Z}_{\mathcal{X}}$.*

PROOF. Let $G \in \mathcal{K}_{\mathcal{X}}$. Let M be a maximal normal $\mathcal{X}$-subgroup of G. Then there exists $K \leq G$ such that K is semisimple and $[M, K] = 1$, in such a way that $E(G) = E(M)K$. As $KM \in \mathcal{X}$ it follows that $K \leq M$ and therefore $E(G) \leq M$.

Thus we have $C_G(M) \trianglelefteq C_G(E(G))$. Therefore $C_{C_G(M)}(F(C_G(M))) \leq F(C_G(M))$ and as $F(C_G(M)) \in \mathcal{X}$ and $[F(C_G(M)), M] = 1$, then $F(C_G(M)) \leq M$. As $C_G(M) = C_{C_G(M)}(F(C_G(M)))$, we obtain $C_G(M) \leq M$.

REMARKS. a) In Theorem 4 the assumption $\mathcal{N} \subseteq \mathcal{X}$ is not superfluous, since if $\mathcal{X} = (G \in \mathcal{E} | G$ is abelian) and $G = SL(2,3)$, then $M = Z(G)$ is the unique maximal normal $\mathcal{X}$-subgroup of G and $C_G(M) \not\leq M$; in consequence $G \notin \mathcal{Z}_{\mathcal{X}}$ and it is obvious that $G \in \mathcal{K}_{\mathcal{X}}$.

b) It is easy to prove that if $\mathcal{X}$ is a class of groups closed for direct products, saturated and such that $\mathcal{K}_{\mathcal{X}} \subseteq \mathcal{Z}_{\mathcal{X}}$, then $\mathcal{N} \subseteq \mathcal{X}$.

c) In Theorem 4 the condition that $\mathcal{X}$ is closed for central products is not superfluous, as can be seen by considering $\mathcal{X} = \mathcal{N} \cup (S_3)$.

Now we will investigate two new classes, which we will denote by $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}'_{\mathcal{X}}$.

DEFINITION 5. *If $\mathcal{X}$ is a class of groups, let*

$$\mathcal{J}_{\mathcal{X}} = (G \in \mathcal{E} | \text{ there exists a maximal normal } \mathcal{X} - \text{subgroup } M \text{ of } G | C_G(M) \text{ is } \mathcal{N} - \text{constrained}).$$

PROPOSITION 6. *Let $\mathcal{X}$ be a class of groups.*

i) If $\mathcal{X}$ is n-closed, then $\mathcal{J}_{\mathcal{X}} \subseteq \mathcal{K}_{\mathcal{X}}$.

ii) If $\mathcal{X}$ is non-empty and closed for central products, then $\mathcal{K}_{\mathcal{X}} \subseteq \mathcal{J}_{\mathcal{X}}$.

PROOF. i) Let $G \in \mathcal{J}_{\mathcal{X}}$. Let Q be a component of G; as $E(G) \in \mathcal{X}$ it follows that $Q \in \mathcal{X}$. Therefore $G \in \mathcal{K}_{\mathcal{X}}$.

ii) Let $G \in \mathcal{K}_{\mathcal{X}}$. Then $E(G) \in \mathcal{X}$ and there exists a maximal normal $\mathcal{X}$-subgroup M of G such that $E(G) \leq M$. Since $C_G(M) \trianglelefteq C_G(E(G))$, then $C_G(M)$ is $\mathcal{N}$-constrained and therefore $G \in \mathcal{J}_{\mathcal{X}}$.

REMARKS. a) There exists a class of groups $\mathcal{X}$ such that $\mathcal{K}_{\mathcal{X}} = \mathcal{J}_{\mathcal{X}}$, and which is neither n-closed nor closed for central products. In consequence the converses of Proposition 6 are false.

It is enough to consider $\mathcal{X} = (1, C_4)$.

b) That $\mathcal{X}$ is an n-closed class of groups does not imply $\mathcal{K}_{\mathcal{X}} = \mathcal{J}_{\mathcal{X}}$.

Consider $\mathcal{X} = (1, A_5, A_6)$.

c) That $\mathcal{X}$ is a non-empty class of groups closed for central products does not imply $\mathcal{K}_{\mathcal{X}} = \mathcal{J}_{\mathcal{X}}$.

Consider $\mathcal{X} = (G \in \mathcal{E} | G = G_1 G_2 \dots G_n$ with $n \in \mathbb{N} \setminus \{0\}$ and where $[G_i, G_j] = 1$ when $1 \leq i \neq j \leq n$ and $G_i \simeq S_5$ for every $i \in \{1, 2, \dots, n\})$.

DEFINITION 7. *If $\mathcal{X}$ is a class of groups, let*

$$\mathcal{J}'_{\mathcal{X}} = (G \in \mathcal{E} | C_G(M) \quad \textit{is} \quad \mathcal{N}-\textit{constrained for every maximal normal} \\ \mathcal{X}-\textit{subgroup of}\, G).$$

REMARKS. a) Obviously if $\mathcal{X}$ is a class of groups closed for central products, then $\mathcal{K}_{\mathcal{X}} \subseteq \mathcal{J}'_{\mathcal{X}}$, though the converse is false; this inclusion is strict in general.

b) There exists an n-closed class of groups $\mathcal{X}$ such that

$$\mathcal{Z}_{\mathcal{X}} \subsetneq \mathcal{J}_{\mathcal{X}} \subsetneq \mathcal{K}_{\mathcal{X}} \quad \text{and} \quad \mathcal{J}'_{\mathcal{X}} \subsetneq \mathcal{J}_{\mathcal{X}}$$

Consider $\mathcal{X} = (1, C_2, A_5, A_6)$.

THEOREM 8. *Let $\mathcal{X}$ be a class of groups. Then*

$$\mathcal{Z}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{Z}_{\mathcal{X}}} = \mathcal{J}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{J}'_{\mathcal{K}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{J}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{X}} = \\ = (G \in \mathcal{E} | C_G(G_{\mathcal{K}_{\mathcal{X}}}) \leq G_{\mathcal{K}_{\mathcal{X}}}) \subseteq \mathcal{K}_{\mathcal{J}'_{\mathcal{X}}}$$

and in general this inclusion is strict. Moreover, if $1 \in \mathcal{X}$, then $\mathcal{K}_{\mathcal{X}} = \mathcal{K}_{\mathcal{J}'_{\mathcal{X}}}$.

PROOF. Let $\mathcal{X}$ be any class of groups.

We have $\mathcal{K}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{X}}$ and also that $\mathcal{K}_{\mathcal{X}}$ is a Fitting class such that $\mathcal{N} \subseteq \mathcal{K}_{\mathcal{X}}$. In consequence we obtain

$$\mathcal{Z}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{J}_{\mathcal{K}_{\mathcal{X}}} = \mathcal{J}'_{\mathcal{K}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{X}} = (G \in \mathcal{E} | C_G(G_{\mathcal{K}_{\mathcal{X}}}) \leq G_{\mathcal{K}_{\mathcal{X}}}).$$

Obviously $\mathcal{K}_{\mathcal{X}} \subseteq \mathcal{K}_{\mathcal{Z}_{\mathcal{X}}} \subseteq \mathcal{K}_{\mathcal{J}_{\mathcal{X}}}$.

Let us see that $\mathcal{K}_{\mathcal{J}_{\mathcal{X}}} \subseteq \mathcal{K}_{\mathcal{X}}$; let $G \in \mathcal{K}_{\mathcal{J}_{\mathcal{X}}}$ and let Q be a component of G; as $Q \in \mathcal{J}_{\mathcal{X}}$ it follows that there exists a maximal normal $\mathcal{X}$-subgroup M of Q such that $C_Q(M)$ is $\mathcal{N}$-constrained and necessarily $M = Q$ and hence $Q \in \mathcal{X}$; therefore $G \in \mathcal{K}_{\mathcal{X}}$.

Hence $\mathcal{K}_{\mathcal{X}} = \mathcal{K}_{\mathcal{Z}_{\mathcal{X}}} = \mathcal{K}_{\mathcal{J}_{\mathcal{X}}}$.

Let us see that $\mathcal{K}_{\mathcal{X}} \subseteq \mathcal{K}_{\mathcal{J}'_{\mathcal{X}}}$; let $G \in \mathcal{K}_{\mathcal{X}}$ and let Q be a component of G; then Q is the unique maximal normal $\mathcal{X}$-subgroup of G and in consequence $Q \in \mathcal{J}'_{\mathcal{X}}$. To see that this inclusion in strict in general consider $\mathcal{X} = (C_7)$.

If $1 \in \mathcal{X}$, then $\mathcal{K}_{\mathcal{J}'_{\mathcal{X}}} \subseteq \mathcal{K}_{\mathcal{J}_{\mathcal{X}}}$ and we obtain $\mathcal{K}_{\mathcal{X}} = \mathcal{K}_{\mathcal{J}'_{\mathcal{X}}}$.

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INDIRIZZO DELL'AUTORE:

M.O. Valero Oltra - Plaza de Honduras, 29 - 46022 Valencia - Spain