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Another characterization of the semi-classical orthogonal polynomials

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RIASSUNTO: *Si stabilisce una nuova caratterizzazione dei polinomi ortogonali semi-classici, generalizzando un risultato di Mc Charty; si esprimono cioè le derivate dei prodotti dei due polinomi consecutivi P_n e P_{n+1} in termini di P_n^2, P_nP_{n+1} e P_{n+1}^2. Con la nuova caratterizzazione si ottengono informazioni sugli zeri dei polinomi. Per i polinomi classici, in particolare, si riesce a dimostrare, indipendentemente dalla forma del funzionale lineare che definisce l'ortogonalità, che gli zeri sono tutti semplici.*

ABSTRACT: *In this paper we give a new characterization of semi-classical orthogonal polynomials, which generalizes Mc Carthy's result, i.e. we express the derivative of the product of two consecutive polynomials, say P_{n+1} an P_n, in terms of P_{n+1}^2, $P_{n+1}P_n$ and P_n^2. This characterization enables us to give information about the zeros of semi-classical orthogonal polynomials. Thus, for classical orthogonal polynomials we can assert - without any reference to the representation of the linear functional with respect to which they are orthogonal - that their zeros are simple.*

KEY WORDS: *Orthogonal polynomials – Semi-classical polynomials – Characterizations.*

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1 – Preliminaries

Let $\mathcal{L}$ be a regular linear functional in $\mathcal{P}'$ (the topological dual of the vector space of all complex polynomials one variable).

By "regular" we mean that the principal minor of the Hankel matrix is non singular i.e. $\det(\mathcal{L}_{k+j})_{k,j=0}^n \neq 0$, $n \geq 0$, where $\mathcal{L}_k = \langle \mathcal{L}, x^k \rangle$ and

$\langle \mathcal{L}, P \rangle$ denotes the value of the linear functional $\mathcal{L}$ applied to P.

It is well known result [5, p.11] that the regularity condition is equivalent to the existence of $\{P_n\}_{n\geq 0}$ a sequence of monic orthogonal polynomials with respect to $\mathcal{L}$, i.e.

- degree of $P_n = n$, $n \geq 0$
- $\langle \mathcal{L}, P_n P_m \rangle = K_n \delta_n$, $K_n \in \mathbb{C}^*$, $n, m \geq 0$.

As well, an orthogonal sequence with respect to a regular linear functional satisfies a three-term recurrence relation, specifically

$$\begin{aligned} &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geq 0 \\ &P_1(x) = x - \beta_0, \ P_0(x) = 1 \\ &(\beta_n, \gamma_n) \in \mathbb{C} \times \mathbb{C}^* . \end{aligned} \tag{1}$$

Let $\mathcal{U}$ be an element of $\mathcal{P}'$, let $\{B_n\}_{n\geq 0}$ be a sequence of monic polynomials. $\{B_n\}_{n\geq 0}$ is said to be quasi-orthogonal of orders s with respect to $\mathcal{U}$ [11] [12] [17] if and only if

$$\begin{aligned} &\langle \mathcal{U}, x^m B_n(x) \rangle = 0 \quad 0 \leq m \leq n-(s+1), \ n \geq s+1 , \\ &\exists r \geq s \quad \text{such that} \quad \langle \mathcal{U}, x^{r-s} B_r(x) \rangle \neq 0 . \end{aligned}$$

If $\langle \mathcal{U}, x^{r-s} B_r(x) \rangle \neq 0$ for any $r \geq s$, $\{B_n\}_{n\geq 0}$ is said to be strictly quasi-orthogonal of order s with respect to $\mathcal{U}$.

REMARK: A strictly quasi-orthogonal sequence of order zero is orthogonal.

2 – Semi-classical orthogonal polynomials

Generatively speaking, the seminal concept of the semi-classical orthogonal polynomials can be dated back to an ingenious yet little quoted paper of JACQUES SHOHAT [18]. He started with an orthogonal sequence $\{P_n\}_{n\geq 0}$ with respect to a weight function w which satisfies an homogenous first order linear differential equation with polynomial coefficients; he proved that $\{P'_{n+1}\}_{n\geq 0}$ is quasi-orthogonal of a certain order, and each P_n satisfies a second order differential equation of Laguerre-Perron type.

DEFINITION. *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with respect to $\mathcal{L}$. $\{P_n\}_{n\geq 0}$ is said to be semi-classical of class s*

(s for Shohat), if and only if the derivative sequence $\{P'_{n+1}\}_{n\geq 0}$ *is quasi-orthogonal of order* s*, with respect to a linear functional* $\tilde{\mathcal{L}}$ [8] [11] [15].

The semi-classical sequences can be characterized as follows:

$\mathbf{C}_1$. *There exists two polynomials* Ψ *of degree* $p \geq 1$ *and* ϕ *of degree* $t \geq 0$ *such that*

$$\Psi\mathcal{L} + D[\phi\mathcal{L}] = 0 \tag{2}$$

$$\prod_{c\in Z_\phi}\left(|r_c| + \left|(\Psi_c\mathcal{L})_0\right|\right) > 0$$

$$s = \max\{p-1, t-2\}$$

where

$$\langle \Psi\mathcal{L}, P\rangle : = \langle \mathcal{L}, \Psi P\rangle$$

$$\langle D[\mathcal{L}], P\rangle : = -\langle \mathcal{L}, P'\rangle\,, \quad P \in \mathcal{P}$$

Z_ϕ *the set of zeros of* ϕ, r_c *and* Ψ_c *are defined in the following way*

$$\phi(x) = (x-c)\phi_c(x)$$

$$\Psi(x) + \phi_c(x) = (x-c)\Psi_c(x) + r_c$$

and

$$(\Psi_c\mathcal{L})_0 = \langle \Psi_c\mathcal{L}, 1\rangle\,.$$

One can prove that $\tilde{\mathcal{L}} = \phi\mathcal{L}$. [3] [4].

$\mathbf{C}_2$. $\{P_n\}_{n\geq 0}$ *satisfies the following structure relation*

$$\begin{aligned} &\phi(x)P'_{n+1}(x) = \sum_{k=n-s}^{n+t} \theta_{n,k}P_k(x), \quad n \geq s, \\ &\exists r \geq s \quad \text{such that} \quad \theta_{r,s} \neq 0 \quad 0 \leq t \leq s+2 \end{aligned} \tag{3}$$

and s *is the smallest integer for which* (3) *holds.* [3] [11].

Relation (3) *can be written in the following compact form* [2] [6] [7] [10]

$$\phi(x)P'_{n+1}(x) = \frac{C_{n+1}(x) - C_0(x)}{2}P_{n+1}(x) - D_{n+1}(x)P_n(x)\,, \quad n \geq 0 \tag{4}$$

where

$$C_{n+1}(x) = -C_n(x) + \frac{2D_n(x)}{\gamma_n}(x-\beta_n)\,, \quad n \geq 0 \tag{5}$$

$$D_{n+1}(x) = -\phi(x) + \frac{\gamma_n}{\gamma_{n-1}} D_{n-1}(x) + \frac{D_n(x)}{\gamma_n}(x-\beta_n)^2 - C_n(x)(x-\beta_n), \quad n \geq 0 \tag{6}$$

$$C_0(x) = -\Psi(x) - \phi'(x) \tag{7}$$

$$D_0(x) = -(\mathcal{L}.\theta_0\Psi)(x) - (\mathcal{L}.\theta_0\phi)'(x) \tag{8}$$

$$D_{-1}(x) = 0$$

$$(\mathcal{L}.\theta_0\Psi)(x) := \left\langle \mathcal{L}_u, \frac{\Psi(x)-\Psi(u)}{x-u} \right\rangle .$$

$\phi(x)$ and $\Psi(x)$ are the same as in $\mathbf{C}_1$. β_n and γ_n are the coefficients of the tree-term recurrence relation (1) satisfied by $\{P_n\}_{n\geq 0}$. Degree of $C_n(x) \leq s+1$ and degree of $D_n(x) \leq s$.

$\mathbf{C}_3$. $\{P_n\}_{n\geq 0}$ *satisfies a linear second order differential equation of Laguerre-Perron type* [2] [9] [16]

$$\phi(x)D_n(x)P_n''(x) + \Big\{C_0(x)D_n(x) - W(\phi(x), D_n(x))\Big\}P_n'(x) + \\ + \Big\{W\Big(\frac{C_n(x)-C_0(x)}{2}, D_n(x)\Big) - D_n(x)\sum_{k=0}^{n-1}\frac{D_k(x)}{\gamma_k}\Big\}P_n(x) = 0$$

where $W(f,g) = fg' - f'g$, ϕ, C_n, *and* D_n *are the parameters introduced in the previous characterization.*

$\mathbf{C}_4$. *The formal Stieltjes function of* $\mathcal{L}$, *namely*

$$S(x) = -\sum_{k\geq 0}\frac{\mathcal{L}_k}{x^{k+1}}$$

satisfies a linear first order differential equation [9] [13]

$$\phi(x)S'(x) - C_0(x)S(x) - D_0(x) = 0\,, \tag{10}$$

where ϕ, C_0 *and* D_0 *are the same polynomials as in* $\mathbf{C}_2$. □

Notice that, if we set $s = 0$, C_1, C_2, C_3 and C_4 are satisfied by the four classical sequences (HERMITE, LAGUERRE, BESSEL, JACOBI) [1]. Therefore the semi-classical theory is a natural extension of the classical one.

In order to complete the analogy we are going to give another characterization of semi-classical polynomials which generalizes the one given by P.J. MC CARTHY [14] in the classical case.

C_5. *$\{P_n\}_{n\geq 0}$ is semi-classical of class s, if and only if $\{P_n\}_{n\geq 0}$ satisfies the following quadratic differential relation*

$$\begin{aligned}\phi(x)\Big[P_{n+1}(x)P_n(x)\Big]' &= \frac{D_n(x)}{\gamma_n}P^2_{n+1}(x)- \\ &-C_0(x)P_{n+1}(x)P_n(x) - D_{n+1}(x)P^2_n(x), \quad n \geq 0\end{aligned} \tag{12}$$

where ϕ, C_0, D_n and γ_n are the same parameters as above.

PROOF. Suppose that $\{P_n\}_{n\geq 0}$ is a semi-classical sequence, thus $\{P_n\}_{n\geq 0}$ satisfies eq.(4). Let us write (4) for the rank $(n-1)$, we have

$$\phi(x)P'_n(x) = \frac{C_n(x) - C_0(x)}{2}P_n(x) - D_n(x)P_{n-1}(x), \quad n \geq 1. \tag{13}$$

We substitute for $P_{n-1}(x)$ from (1) into (13), we get

$$\begin{aligned}\phi(x)P'_n(x) &= \frac{D_n(x)}{\gamma_n}P_{n+1}(x)+ \\ &+\left[\frac{C_n(x) - C_0(x)}{2} - \frac{D_n(x)}{\gamma_n}(x-\beta_n)\right]P_n(x).\end{aligned} \tag{14}$$

Using eq. (5), (14) becomes

$$\phi(x)P'_n(x) = \frac{D_n(x)}{\gamma_n}P_{n+1}(x) - \frac{C_n(x) + C_0(x)}{2}P_n(x), \quad n \geq 1. \tag{15}$$

Taking in account (5) for $n = 0$, (15) is still valid for $n = 0$, thus the range of validity of (15) is $n \geq 0$. If we multiply (4) and (15) by $P_n(x)$ and $P_{n+1}(x)$ respectively and we add the resulting equations, we get

$$\begin{aligned}\phi(x)\Big[P_{n+1}(x)P_n(x)\Big]' &= \frac{D_n(x)}{\gamma_n}P^2_{n+1}(x) \\ &- C_0(x)P_{n+1}(x)P_n(x) - D_{n+1}(x)P^2_n(x), \qquad n \geq 0\end{aligned}$$

which is exactly (12).

Conversely, suppose that $\{P_n\}_{n\geq 0}$ is a sequence of monic orthogonal polynomials satisfying (12), where $\{D_n\}_{n\geq 0}$ is a given sequence of bounded degree polynomials and satisfying

$$D_{n+2}(\beta_{n+1}) = -\phi(\beta_{n+1}) + \frac{\gamma_{n+1}}{\gamma_n} D_n(\beta_{n+1})$$

Let us make $n \longrightarrow n+1$ in (12), we have

$$\begin{aligned}\phi(x)\Big[P_{n+2}(x)P_{n+1}(x)\Big]' &= \frac{D_{n+1}(x)}{\gamma_{n+1}} P^2_{n+2}(x)\\ &- C_0(x)P_{n+2}(x)P_{n+1}(x) - D_{n+2}(x)P^2_{n+1}(x), \quad n \geq 0 .\end{aligned}$$

By using the three-term recurrence relation, we express $P_{n+2}(x)$ in terms of $P_{n+1}(x)$ and $P_n(x)$, after differentiating the new expression, we have

$$\begin{aligned}&\phi(x)P^2_{n+1}(x) + 2(x-\beta_{n+1})\phi(x)P_{n+1}(x)P'_{n+1}(x)\\ &-\gamma_{n+1}\phi(x)\Big[P_{n+1}(x)P_n(x)\Big]' = (x-\beta_{n+1})^2\frac{D_{n+1}(x)}{\gamma_{n+1}}P^2_{n+1}(x)\\ &-2(x-\beta_{n+1})D_{n+1}(x)P_{n+1}(x)P_n(x) - (x-\beta_{n+1})C_0(x)P^2_{n+1}(x)\\ &-D_{n+2}(x)P^2_{n+1}(x)+\gamma_{n+1}D_{n+1}(x)P^2_n(x)+\gamma_{n+1}C_0(x)P_{n+1}(x)P_n(x), n \geq 0 .\end{aligned}$$

We substitute for $\phi(x)[P_{n+1}(x)P_n(x)]'$ from (12) into the previous identity, then we cancel the term $P_{n+1}(x)$ and we re-arrange the terms, we get

$$\begin{aligned}&2(x-\beta_{n+1})\phi(x)P'_{n+1}(x)\\ &=\Big\{-D_{n+2}(x) - \phi(x) + \frac{\gamma_{n+1}}{\gamma_n}D_n(x)\\ &+\frac{D_{n+1}}{\gamma_{n+1}}(x-\beta_{n+1})^2 - (x-\beta_{n+1})C_0(x)\Big\}P_{n+1}(x)\\ &-2(x-\beta_{n+1})D_{n+1}(x)P_n(x), \quad n \geq 0 .\end{aligned} \tag{16}$$

If we set

$$\begin{aligned}(x-\beta_{n+1})C_{n+1}(x) &= -D_{n+2}(x) - \phi(x) + \frac{\gamma_{n+1}}{\gamma_n}D_n(x)\\ &+ \frac{D_{n+1}(x)}{\gamma_{n+1}}(x-\beta_{n+1})^2, \quad n \geq 0 .\end{aligned}$$

we get a new sequence $\{C_n\}_{n\geq 0}$ and the previous identity is exactly (6). If the degree of $D_n(x)$ is at most s, the degree of $C_n(x)$ does not exceed $s+1$.

Hence (16) becomes

$$\phi(x)P'_{n+1}(x) = \frac{C_{n+1}(x) - C_0(x)}{2}P_{n+1}(x) - D_{n+1}(x)P_n(x)\,, \quad n \geq 0\,,$$

which is exactly (4); we already see that identity (6) is satisfied, to complete the proof we have to show that identity (5) is also satisfied.

Let us expand the left member of (12), we have

$$\begin{aligned}&\phi(x)P'_{n+1}(x)P_n(x) + \phi(x)P_{n+1}P'_n(x)\\ &=\frac{D_n(x)}{\gamma_n}P^2_{n+1}(x) - C_0(x)P_{n+1}(x)P_n(x) - D_{n+1}(x)P^2_n(x)\,, \quad n \geq 0\,.\end{aligned}$$

We substitute for $\phi(x)P'_{n+1}(x)$ from (17) into the previous relation, we get

$$\phi(x)P'_n(x) = \frac{D_n(x)}{\gamma_n}P_{n+1}(x) - \frac{C_{n+1}(x) + C_0(x)}{2}P_n(x)\,, \quad n \geq 0\,, \tag{18}$$

in (18) we change n into $n+1$, we obtain

$$\begin{aligned}\phi(x)P'_{n+1}(x) = &\frac{D_{n+1}(x)}{\gamma_{n+1}}P_{n+2}(x)\\ &- \frac{C_{n+2}(x) + C_0(x)}{2}P_{n+1}(x)\,, \qquad n \geq -1\,,\end{aligned}$$

Using the three-term recurrence relation, which gives $P_{n+2}(x)$ in terms of $P_{n+1}(x)$ and $P_n(x)$, we have

$$\begin{aligned}\phi(x)P'_{n+1}(x) = &\left[\frac{D_{n+1}(x)}{\gamma_{n+1}}(x - \beta_{n+1}) - \frac{C_{n+2}(x) + C_0(x)}{2}\right]P_{n+1}(x)\\ &- D_{n+1}(x)P_n(x)\,, \quad n \geq -1\,.\end{aligned}$$

By comparison with (17), we get

$$\begin{aligned}C_{n+2}(x) &= -C_{n+1}(x) + 2\frac{D_{n+1}(x)}{\gamma_{n+1}}(x - \beta_{n+1})\,, \quad n \geq 0\\ C_1(x) &= -C_0(x) + 2\frac{D_0(x)}{\gamma_0}(x - \beta_0)\,.\end{aligned}$$

Which is exactly (5). □

3 – Applications

1. From characterization $\mathbf{C}_6$, we can extract informations about the zeros of $P_n(x)$. Suppose that $\zeta_{n,0}$ is a zero of $P_n(x)$ with multiplicity $m \geq 2$, it is straightforward that $\zeta_{n,0}$ is a zero of $D_n(x)$ with multiplicity $m-1$. Since the degree of $D_n(x)$ is bounded by s, m is less than or equal to $s+1$: furthermore, when n is large and s small almost all the zeros of $P_n(x)$ are simple.
2. We can specialize the result of $\mathbf{C}_6$ to various classical polynomials. The values of the ingredients involved in (12) are summarized in the following table:

	Hermite	Laguerre
β_n	0	$2n+1+\alpha$
γ_{n+1}	$\frac{n+1}{2}$	$(n+1)(n+1+\alpha)$
ϕ	1	x
Ψ	$2x$	$x-\alpha-1$
C_0	$-2x$	$-x+\alpha$
$\frac{C_{n+1}-C_0}{2}$	0	$n+1$
$\frac{D_n}{\gamma_n}$	-2	-1
	Bessel	**Jacobi**
β_n	$\frac{1-a}{(n-1+a)(n+a)}$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}$
γ_{n+1}	$-\frac{(n+1)(n-1+2a)}{(2n-1+2a)(n+a)^2(2n+1+2a)}$	$4\frac{(n+1)(n+1+\alpha+\beta)(n+1+\alpha)(n+1+\beta)}{(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)^2(2n+3+\alpha+\beta)}$
ϕ	x^2	x^2-1
Ψ	$-2(ax+1)$	$-(\alpha+\beta+2)x-\alpha+\beta$
C_0	$2(a-1)x+2$	$(\alpha+\beta)x+\alpha-\beta$
$\frac{C_{n+1}-C_0}{2}$	$(n+1)\left(x-\frac{1}{n+a}\right)$	$(n+1)\left(x+\frac{\beta-\alpha}{2n+2+\alpha+\beta}\right)$
$\frac{D_n}{\gamma_n}$	$2n-1+2a$	$2n+1+\alpha+\beta$

3. Let $\mathcal{H}_\mu$ be the linear functional with respect to which the generalized Hermite polynomials are orthogonal [5. p.157], $\mathcal{H}_\mu$ satisfies

$$(2x^2 - 2\mu - 1)\mathcal{H}_\mu + D[x\mathcal{H}_\mu] = 0$$

Let us note $H_n(x;\mu)$ the monic generalized Hermite polynomials, the three-term recurrence coefficients read as:

$$\beta_n = 0\,, \quad \gamma_{n+1} = \frac{1}{2}\Big\{n + 1 + \mu[1 + (-1)^n]\Big\}, \quad n \geq 0\,.$$

Thus $\mathcal{H}_\mu$ is regular if and only if $\mu \neq -n - \frac{1}{2}$, $n \geq 0$.
One can easily give all the elements quoted in $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_6$, namely

$$\begin{aligned} &\phi(x) = x, && \Psi(x) = 2x^2 - 2\mu - 1 \\ &C_0(x) = -2(x^2 - \mu), && D_0(x) = -2x \\ &\frac{C_{n+1}(x) - C_0(x)}{2} = -\mu[1 + (-1)^n], && \frac{D_{n+1}}{\gamma_{n+1}} = -2x, \quad n \geq 0\,. \end{aligned}$$

Hence (12) can be written as

$$\begin{aligned} x\Big[H_{n+1}(x;\mu)H_n(x;\mu)\Big]' &= -2xH_{n+1}^2(x;\mu) \\ &+2(x^2 - \mu)H_{n+1}(x;\mu)H_n(x;\mu) \\ &+\Big\{n + 1 + \mu[1 + (-1)^n]\Big\}xH_n^2(x;\mu), \quad n \geq 0\,. \end{aligned}$$

When μ tends to zero, $H_n(x;\mu)$ are reduced to the monic Hermite polynomials, the previous identity is reduced to the one which is satisfied by the monic Hermite polynomials.

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