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Exponential sums connected with Möbius function

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RIASSUNTO: *In questo lavoro si ottengono alcune formule di maggiorazione per le somme esponenziali della funzione aritmetica* $\mu_r^*(n)$, *che generalizza la funzione di Möbius, e della funzione* $h_r(n) = \sum_{d|n} \mu_r^*(d)$.

ABSTRACT: *We obtain upper bounds of an exponential sum of the arithmetical function* $\mu_r^*(n)$ *which generalizes the Möbius function and the functions* $h_r(n) = \sum_{d|n} \mu_r^*(d)$.

KEY WORDS: *Exponential sums - Möbius function - Summatory function.*

A.M.S. CLASSIFICATION: 11L07

1 – Introduction

H. DAVENPORT [2] obtained the following estimate: "If k is a positive number, and α is real, then

$$\sum_{n\le x} \mu(n)e(n\alpha) \ll x\log^{-k} x \tag{1.1}$$

uniformly in α as $x \to \infty$". Assuming that there are no Siegel zeros ([3] p.88-96), D. HAJELA and B. SMITH [4] obtained

$$\sum_{n\le x} \mu(n)e(n\alpha) \ll xe^{-c\sqrt{\log x}} \tag{1.2}$$

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H. WALUM [8] studying the formulae for periodic functions obtained

$$\sum_{n\le x}\lambda(n)e(n\alpha)\ll x\log^{-k}x \tag{1.3}$$

where $\lambda(n)=\sum_{u^2v=n}\mu(v)$. The purpose of this paper is to obtain similar estimates for the functions of type Möbius $\mu_r^*(n)$ and $h_r(n)$. In the case of Möbius function, this problem is concerned to estimate exponential sum over prime numbers on minor arcs. The function $\mu_r^*(n)$ is a generalization of the Möbius function, that is, if $r=1$ $\mu_1^*(n)=\mu(n)$ and for each $r\ge 1$ is $\mu_r^*(1)=1$, $\mu_r^*(n)=0$ if $p^{r+1}|n$ and $\mu_r^*(n)=(-1)^{\Omega(n)}$ if $n=\prod p_i^{\alpha_i}$, $0\le\alpha_i\le r$,$\Omega(n)=\sum\alpha_i$. Moreover $\mu_r^*(n)$ coincides with the function $\mu_{r+1}(n)$ defined by SASTRY [6]. The function $h_r(n)$ is for each odd integer $r\ge 1$, $h_r(1)=1$ and when $n>1$, $n=p_1^{\alpha_1}\dots p_s^{\alpha_s}$

$$h_r(n)=\begin{cases}1, & \text{if } \alpha_i<r,\alpha_i\equiv 0 \pmod 2), i=1,\dots s\\ 0, & \text{otherwise}.\end{cases}$$

And for each even integer $r\ge 0$, $h_r(1)=1$ and when $n>1$, $n=p_1^{\alpha_1}\dots p_s^{\alpha_s}$

$$h_r(n)=\begin{cases}1, & \text{if } \alpha_i\ge r \text{ or } (\alpha_i<r,\alpha_i\equiv 0 \pmod 2))\ i=1,\dots s\\ 0, & \text{otherwise}\end{cases}$$

$$F(s)=\sum_{n=1}^{\infty}\frac{\mu_r^*(n)}{n^s}=\frac{1}{\zeta(s)}\sum_{n=1}^{\infty}\frac{h_r(n)}{n^s}=\frac{\zeta(2s)}{\zeta(s)}\gamma_{r+1}(s)\quad \mathrm{Re}\, s>1 \tag{1.4}$$

with

$$\gamma_{r+1}(s)=\begin{cases}\dfrac{\zeta((r+1)s)}{\zeta(2(r+1)s)} & r \text{ even}\\[2ex] \dfrac{1}{\zeta((r+1)s)} & r \text{ odd}\end{cases} \tag{1.5}$$

and

$$h_r(n)=\sum_{d|n}\mu_r^*(d)=\begin{cases}\displaystyle\sum_{d^2m^{r+1}=n}|\mu(m)|, & \text{if } r \text{ even}\\ \displaystyle\sum_{d^2m^{r+1}=n}\mu(m), & \text{if } r \text{ odd}.\end{cases} \tag{1.6}$$

Thus we prove the following theorems.

THEOREM 1. *Let us suppose* $(a,q)=1$, *and* $|\alpha-\frac{a}{q}|<\frac{1}{q^2}$. *Then,*

(1.7) $$\sum_{n\le x}\mu_r^*(n)\ e(n\alpha)\ll \ln^3 x\left\{\frac{x}{q^{\frac{1}{2}}}+x^{\frac{1}{2}}q^{\frac{1}{2}}+x^{\frac{4}{5}}\right\}.$$

THEOREM 2. *If there are no Siegel zeros there is a positive constant* c_7 *such that*

(1.8) $$\|\sum_{n\le x}\mu_r^*(n)\ e(n\alpha)\|_\infty\ll x\ e^{-c_7\sqrt{\ln x}}.$$

THEOREM 3. i) *Let* $r\ge 0$ *and* α *real, then*

(1.9) $$\sum_{n\le x}h_r(n)e(n\alpha)\ll\begin{cases}\min\{x,\|\alpha\|^{-1}\}, & \textit{if}\quad r=0\\ e(\alpha), & \textit{if}\quad r=1.\end{cases}$$

ii) *For all* $r>1$ *and* α *real* $|\alpha-\frac{a}{q}|<x^{-5/4+1/2(r+1)}$

(1.10) $$\sum_{n\le x}h_r(n)e(n\alpha)\ll_r x^{1/2}q^{-1/2+\epsilon}+x^{\frac{3}{4}+\frac{1}{2(r+1)}}.$$

For $r>1$ odd and $\alpha=0$ from Theorem 14-2 [5] we deduce

$$\sum_{n\le x}h_r(n)=\gamma_{r+1}(1)x^{1/2}+O(x^{1/(r+1)}\exp\{-C(\log x)^{3/5}(\log\log x)^{-1/5}\})$$

being C a positive constant.

2 – Previous Lemmas

We will need the following auxiliary lemmas. To obtain estimates of the sum on the minor arcs, we begin using a property of $\mu_r^*(n)$ which is given in Lemma 1 and it has been obtained using the VAUGHAN's technique [7], p-27 .This is, beginning with a functional identity between the summatory function of certain Dirichlet series and using the unicity theorem of Dirichlet series, we can deduce other identity on the series coefficients. Then if $M(s) = \sum_{n\le U} \frac{\mu_r^*(n)}{n^s}$ and $F(s)$ is given by (1.4) we can write the following identity

$$F(s) = 2M(s) - \frac{M^2(s)}{F(s)} + \left(1 - \frac{M(s)}{F(s)}\right)(F(s) - M(s)). \tag{2.1}$$

For $\sigma > 1$ these functions can be expanded as Dirichlet series and such expansions are unique , thus we may compare coefficients . Hence the following lemma holds

LEMMA 1. *The function $\mu_r^*(n)$ is $\mu_r^*(n) = a_0(n) - a_1(n) - a_2(n)$ where*

$$\begin{aligned} a_0(n) &= \begin{cases} 2\,\mu_r^*(n) & n \le U \\ 0 & n > U \end{cases} \\ a_1(n) &= \sum_{\substack{mke=n \\ m\le U \\ k\le U}} \mu_r^{*-1}(e)\,\mu_r^*(m)\,\mu_r^*(k) \\ a_2(n) &= \sum_{\substack{mk=n \\ m>U \\ k>U}} \mu_r^*(m) \sum_{\substack{d/k \\ d\le U}} \mu_r^*(d)\,\mu_r^{*-1}\left(\frac{k}{d}\right) \end{aligned} \tag{2.2}$$

LEMMA 2. [4] *Let χ a character (mod)q. If there are no Siegel zeros:*

i) *there are constants $c > 0, c_0 > 0$ such that*

$$\sum_{n\le x} \mu(n)\chi(n) \ll x\, e^{-c\sqrt{\ln x}}, \qquad q \le e^{c_0\sqrt{\ln x}} \quad , \tag{2.3}$$

ii) *if* $(\ell, q) = 1$ *there is constant* c' *such that* ,

$$\sum_{\substack{m \le x \\ m \equiv \ell (mod q)}} \mu(m) \ll x\, e^{-c'\sqrt{\ln x}} \qquad q \le e^{c'\sqrt{\ln x}}. \tag{2.4}$$

LEMMA 3. *In the premises of Lemma 2 there are constants* c_1, c_2 *such that*

$$\sum_{n \le x} \mu_r^*(n)\chi(n) \ll x\, e^{-c_1\sqrt{\ln x}} \quad \text{for} \quad q \le e^{c_2\sqrt{\ln x}}. \tag{2.5}$$

PROOF. We note that

$$\zeta(2s)\gamma_{r+1}(s) = \sum_{n=1}^{\infty} \frac{h_r(n)}{n^s} \quad , \quad \mu_r^*(n) = \sum_{d/n} h_r(d)\, \mu(\frac{n}{d}). \tag{2.6}$$

then we have

$$\sum_{n \le x} \mu_r^*(n)\chi(n) = \sum_{d \le x} h_r(d)\chi(d) \sum_{\delta \le \frac{x}{d}} \mu(\delta) = \sum_{d \le Y} + \sum_{Y < d \le x} = S_1 + S_2.$$

In S_2 using the bound trivial for the inner sum and the propertie (see T-1 [1])

$$\sum_{n \le x} \frac{h_r(n)}{n} = \zeta(2)\gamma_{r+1}(1) - \gamma_{r+1}(1/2)x^{-1/2} + O(x^{-r/(r+1)}) \tag{2.7}$$

we obtain $S_2 \ll x^{\frac{3}{4}}$. In S_1, we take $Y = x^{\frac{1}{2}}$ and applying Lemma 2-i) we obtain the bound (2.5). Thus the Lemma is proved. □

LEMMA 4. *Let* $q \le \delta$ *and* $|\alpha - \frac{a}{q}| \le \frac{1}{qQ}$. *There are constants* $A > 0$, $B > 0$ *such that*

$$|\sum_{n \le x} \mu_r^*(n)\; e(\alpha n)| \ll (\delta + \frac{x}{Q}) x e^{-A\sqrt{\ln x}} \quad \text{for} \quad q \le e^{B\sqrt{\ln x}}. \tag{2.8}$$

PROOF. We denote by $S(\alpha)$ the sum in (2.8). Let $\alpha = \frac{a}{q} + z$ and M_n the sum:

$$M_n = \sum_{m \leq n} \mu_r^*(m)\, e(\frac{ma}{q}) = \sum_{\ell \leq q} e(\frac{\ell a}{q}) \sum_{\substack{m \leq n \\ m \equiv \ell (\mathrm{mod}\, q)}} \mu_r^*(m)$$

then by partial summation $|S(\alpha)| = |M_x\, e(xz) + \sum_{n \leq x-1} M_n\, e(nz)(1 - e(z))|$. Hence

$$|S(\alpha)| \leq (x|z| + 1) \max_{n \leq x} |M_n| \leq (\delta + \frac{x}{Q}) \max_{n \leq x} \max_{\ell \leq q} |\sum_{\substack{m \leq n \\ m \equiv \ell (\mathrm{mod}\, q)}} \mu_r^*(m)|$$

i) If $(\ell, q) = 1$ and

$$\sum_{\substack{m \leq x \\ m \equiv \ell (\mathrm{mod}\, q)}} \mu_r^*(m) = \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi}(\ell) \sum_{m \leq x} \mu_r^*(m) \chi(m)$$

by Lemma 3, there are positive constants c_3 and c_4 such that

$$(2.9) \qquad |\sum_{\substack{m \leq x \\ m \equiv \ell (\mathrm{mod}\, q)}} \mu_r^*(m)| \ll x\, e^{-c_3 \sqrt{\ln x}} \quad \text{for} \quad q \leq e^{c_4 \sqrt{\ln x}}$$

ii) If $(\ell, q) > 1$ we write

$$S^*(\ell) = \sum_{\substack{m \leq x \\ m \equiv \ell (\mathrm{mod}\, q)}} \mu_r^*(m) = \sum_{\substack{\delta \leq x \\ (\delta, q)/\ell}} h_r(\delta) \sum_{\substack{d \leq \frac{x}{\delta} \\ d \equiv \ell' \overline{\delta'} (\mathrm{mod}\, q')}} \mu(d) = \sum_{\delta \leq Y} + \sum_{Y < \delta \leq x} = S_1 + S_2.$$

being $q' = q/D$ and $\ell' = \ell/D$, $D = (\delta, q)/\ell \quad \delta\overline{\delta'} \equiv 1 (\mathrm{mod}\, q')$. We take $Y = x^{\frac{1}{2}}$ then by Lemma 2-i)

$$S_1 \ll \sum_{\substack{\delta \leq Y \\ (\delta, q)/\ell}} h_r(\delta) \frac{x}{\delta} e^{-c' \sqrt{\ln \frac{x}{\delta}}}, \quad \text{for} \quad q' \leq e^{c' \sqrt{\ln \frac{x}{\delta}}}.$$

Moreover the function $\frac{h_r(\delta)}{\delta}$ is non negative and $q' \le e^{c'\sqrt{\ln\frac{x}{Y}}}$ $\forall \delta \le Y$, that is $q' \le e^{c_6\sqrt{\ln x}}$. Then by (2.7) we deduce $S_1 \ll xe^{-c'\sqrt{\ln\frac{x}{Y}}} = xe^{-c_5\sqrt{\ln x}}$.

Also, from the bound trivial and (2.7) we obtain $S_2 \ll xY^{\frac{-1}{2}} = x^{\frac{3}{4}}$. Hence if there are no Siegel zeros then there are positive constants c_5 and c_6 so that

$$S^*(\ell) \ll xe^{-c_5\sqrt{\ln x}}, \text{for} \quad q \le e^{c_6\sqrt{\ln x}}. \tag{2.10}$$

Thus from (2.9) and (2.10) the lemma is deduced. □

3 – Proof of theorems

PROOF OF THEOREM 1. Using (2.1) we can write $S(\alpha)$ as $S = S_0 - S_1 - S_2$ where each S_i is

$$S_i = \sum_{n\le x} a_i(n)\ e(n\alpha) \quad , i = 0, 1, 2 \tag{3.1}$$

By (3.1) , (2.2) we trivially have $|S_0| \ll U$. To estimate S_1 , by (3.1) , (2.2) we have

$$S_1 = \sum_{\substack{mkt\le x\\ m\le U\\ k\le U}} \mu_r^{*-1}(t)\ \mu_r^*(m)\ \mu_r^*(k)\ e(mkt\alpha) =$$

$$= \sum_{n\le U^2} \sum_{t\le\frac{x}{n}} \mu_r^{*-1}(t) e(tn\alpha) \left(\sum_{\substack{m\le U\\ mk=n}} \sum_{k\le U} \mu_r^*(m)\ \mu_r^*(k) \right)$$

Hence we have

$$S_1 \ll \sum_{n\le U^2} d(n) \left| \sum_{t\le\frac{x}{n}} \mu_r^{*-1}(e)\ e(tn\alpha) \right| = \sum_{n\le U^2} d(n)|S_1^*| \tag{3.2}$$

being the inner sum for r odd

$$S_1^* = \sum_{t\le\frac{x}{n}} \mu_r^{*-1}(t)\ e(tn\alpha) = \sum_{h\le\frac{x}{n}} |\mu(h)| \sum_{m\le(\frac{x}{nh})^{\frac{1}{r+1}}} e(nhm^{r+1}\alpha)$$

Since $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, $|\theta| < 1$ we can write the sum S_1 in the form $S_1 = S_{1_{q\nmid n}} + S_{1_{q/n}}$.

To estimate these sums we observe that if $q \nmid n$ then by the bound trivial we deduce
$|S_1^*| \leq \frac{x}{n}$. Hence

$$S_{\substack{1\\ q\nmid n}} \ll x \sum_{\substack{n\leq U^2\\ q\nmid n}} \frac{d(n)}{n} \ll x\frac{d(q)}{q}\ln^2(U^2). \tag{3.3}$$

If $q|n$ and $\alpha = \frac{a}{q} + z$ with $|z| < \frac{1}{q^2}$ then by the Van der Corput Lemma

$$S_1^* \ll \sum_{h\leq \frac{x}{n}} |\mu(h)| \min\left\{\left(\frac{x}{nh}\right)^{\frac{1}{r+1}}, (|z|nh)^{-\frac{1}{r+1}}\right\} + O(\frac{x}{n})$$

Thus

$$S_{\substack{1\\ q/n}} \ll \sum_{\substack{n\leq U^2\\ q|n}} d(n) \sum_{h\leq \frac{x}{n}} |\mu(h)| \min\left\{\left(\frac{x}{nh}\right)^{\frac{1}{r+1}}, (|z|nh)^{-\frac{1}{r+1}}\right\} +$$

$$O\left(x \sum_{\substack{n\leq U^2\\ q|n}} \frac{d(n)}{n}\right). \tag{3.4}$$

When $U = x^{\frac{2}{5}}$ and using the estimates on $S_{1_{q\nmid n}}$, $S_{1_{q/n}}$ together, we get for S_1 and r odd

$$S_1 \ll \frac{x\ln^2 x}{q^{1-\epsilon}}. \tag{3.5}$$

The same bound is obtained if r even . Now we estimate $S_2, \forall r \geq 1$, by (2.3)

$$S_2 = \sum_{\substack{mk\leq x\\ m>U,k>U}} \mu_r^*(m) \sum_{\substack{d/k\\ d\leq U}} \mu_r^*(d)\mu_r^{*-1}(\frac{k}{d})\ e(mk\alpha) =$$

$$= \sum_{k>U} \left(\sum_{\substack{d/k\\ d\leq U}} \mu_r^*(d)\mu_r^{*-1}(\frac{k}{d})\right) \sum_{U<m\leq \frac{x}{k}} \mu_r^*(m)\ e(mk\alpha)$$

Since $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, $|\theta| < 1$ we can write the sum S_1 in the form $S_1 = S_{1_{q\nmid n}} + S_{1_{q/n}}$.

To estimate these sums we observe that if $q \nmid n$ then by the bound trivial we deduce
$|S_1^*| \leq \frac{x}{n}$. Hence

$$S_{1_{q\nmid n}} \ll x \sum_{\substack{n \leq U^2 \\ q\nmid n}} \frac{d(n)}{n} \ll x \frac{d(q)}{q} \ln^2(U^2). \tag{3.3}$$

If $q|n$ and $\alpha = \frac{a}{q} + z$ with $|z| < \frac{1}{q^2}$ then by the Van der Corput Lemma

$$S_1^* \ll \sum_{h \leq \frac{x}{n}} |\mu(h)| \min\left\{\left(\frac{x}{nh}\right)^{\frac{1}{r+1}}, (|z|nh)^{-\frac{1}{r+1}}\right\} + O(\frac{x}{n})$$

Thus

$$S_{1_{q/n}} \ll \sum_{\substack{n \leq U^2 \\ q|n}} d(n) \sum_{h \leq \frac{x}{n}} |\mu(h)| \min\left\{\left(\frac{x}{nh}\right)^{\frac{1}{r+1}}, (|z|nh)^{-\frac{1}{r+1}}\right\} + \tag{3.4}$$
$$O\left(x \sum_{\substack{n \leq U^2 \\ q|n}} \frac{d(n)}{n}\right).$$

When $U = x^{\frac{2}{5}}$ and using the estimates on $S_{1_{q\nmid n}}$, $S_{1_{q/n}}$ together, we get for S_1 and r odd

$$S_1 \ll \frac{x \ln^2 x}{q^{1-\epsilon}}. \tag{3.5}$$

The same bound is obtained if r even . Now we estimate $S_2, \forall r \geq 1$, by (2.3)

$$S_2 = \sum_{\substack{mk \leq x \\ m>U, k>U}} \mu_r^*(m) \sum_{\substack{d/k \\ d \leq U}} \mu_r^*(d) \mu_r^{*-1}(\frac{k}{d})\ e(mk\alpha) =$$
$$= \sum_{k>U} \left(\sum_{\substack{d/k \\ d \leq U}} \mu_r^*(d) \mu_r^{*-1}(\frac{k}{d}) \right) \sum_{U<m\leq \frac{x}{k}} \mu_r^*(m)\ e(mk\alpha)$$

By decomposing the range of summation on k as follows

$$A = \{2^j U \;/ j = 0, \dots i \; ; \; 2^i U^2 < x \le 2^{i+1} U^2\} \tag{3.6}$$

we write $S_2 = \sum_{Y \in A} S(Y)$ where

$$S(Y) = \sum_{Y<k\le 2Y} \left(\sum_{\substack{d/k \\ d\le U}} \mu_r^*(d) \mu_r^{*-1}\left(\frac{k}{d}\right) \right) \sum_{U<m\le \frac{x}{k}} \mu_r^*(m)\; e(mk\alpha)$$

By Hölder's inequality :

$$|S(Y)|^2 \ll \sum_{Y<k\le 2Y} d^2(k) \sum_{Y<k\le 2Y} |\sum_{U<m\le \frac{x}{k}} \mu_r^*(m) e(mk\alpha)|^2 \ll$$

$$\ll \sum_{Y<k\le 2Y} d^2(k) \sum_{U<m\le \frac{x}{Y}} \sum_{U<\ell\le \frac{x}{Y}} \min\{Y, \|(m-\ell)\alpha\|^{-1}\} \ll$$

$$\ll x \ln^4 x \left(\frac{x}{q} + Y + \frac{x}{Y} + q \right)$$

Thus the sum S_2 is

$$S_2 \ll (\ln x)^3 \left(\frac{x}{q^{1/2}} + x^{1/2} q^{1/2} + \frac{x}{U^{1/2}} \right). \tag{3.7}$$

Replacing (3.1) , (3.5) and (3.7) on S and taking $U = x^{2/5}$ we obtain

$$\sum_{n\le x} \mu_r^*(n)\; e(n\alpha) \ll \ln^3 x \left\{ \frac{x}{q^{\frac12}} + x^{\frac12} q^{\frac12} + x^{\frac45} \right\}.$$

and Theorem is proved. □

PROOF OF THEOREM 2. Let $Q = x\; e^{-c_8\sqrt{\ln x}}$ and $\delta = e^{c_8\sqrt{\ln x}}$, with $c_8 < c_6$. If $\delta \le q \le Q$ we can apply Theorem 1 and we have

$$|\sum_{n\le x} \mu_r^*(n)\; e(n\alpha)| \ll x\; e^{-c_9\sqrt{\ln x}}.$$

If $q \le \delta$ we apply Lemma 4 to get

$$|\sum_{n\le x} \mu_r^*(n)\; e(n\alpha)| \ll x\; e^{-c_{10}\sqrt{\ln x}}$$

for some constant c_7. □

PROOF OF THEOREM 3. If $r = 0$ or $r = 1$ the formula (1.9) is trivial. Let $r > 1$ be an odd integer, by (1.6) we can write

$$S = \sum_{n\leq x} h_r(n)e(n\alpha) = \sum_{d\delta\leq x} \mu_r^*(d)e(d\delta\alpha) = \sum_{d^2m^{r+1}\leq x} \mu(m)e(d^2m^{r+1}\alpha) \tag{3.9}$$

and for $r \geq 2$ even integer we have

$$S = \sum_{n\leq x} h_r(n)e(n\alpha) = \sum_{d^2m^{r+1}\leq x} |\mu(m)|e(d^2m^{r+1}\alpha) \tag{3.10}$$

The sum on the right-hand may be regarded as a sum over the lattice points satisfying $H \geq 1, K \geq 1$, $H^2K^{r+1} \leq x$. This region may be broken into three parts by a point (H, K) on $H^2K^{r+1} = x$ in the standard way of the elementary theories of the divisor problem. One obtains

$$S = \sum_{d\leq H} \sum_{m^{r+1}\leq(x/d^2)} \mu(m)e(d^2m^{r+1}\alpha) + \sum_{m\leq K} \mu(m) \sum_{d^2\leq(x/m^{r+1})} e(d^2m^{r+1}\alpha) -$$

$$- \sum_{\substack{d\leq H\\ m\leq K}} \mu(m)e(d^2m^{r+1}\alpha) = S_1 + S_2 - S_3 \tag{3.11}$$

Analogously if $r \geq 2$ even integer.The terms in these sums are one in absolute value, so S_3 may be estimated by their number of terms and $S_3 \ll HK$. Also $S_1 \ll x^{\frac{1}{r+1}}H^{1-\frac{2}{r+1}}$, $r > 1$. We take $H = x^{1/4}$, $K = x^{1/2(r+1)}$. To evaluate S_2 we suppose in the first place $\alpha = a/q$, $(a, q) = 1$. Hence

$$S_2 \ll \frac{x^{1/2}}{q^{1/2}} \sum_{m\leq K} \frac{(m^{r+1}, q)^{1/2}}{m^{\frac{r+1}{2}}} = \frac{x^{1/2}}{q^{1/2}} \sum_{d|q} \sum_{\substack{m\leq K\\ (m,q)=q}} \frac{(m^{r+1}, q)^{1/2}}{m^{\frac{r+1}{2}}} \ll \frac{x^{1/2}}{q^{1/2-\epsilon}}$$

We have deduced (1.10) . If $\alpha = \frac{a}{q} + \beta$ with $|\beta| \leq x^{-5/4+1/2(r+1)}$, then

$$\left|\sum_{d\leq(x/m^{r+1})^{1/2}} e(d^2m^{r+1}\alpha)\right| \leq$$

$$\leq \left| \sum_{d \leq (x/m^{r+1})^{1/2}} e(d^2 m^{r+1} \frac{a}{q}) \right| + \left| \sum_{d \leq (x/m^{r+1})^{1/2}} e(d^2 m^{r+1} \frac{a}{q}) \left(e(d^2 m^{r+1} \beta) - 1\right) \right|$$

$$= |S(\frac{a}{q})| + |S(\frac{a}{q} + \beta) - S(\frac{a}{q})| \quad ,$$

$$|S(\frac{a}{q} + \beta) - S(\frac{a}{q})| \ll m^{r+1} |\beta| (x/m^{r+1})^{3/2}$$

Thus, we have $S_2 \ll x^{1/2} q^{-1/2+\epsilon} + x^{1/4+1/2(r+1)}$ and (1.10) is deduced. The theorem is proved. □

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