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Some sensitivity conditions for infinite groups

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RIASSUNTO – *Si caratterizzano i gruppi risolubili nei quali ogni sottogruppo normale è pronormale-sensitivo. Tali gruppi risultano essere T-gruppi. Inoltre si descrivono i gruppi risolubili i cui sottogruppi infiniti godono della proprietà della χ-sensitività, quando χ è la normalità, la pronormalità, la massimalità.*

ABSTRACT – *We characterize soluble groups in which all normal subgroups are pronormal-sensitive. Such groups turn out to be T-groups. Moreover soluble groups with the χ-sensitivity condition for infinite subgroups are described, where χ is normality, pronormality, maximality.*

KEY WORDS – *Sensitivity - Infinite subgroup - Pronormal subgroup.*

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1 – Introduction

Let χ be a subgroup property and write $H \leq_\chi G$ if H is a χ-subgroup of the group G. Following P. VENZKE [18] we say that a subgroup K of the group G is *χ-sensitive* if the following holds:

(*i*) $H \leq_\chi K \implies \exists X \leq_\chi G : X \cap K = H$

(*ii*) $H \leq_\chi G$ and $K \not\leq H \implies H \cap K \leq_\chi K$.

In fact the property χ considered by VENZKE in [18] is that of being a maximal subgroup. From results in his work and [14] it follows that in a finite soluble group the normal subgroups are maximal-sensitive if and only if all (proper) subgroups are intersection of maximal subgroups.

Later M. EMALDI [6] has shown that the hypothesis of finiteness is unnecessary. Several authors have studied groups with sensitivity conditions for various properties χ, among them we quote [19], [12] and [1]. In the last work S. BAUMAN has considered normality as χ and shown that G is a finite group then every subgroup of G is normal-sensitive if and only if G is a soluble T-group. Here, as usual, by a *T-group* we mean a group whose subnormal subgroups are normal. Much work has been done on T-groups; our general reference on this topic will be [16], from which we will borrow terminology. Bauman's result has been extended by B. BRUNO and M. EMALDI in [2]. They have shown that if G is locally finite or soluble group, it has the property considered by Bauman if and only if it is a $\overline{T}$-group, i.e., all of its subgroups are T-groups. Furthermore in [3] groups with quasinormal-sensitive subgroups have been studied.

A subgroup H of a group G is said *pronormal* if for any $x \in G$, H is conjugate to H^x in $\langle H, H^x \rangle$ or, in other words, x belongs to the set $\langle H, H^x \rangle N_G(H)$. Clearly normal subgroups and maximal subgroups are pronormal. Also Sylow p-subgroups of a finite group G are pronormal and it is well-known that, if G is soluble, the prime p may be replaced by any set π of primes. Recently there has been some interest in pronormal subgroups. Generalizing a previous result of T.A. PENG [15], N.F. KUZENNYI and I.YA. SUBBOTIN [13] have characterized locally soluble groups in which all subgroups are pronormal (see below). Later EMALDI in [8] has shown that for soluble groups the properties of having all subgroups pronormal or that of having all subgroups pronormal-sensitive coincide.

In this work we characterize soluble groups in which all *normal* subgroups are pronormal-sensitive. They turn out to have property T but not necessarily $\overline{T}$, (see Theorem 2.2). Some results on this problem can also be found in [4]. Moreover we consider group classes obtained by imposing the χ-sensitivity condition (only) to *infinite* subgroups, where χ is normality, pronormality, maximality. Also soluble groups in which all infinite subgroup are pronormal are characterized.

Our notation is mostly standard; we refer in particular to [16] and [17].

2 – Groups whose normal subgroups are pronormal-sensitive

In this section we will study groups in which the condition of being pronormal-sensitive is imposed to the normal subgroups of the group. For the sake of compactness a group with such a property will be sometimes called $\mathcal{X}$-*group*. Clearly simple groups are $\mathcal{X}$-groups. Observe that if G is a hypercentral group then pronormal and normal subgroups coincide. In fact an ascendent pronormal subgroup is normal and so the hypercentre of a group normalizes every pronormal subgroup. Thus for a hypercentral group also properties T and $\mathcal{X}$ coincide. In Theorem 2.2 we will characterize soluble $\mathcal{X}$-groups. The proof of the theorem will be accomplished through a series of lemmas and propositions, some of which give characterizations of pronormal subgroups of soluble T-groups. The first proposition handles also the non-soluble case. If H is a pronormal subgroup of G we will write $H\, pn\, G$.

PROPOSITION 2.1. *Let G be a $\mathcal{X}$-group, $G' \leq N \leq G$ and $H \leq G$; then*

(i) $H\, pn\, N$ implies $H\, pn\, G$;
(ii) $H \lhd N$ implies $H \lhd G$;
(iii) N is a $\mathcal{X}$-group.

PROOF. (*i*) By Zorn's lemma there is a subgroup M maximal subject to the condition $N \leq M$ and $H\, pn\, M$. Since M is normal in G there is a pronormal subgroup P of G such that $H = P \cap M$. Then $H \lhd P$ and so $H\, pn\, MP$. It follows that $P \leq M$ and $H = P\, pn\, G$.

(*ii*) If $H \lhd N$ then by (*i*) H is at the same time pronormal and subnormal in G.

(*iii*) Let $H\, pn\, K \lhd N$. Then by the above $K \lhd G$ and so there is a pronormal subgroup P of G such that $H = P \cap K = (P \cap N) \cap K$ and $P \cap N\, pn\, N$. Furthermore if $P_1\, pn\, N$ and $K \lhd N$ then $P_1 = P \cap N$ where $P\, pn\, G$; hence $P_1 \cap K = P \cap K\, pn\, K$, as $K \lhd G$. □

We observe that if G is a T-group then the statement of Proposition 2.1 remains true even if we replace G' with $\gamma_3(G)$.

Before stating the main result of the section we give some definitions. A group G is said to have property $\mathcal{P}$ if and only if for any $H \leq G$ it holds:

(j) $H\, pn\, H^G \implies H\, pn\, G$
(jj) $H\, pn\, G$ and $N \lhd G \implies H \cap N\, pn\, G$

Of course, in view of (j), (jj) may be equivalently stated with $H \cap N\,pn\,N$ instead of $H \cap N\,pn\,G$. Moreover it is clear that $\mathcal{P}$ implies $\mathcal{X}$ and (j) implies T. Actually for soluble groups even $\mathcal{X}$ implies T because it turns out to be equivalent to $\mathcal{P}$, as we will see in the statement of Theorem 2.2.

We will call *KS-group* a soluble group all of whose subgroups are pronormal. By results in [13], non-periodic KS-groups are abelian and a periodic group G is a KS-group if and only if $G = B \ltimes A$, where $A = \gamma_3(G)$ is an abelian Hall subgroup without elements of order 2, every subgroup of A is normal in G, B is a Dedekind group and *any Sylow* $\pi(B)$-*subgroup of* G *complements* A. Moreover if C is a subgroup of the above KS-group G, then there is a subgroup B_1 such that $G = B_1 \ltimes A$ and $C = (B_1 \cap C) \ltimes (A \cap C)$. By results in [16] it follows that if G is a soluble $\overline{T}$-group and $\pi(\gamma_3(G))$ is finite, then G is a KS-group.

In fact KS-groups fall into the class of T-groups with the *splitting property*, i.e., periodic soluble T-groups G which split over the odd component of $\gamma_3(G)$, which coincides with $\gamma_3(G)$ if and only if the group has also $\overline{T}$. A periodic soluble T-group with the splitting property has the form $G = (B \times D) \ltimes_\theta A$, where A and B are abelian coprime $2'$-groups, D is a Sylow 2-subgroup and the action by conjugation of $B \times D$ on A is realized by $\theta : B \times D \longrightarrow Paut\,A$, where $Paut\,A$ is the group of power automorphism of A (see Theorem 5.2.1 of [16]).

THEOREM 2.2. *Let G be a soluble group. Then the normal subgroups of G are pronormal-sensitive if and only if G is a T-group and one of the following holds:*

(i) G is hypercentral;

(ii) G has an abelian non-periodic subgroup of index 2;

(iii) $G = (B \times D) \ltimes A$, where A and B are abelian coprime $2'$-groups, D is a Sylow 2-subgroup of G and $G/\gamma_3(D)$ is a KS-group.

Furthermore G has all normal subgroups pronormal-sensitive if and only if G has $\mathcal{P}$.

About case *(iii)* we note that clearly $\gamma_3(D)$ is the 2-component of $\gamma_3(G)$ and if $G = (B \times D) \ltimes_\theta A$ then $\gamma_3(D) \leq \ker\theta$ and $G/\gamma_3(D)$ is canonically isomorphic to $(B \times (D/\gamma_3(D)) \ltimes_{\bar\theta} A$, where $\bar\theta$ is induced by θ. Moreover such a group G is a T-group whenever A is *quasi-central* in G, i.e., every subgroup of A is normal in G.

LEMMA 2.3. *Let G be a soluble $\mathcal{X}$-group. Then G is a T-group.*

PROOF. Let first G be nilpotent. Then pronormal subgroups are normal and from $H \lhd K \lhd G$ it follows $H = P \cap K \lhd G$, where $P \lhd G$. Hence G is a T-group in this case.

Let now G be any soluble $\mathcal{X}$-group. Clearly the derived series $\{G^{(i)}\}_{i\geq 0}$ of G is quasi-central; in fact by Proposition 2.1 any of its members has $\mathcal{X}$, so that, if $G^{(i+1)} \leq H \leq G^{(i)}$, from $H \lhd G^{(j)}$ it follows $H \lhd G^{(j-1)}$ for any j, where $1 \leq j \leq i$. For any i, let $C_i = C_G(G^{(i)}/G^{(i+1)})$. Thus G/C_i is isomorphic to a group of power automorphisms of $G^{(i)}/G^{(i+1)}$ and so is abelian. Hence $G' \leq C_i$ and the series $\{G^{(i)}\}_{i>0}$ is a central series for G', which is therefore nilpotent. By the above and Proposition 2.1 any subgroup of G' is normal in G; in particular G' is a Dedekind group.

Let finally $H \lhd H^G \lhd G$. There is no loss of generality in supposing $H \cap G' = 1$. Then H is abelian and $H^G = H \times (H^G \cap G')$ is a Dedekind group; hence HG' is nilpotent by Fitting's Theorem. Since HG' is a $\mathcal{X}$-group we get $H \lhd HG'$ and so $H \lhd G$, again by Proposition 2.1. □

The next lemma gives a necessary condition for a subgroup of a soluble periodic T-group to be pronormal. Then Proposition 2.5 will settle the periodic case.

LEMMA 2.4. *Let G be a periodic soluble T-group, A_2 the 2-component of an abelian normal subgroup A of G such that $\pi(A) \cap \pi(G/A) \subseteq \{2\}$ and H any subgroup of G. Then H pn HA if and only if $H \lhd HA_2$.*

PROOF. If H *pn* HA we have $H \lhd HA_2$, since A_2 is contained in the hypercentre of G. Conversely let $H \lhd HA_2$ and, without loss of generality, suppose $H \cap A = 1$. Then it is enough to show that, for any $a \in A_{2'}$, H is conjugate to H^a in $\langle H, H^a \rangle$. On the other hand $\langle H, H^a \rangle = H \ltimes A_1 = H^a \ltimes A_1 \leq H \ltimes \langle a \rangle$, where $A_1 \leq \langle a \rangle$. The conjugacy follows now from the Schur-Zassenhaus Theorem, since $\langle a \rangle$ is finite. □

PROPOSITION 2.5. *Let G be a periodic soluble group. Then the following are equivalent:*

(a) G is a $\mathcal{X}$-group;

(b) $G = (B \times D) \ltimes A$, where A and B are abelian coprime $2'$-groups, A is quasi-central in G, D is a Sylow 2-subgroup of G which is a T-group and $G/\gamma_3(D)$ is a KS-group;

(c) G is a T-group and the subgroups of G normalized by the 2-component of $\gamma_3(G)$ are pronormal;
(d) G has $\mathcal{P}$.

PROOF. Let L_2 be a Sylow 2-subgroup of $L = \gamma_3(G)$.

$(a) \implies (b)$ By Lemma 2.3, G is a T-group. Let $\bar{G} = G/L_2$. Then by Lemma 2.4 for any $\bar{H} \leq \bar{G}$ it holds $\bar{H}\, pn\, \bar{H}\bar{L}$; therefore $\bar{H}\, pn\, \bar{G}$ by Proposition 2.1. Thus $\bar{G}$ is a KS-group. Then $\bar{G}$ splits over $\bar{L}$, say $\bar{G} = \bar{K} \ltimes \bar{L}$. If A is the $2'$-component of L then $G = K \ltimes A$ and G has the splitting property. Therefore $G = (B \times D) \ltimes_\theta A$, where clearly $\gamma_3(D) = L_2$ and also the requirements on A, B, D are satisfied.

$(b) \implies (c)$ By results in [16] G is a T-group. We observe that $L = \gamma_3(G) = \gamma_3(D)[BD, A]$; in particular $\gamma_3(D) = L_2$. Let now $H \leq G$ and $H \triangleleft HL_2$. By Lemma 2.4 we get at once $H\, pn\, HA$. Let $\bar{G} = G/H \cap A$. Then $\bar{G} = (\bar{B} \times \bar{D}) \ltimes \bar{A}$ has the same form as G. Thus there is no loss of generality in assuming $H \cap A = 1$.

Observe that any Sylow $\pi(BD)$-subgroup K of G complements A. In fact $\gamma_3(D) \triangleleft G$ and $\gamma_3(D) \leq K$, hence it follows that $K/\gamma_3(D)$ complements $A\gamma_3(D)/\gamma_3(D)$ in the KS-group $G/\gamma_3(D)$ and $KA = G$. Now take K such that $H \leq K$. It follows $H \triangleleft H\gamma_3(D) = H\gamma_3(K)\, sn\, K$ and $H \triangleleft K$, for $K \simeq G/A$ is a T-group. Thus $H\, pn\, KA = G$.

$(c) \implies (d)$ Let $H \leq G$ and $N \triangleleft G$. If $H\, pn\, G$ then, by Lemma 2.4, $[H, L_2] \leq H$ and consequently $[H \cap N, L_2] \leq H \cap N$ which implies $H \cap N\, pn\, N$. On the other hand, if $H\, pn\, H^G$ then we have $HL_{2'} \triangleleft H^G L_{2'} \triangleleft G$, since $G/L_{2'}$ is hypercentral. Hence $HL_{2'} = H^G L_{2'}$ and $[H, L_2] \leq HL_{2'} \cap L_2 = H \cap L_2$ so that H is pronormal in G.

$(d) \implies (a)$ This is clear. □

As costumed in the treatment of soluble T-groups we will study the non-periodic case by distinguishing whether $Fit\, G = C_G(G')$ is periodic or not (type 1 or type 2, respectively). The next two statements settle the non-periodic case and complete the proof of Theorem 2.2.

PROPOSITION 2.6. *Let G be a non-abelian T-group of the form $G = \langle z, C \rangle$, where C is a non-periodic abelian subgroup with index 2. Then G has $\mathcal{P}$ and for any subgroup H of G not contained in C the following are equivalent:*
(a) H is pronormal in G;

(b) H is pronormal in H^G;
(c) $C/H \cap C$ is periodic and its 2-component has exponent at most 2.

PROOF. We first show the equivalence of (a), (b) and (c). Recall that z acts by conjugation on C as the inversion map.

$(a) \implies (b)$ This is obvious.

$(b) \implies (c)$ Let C_0 and C_1 be subgroups of G such that $H \cap C^2 \leq C_0 \leq C_1 \leq C^2$ and C_1/C_0 is a 2-group. Then C_1/C_0 is contained in the hypercentre of G/C_0 and $HC_0\, pn\ HC_1 \leq HC^2 = H^G$. It follows $HC_0 \triangleleft HC_1$ and so ${C_1}^2 = [H, C_1] \leq HC_0 \cap C^2 = C_0$. Therefore $C^2/H \cap C^2$ is periodic with no element of order 4. On the other hand, since $C^2 = \gamma_3(G)$, the 2-component of C^2 is radicable. Thus $C^2/H \cap C^2$ has no elements of order 2 and (c) is satisfied.

$(c) \implies (a)$ This follows from Lemma 2.4, since the 2-component of $C/H \cap C$ lies in the centre of $G/H \cap C$.

Finally, let us to show that G has $\mathcal{P}$. In fact we must only verify condition (jj). Then let $H\, pn\ G$ and $N \triangleleft G$. If N is contained in C there is nothing to prove. Otherwise N has the same form of G with $C^* = N \cap C$ in the role of C, since $C^*/H \cap C^*$ is periodic and has elementary abelian 2-component, by (c). It follows $H \cap N\, pn\ G$. □

LEMMA 2.7. *Let G be a non-periodic soluble T-group such that $C = C_G(G')$ is periodic. Then G is a $\mathcal{X}$-group if and only if it is hypercentral.*

PROOF. Let G be a $\mathcal{X}$-group. Since we just have to show that G' is contained in the hypercentre of G there is no loss of generality if we assume that G' is a p-group. Moreover, since G is generated by its elements of infinite order we may also assume $G = \langle x \rangle \ltimes G'$, where x is one of such elements. Let α the p-adic integer such that x acts on G' as the power automorphism of exponent α.

If $\langle x \rangle$ is pronormal in G, then so is $\langle x^{p-1} \rangle = \langle x \rangle \cap \langle x^{p-1} \rangle G'$ in $\langle x^{p-1} \rangle G'$, as the latter is normal in G. On the other hand $\langle x^{p-1} \rangle G'$ is hypercentral because x induces on G' the power automorphism of exponent α^{p-1}, which is congruent to 1 modulo p. Hence G' normalizes $\langle x^{p-1} \rangle$ and so $[x^{p-1}, G'] = 1$, which is a contradiction because no element of infinite order centralizes G'.

Thus $\langle x\rangle$ is not pronormal and so there is $a \in G'$ such that $a \notin \langle x, x^a\rangle = \langle x, xa^{1-\alpha}\rangle = \langle x\rangle\langle a^{1-\alpha}\rangle$. It follows that p divides $\alpha - 1$ and so $\langle x\rangle G' = G$ is hypercentral. The sufficency of the condition is evident. □

We make a final remark about the fact that the sensitivity property $\mathcal{X}$ together with solubility implies T. Actually what we really have used in the proof of Lemma 2.3 is that subnormal pronormal subgroups are normal. So let us give a definition. Let G be a group and $\mathcal{S}(G)$ a subset of the subgroup lattice $\ell(G)$ of G. We say that the $\mathcal{S}(G)$ is a *transitivity system* for G if the following hold:

(i') $H \triangleleft K \triangleleft G \implies \exists\, S \in \mathcal{S}(G) : S \cap K = H$

(ii') if $N \triangleleft G$ and $S \in \mathcal{S}(G)$, $S \cap N \; sn \; N \implies S \cap N \triangleleft N$.

Observe that, if $N = G$, condition (ii') states $\mathcal{S}(G) \cap sn(G) \subseteq n(G)$. Moreover, if $\mathcal{S}(G)$ is a transitivity system for G and $G' \leq G_1 \leq G$, by a slight modification of the proof of Proposition 2.1(i) one can prove that the normal subgroups of G_1 are normal in G and that the set $\mathcal{S}(G_1) = \{S \cap G_1 \mid S \in \mathcal{S}(G)\}$ is a transitivity system for G_1. Finally, by the same proof of Lemma 2.3 and the trivial observation that the set of normal subgroups is a transitivity system if and only if the group is a T-group, one gets the following:

THEOREM 2.8. *A soluble group is a T-group if and only if it has a transitivity system.*

3 – Conditions on infinite subgroups

In this section we shall study soluble groups in which *infinite* subgroups are χ-sensitive, where χ is normality, pronormality and maximality. Consequently we divide the section in three parts.

3.1 – Normality

We shall say that a group G has property B (respectively: property IB), or that G is a B-group (respectively: IB-group), if all its subgroups (respectively: infinite subgroups) are normal-sensitive. Furthermore a group whose infinite subgroups are T-groups is said a $\tilde{T}$-*group*. As quoted above, for soluble groups properties B and $\overline{T}$ coincide. A corresponding results hold for the classes IB and $\tilde{T}$.

PROPOSITION 3.1. *The following hold:*
(i) If G is an IB-group, then it is a $\tilde{T}$-group;
(ii) If G is a soluble $\tilde{T}$-group, then it is an IB-group.

Before proving Proposition 3.1 we have a look at $\tilde{T}$-groups. Clearly all subgroups of $\tilde{T}$-groups are IT-groups, i.e.,groups in which infinite subnormal subgroups are normal. The class of IT-groups has been studied by S. Franciosi and F. de Giovanni [9] and H. Heineken [10]. It comes out that infinite periodic soluble IT-groups are either T-groups or Prüfer-by-finite. Similarly, infinite soluble groups whose subgroups are IT-groups (say $\overline{IT}$*-groups*) are either $\overline{T}$-groups or Prüfer-by-finite; moreover a Prüfer-by-finite T-group is a $\overline{T}$-group, unless its finite residual is a 2-group, by Theorem 6.1.1 in [16]. Thus *an infinite soluble group is a $\tilde{T}$-group if and only if it is either a $\overline{T}$-group or a T-group which is a finite extension of a Prüfer 2-group*, where the sufficency of the condition is ensured by a result of H. Heineken and J. C. Lennox [11], from which it follows that subgroups with finite index of a soluble T-group are T-groups. Note that this implies that *an infinite soluble group has property $\tilde{T}$ if and only if it has both properties T and $\overline{IT}$.*

PROOF OF PROPOSITION 3.1. (*i*) Clearly G may be assumed infinite. Let $H \triangleleft K \triangleleft G$ and N be the normal core in G of $N_G(H)$. If K is finite, then N has finite index in G. Thus in any case we have $H \triangleleft KN$ and KN is infinite. It follows that there is a normal subgroup N_1 of G such that $H = N_1 \cap KN \triangleleft G$. We have seen that G is a T-group. Since subgroups of an IB-group are IB-groups G is a $\tilde{T}$-group.

(*ii*) Again assume G infinite. If G is a $\overline{T}$-group it is a B-group, (see [2]). Then let G be not a $\overline{T}$-group. By the results in the previous paragraph G is finite extension of a Prüfer 2-group R (by a finite $\overline{T}$-group, of course). Let $H \triangleleft K \leq G$, where K is infinite and so $R \leq K$. If H is infinite there is nothing to prove, since G/R is a B-group. Let then H be finite and U be the subgroup of G formed by the elements of odd order. Since G/UR is a Dedekind group it holds $HU \triangleleft KU \triangleleft G$ and HU is a finite B-group. It follows the existence of a normal subgroup N of

HU (and hence of G) such that $H = N \cap (K \cap HU) = N \cap K$, what we wanted. □

Because of Proposition 3.1 it is desirable to give an explicit description of soluble $\tilde{T}$-groups which are not $\overline{T}$-groups. This follows easily from what we observed above, as they have the splitting property, for they have only finitely many elements of odd order.

THEOREM 3.2. *Let G be an infinite soluble group. Then G has property $\tilde{T}$ and not $\overline{T}$ if and only if $G = (S \times E \times B) \rtimes A$, where:*

(i) A and B are abelian groups with finite coprime odd orders;

(ii) E is an elementary abelian 2-group;

(iii) every subgroup of A is normal in G;

(iv) either $S = \langle z, D\rangle$ or $S = Q \rtimes D$, where D is a Prüfer 2-group, z has order 2 or 4, Q is isomorphic to the quaternion group of order 8 and $[S, D] \neq 1$.

3.2 – Pronormality

We begin by focusing our attention on groups in which every infinite subgroup is pronormal. Since pronormality and subnormality together imply normality it is clear that these groups are $I\overline{T}$-groups. On the other hand, in a finite $\overline{T}$-group all subgroups are pronormal. Hence in a locally finite $\overline{T}$-group all finite subgroups are pronormal, (recall that for non-abelian $\overline{T}$-groups local finiteness and solubility are equivalent). Indeed this property characterizes $\overline{T}$-groups among locally finite groups (see [7]).

PROPOSITION 3.3. *Let G be an infinite soluble group. Then the following are equivalent:*

(a) all infinite subgroups of G are pronormal;

(b) all subgroups of G are pronormal or G is an extension of a Prüfer p-group by a finite T-group.

PROOF. (a) $\Longrightarrow$ (b) Clearly G is an $I\overline{T}$-group. Let G be not Prüfer-by-finite. Then G is a $\overline{T}$-group and all finite subgroups of G are also pronormal.

$(b) \implies (a)$ Let G be Prüfer-by-finite. Then the infinite subgroups of G contain the finite residual R of G and hence are pronormal, as G/R is a finite $\overline{T}$-group. □

LEMMA 3.4. *If every infinite subgroup of the group G is pronormal-sensitive, then G is a $\tilde{T}$-group.*

PROOF. Since the class under consideration is subgroup closed it is enough to show that G is a T-group if it is infinite. Let $H \triangleleft K \triangleleft G$. As in the proof of Proposition 3.1, $N_G(H)$ is infinite and there is a pronormal subgroup P of G such that $H = P \cap N_G(H) = N_P(H)$. Thus H is a self-normalizing subnormal subgroup of P. It follows that $H = P \triangleleft G$. □

We can now characterize soluble groups whose infinite subgroups are pronormal.

THEOREM 3.5. *Let G be an infinite soluble group. Then the following are equivalent:*
(a) every infinite subgroup of G is pronormal-sensitive;
(b) all infinite subgroups of G are pronormal and G is a T-group;
(c) all subgroups of G are pronormal or G is a Prüfer-by-finite T-group.

PROOF. By Proposition 3.3 (b) and (c) are equivalent. Let us show that also (a) and (c) are.

$(a) \implies (c)$ By the previous lemma G is a $\tilde{T}$-group. If it is not Prüfer-by-finite then it is a $\overline{T}$-group and so its finite subgroups are pronormal. Let H be an infinite subgroup of G. By Lemma 2.4, $H \, pn \, HG'$ and so as in the proof of part (i) of Proposition 2.1 we conclude $H \, pn \, G$.

$(c) \implies (a)$ If all subgroups of G are pronormal there is nothing to prove. Let then G be a Prüfer-by-finite T-group. If the finite residual R of G is not a 2-group, then G is a $\overline{T}$-group and all finite subgroups of G are pronormal, hence all subgroups of G are pronormal. Therefore we can assume that R is the 2-component of $L = \gamma_3(G)$. In this case we have that *a subgroup H of G is pronormal if and only if it is normalized by the finite residual R*. In fact, let H be normalized by R and, without loss of generality $H \cap L = 1$. Since G has a finite number of elements of odd order, there is a complement K of $L_{2'}$ in G containing H. We get

$H\,pn\ HL_{2'}$ and $H \triangleleft K$, by Lemma 2.4 and for $H \triangleleft HR = H\gamma_3(K) \triangleleft K$ Thus $H\,pn\ KL_{2'} = G$, as we claimed.

Verifying that (a) holds is now easy. In fact from $H\,pn\ K$ and $R \leq K$ it follows that $[H, R] \leq H$, since R is contained in the hypercentre of G. Moreover from $H\,pn\ G$ and $R \leq K$ it follows $[H \cap K, R] \leq [H, R] \cap K \leq H \cap K$ and so $H \cap K\,pn\ G$. □

About Theorem 3.5 we observe that in the proof we have also seen that in the statement of (c) the Prüfer subgroup may be taken a 2-group.

3.3 – Maximality

In this last part we study soluble groups whose infinite subgroups are maximal-sensitive. As we said in the introduction, a soluble group G has all normal subgroups maximal-sensitive if and only if G is an *IM-group*, i.e., a group in which every subgroup is intersection of maximal subgroups. In [14] F. MENEGAZZO has showed that a soluble group is an *IM*-group if and only if it is periodic and has a normal Hall subgroup N such that both N and G/N are elementary abelian and every subgroup of N is normal in G. In particular a periodic soluble T-group has property *IM* if and only if all its Sylow subgroups have prime exponent. Moreover soluble groups whose infinite subgroups are intersection of maximal subgroups coincide with those soluble groups which are either *IM*-groups or extensions of a Prüfer group by a finite *IM*-group, see Theorem 2.6 in [5].

We first state a lemma on maximal subgroups and then show that the groups under consideration are IT-groups.

LEMMA 3.6. *Let G be a group, M a maximal subgroup of G and $H \leq M$. If H is a subnormal subgroup of G with defect 2, then $H^G \leq M$. In particular the intersection of all maximal subgroups of G containing H is a normal subgroup of G.*

PROOF. Let by contradiction $G = H^G M$ and $x \in G$. Then there are $k \in H^G$ and $y \in M$ such that $x = ky$. If follows $H^x = H^y \leq M$ and $H^G \leq M$, the contradiction we wanted. The rest is now clear, since the intersection of all maximal subgroups of G containing H is the preimage of the Frattini subgroup of G/H^G. □

LEMMA 3.7. *Let G be a group .Then the following hold:*
(i) if all subgroups of G are maximal-sensitive, then G is a T-group;
(ii) if all infinite subgroups of G are maximal-sensitive, then G is an IT-group

PROOF. Let $H \triangleleft H^G \triangleleft G$ and $H \neq H^G$. Furthermore let $H \leq H_1 \leq H_2 \leq H^G$, where H_1 is maximal in H_2. Then there is a maximal subgroup M of G such that $H_1 = M \cap H_2$. From the previous lemma it follows $H_2 \leq H^G \leq M$ and so $H_1 = H_2$, a contradiction.

The second part of the statement has the same proof, where H is chosen infinite. □

We can now pass to the characterization of the groups under consideration.

THEOREM 3.8. *Let G be a soluble infinite group. Then the following are equivalent:*
(a) every infinite subgroup of G is maximal-sensitive;
(b) every infinite subgroup of G is intersection of maximal subgroups;
(c) G is either an IM-group or an extension of a Prüfer group by a finite IM-group.

PROOF. We have already quoted that (*b*) and (*c*) are equivalent. Let R be the finite residual of G from now on.

(*a*) $\Longrightarrow$ (*b*) If G is Prüfer-by-finite we have that G/R is an *IM*-group by the characterization of Emaldi-Venzke. So assume G be not Prüfer-by-finite.

Let first G be a T-group. We show that G is periodic. On one hand, if there is an element of infinite order x in $C = C_G(G') = Fit\, G$ then $G/\langle x^4 \rangle$ is an *IM*-group with an element of order 4, a contradiction. On the other hand, if C is periodic then the set P of all elements of finite order is a subgroup of G. Thus every subgroup of the torsion-free group G/P is maximal-sensitive and so $G = P$ is periodic. If by contradiction G has the minimal condition on (sub)normal abelian subgroups, then it is a Chernikov group (see [17]). In this case, if R has rank greater than 1 it has an infinite proper subgroup R_1. Thus G/R_1 is an *IM*-group with a Sylow subgroup of unbounded exponent, a contradiction. Therefore G has a normal abelian subgroup $A = \mathrm{Dr}_{i \in \mathbb{Z}} A_i$, where each A_i has prime order. Then for any primary cyclic subgroup H of G there is a proper subgroup B of A with finite index (in A) such that $H \cap B = 1$. Since G/B is an *IM*-group, $H \simeq HB/B$ has prime order. Hence G itself is an *IM*-group.

Assume finally G is not a T-group. Then by results in [9] it is non-periodic and G' is a Prüfer group. This yields the contradiction that G is periodic, since G/G' is an *IM*-group.

$(c) \implies (a)$ If G is an *IM*-group there is nothing to prove. Let then G/R be a finite *IM*-group . Clearly any subgroup of G is infinite if and only if it contains R. Moreover if M is an maximal subgroup of an infinite subgroup of G then $R \leq M$, as maximal subgroups of Chernikov groups have finite index. Since every subgroup of G/R is maximal-sensitive, the proof is complete. □

Note that there exist infinite soluble non-T groups whose infinite subgroups are intersection of maximal subgroups. An example is the group $G = \langle \alpha \rangle \ltimes (A \times B)$, where A is a Prüfer p-group, $B = \langle b \rangle$ has order p and α is the automorphism of $A \times B$ which fixes A pointwise and maps b to $a_1 b$, where a_1 is an element of A with order p. Observe that the quotient G/A has exponent p.

We finally exhibit a metabelian group G, whose infinite normal subgroup are maximal-sensitive, which is not an *IT*-group. Let A and B be Prüfer 2-groups with the following standard presentations:

$$A = \langle a_n \mid n \in N_0,\ a_0 = 1,\ a_{n+1}^2 = a_n \rangle$$
$$B = \langle b_n \mid n \in N_0,\ b_0 = 1,\ b_{n+1}^2 = b_n \rangle .$$

Let α be the automorphism of $A \times B$ defined by $a_n^\alpha = b_n$ and $b_n^\alpha = a_n^{-1} b_n^{-1}$ and let $G = \langle \alpha \rangle \ltimes (A \times B)$. Then it can be verified that the only proper infinite normal subgroup of G is $A \times B$, which has no maximal subgroup.

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