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Some asymptotic properties of orthogonal polynomials

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RIASSUNTO: *Sia* $\{\phi_n(z)\}_{n=0}^{\infty}$ *un sistema di polinomi monici ortogonali rispetto ad una misura di Borel positiva* $d\mu$, *con supporto sul cerchio unitario, ed* A *l'insieme degli zeri dei polinomi* $\phi_n(z)$. *In questo lavoro vengono mostrate alcune connessioni tra il comportamento dei parametri associati di Szegö* $\{a_n\}_{n=1}^{\infty}$ ($\overline{a_{n-1}} = -\phi_n(0)$) *e la posizione limite dei punti di* A. *Da ciò si deducono alcune informazioni relative alla posizione degli zeri.*

ABSTRACT: *Let* $\{\phi_n(z)\}_{n=0}^{\infty}$ *be a system of monic polynomials orthogonal with respect to a positive Borel measure* $d\mu$, *supported on the unit circle, and* A *the set of the zeros of the polynomials* $\phi_n(z)$. *Some connections between the behaviour of the associated Szegö parameters* $\{a_n\}_{n=1}^{\infty}$ ($\overline{a_{n-1}} = -\phi_n(0)$) *and the location of the limit points of* A *are obtained here. From them, some information about the location of the zeros is deduced.*

KEY WORDS: *Orthogonal polynomials – Zeros – Szegö parameters.*

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1 – Introduction

Let $\{\phi_n(z)\}_{n=0}^{\infty}$ be a sequence of monic polynomials orthogonal with respect to a finite positive Borel measure, $d\mu$, on the interval $[0, 2\pi)$, such that its support is an infinite set of $T = \{z \in \mathbb{C}\colon |z| = 1\}$. Let Δ_n be

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the determinant of its $(n+1)\times(n+1)$ associated moment matrix and $e_n = \Delta_n\Delta_{n-1}^{-1}$ if $n \geq 1$, $e_0 = 1$.

For every nth degree polynomial Q, the reverse polynomial Q^* is defined by $Q^*(z) = z^n\overline{Q\left(\frac{1}{\bar{z}}\right)}$. The polynomials $\phi_n(z)$ and $\phi_n^*(z)$ are related through the recurrence formula ([7], form.(8,1), p.155)

$$\text{(R.I)} \qquad z\phi_{n-1}(z) = \phi_n(z) + \overline{a_{n-1}}\phi_{n-1}^*(z)$$

or equivalently ([3], corol. I, 3.1, p.8)

$$\text{(R.II)} \qquad \phi_n(z) = \frac{e_n}{e_{n-1}}z\phi_{n-1} - \overline{a_{n-1}}\phi_n^*(z)$$

where $\{a_n\}_{n=1}^{\infty}$ is the family of Szegö (or reflection) parameters ($a_{n-1} = = \overline{-\phi_n(0)}$); therefore, the verify $|a_n| < 1\ \forall n$.

It is well known (see [2], [6], [7]) that

$$\lim_n e_n > 0 \iff \{a_n\} \in l^2 \iff \log\mu' \in L^1[0,2\pi)$$
$$\lim_n e_n = 0 \iff \{a_n\} \notin l^2 \iff \log\mu' \notin L^1[0,2\pi)$$

The present paper deals with some questions related to the zeros of the polynomials $\phi_n(z)$. On that respect, we must emphasize the fact that only a few results are known about the location of the zeros of orthogonal polynomials on the unit circle and, even about the properties of the zeros of such polynomials, in spite of the great variety of results of such nature for orthogonal polynomials on the real line. Next, we summarize some of the main results on this topic.

In [4] we proved the following proposition, which improves a previous result:

PROPOSITION 1.1. *For every sequence of complex numbers* $\{\alpha_n\}_{n=1}^{\infty}$ *with* $|\alpha_n| < 1$ *for* $n = 1, 2, \dots,$ *there exists only one family* $\{\phi_n(z)\}_{n=0}^{\infty}$ *of monic orthogonal polynomials on* T *such that* $\phi_0(z) = 1$ *and* $\phi_n(\alpha_n) = 0$, $n \geq 1$.

Sometime later in [10] NEVAI and TOTIK revealed a connection between orthogonal polynomials, their zeros and the family $\{\phi_n(0)\}_{n=0}^{\infty}$, through the quantities r_i $(i = 1, 2, 3, 4)$ defined as follows:

$$r_1(d\mu) = \limsup_{n\to\infty} \left|\phi_n(0)\right|^{1/n}$$
$$r_2(d\mu) = \inf_k \limsup_{n\to\infty} |\alpha_{kn}|$$
$$r_3(d\mu) = \left\{ \inf r\colon \sup_n \max_{|z|=r^{-1}} \left|\phi_n^*(z)\right| < +\infty \right\}$$
$$r_4(d\mu) = \{\inf r\colon D(d\mu, z)^{-1} \quad \text{is analytic for} \quad |z| < r^{-1}\}$$

where $D(d\mu, z)$ is the Szegö function defined by

$$D(d\mu, z) = \begin{cases} \exp\left(\frac{1}{4\pi}\int\limits_0^{2\pi} \log \mu'(t) \frac{e^{it}+z}{e^{it}-z} dt\right) & |z| < 1 \\ 0 \quad \text{otherwise} & \end{cases}$$

and α_{kn} denote the zeros of $\phi_n(z)$ ordered in such a way that

$$(1) \qquad |\alpha_{nn}| \leq |\alpha_{n-1,n}| \leq \ldots \leq |\alpha_{1n}| < 1$$

The main result is:

PROPOSITION 1.2. *For every measure $d\mu$ we have $r_1(d\mu) = r_2(d\mu)$. If there is $j \in \{1, 2, 3, 4\}$ such that $r_j(d\mu) < 1$, then $r_1(d\mu) = r_3(d\mu) = r_4(d\mu)$*

Besides if $r_1 < 1$, for each $1 > \sigma > r_1$ the number of zeros of $\phi_n(z)$ outside of $C_\sigma = \{z\colon |z| < \sigma\}$ is bounded independently of n.

The last result is used by MHASKAR and SAFF in [9] to give the limiting distribution of the zeros, under some mild conditions on $d\mu$.

Connections between properties of the sequence $\phi_n(0)$ and properties of the measure $d\mu$ have been examined in some recent papers like [1] or [8]. We are planning to study in the near future how far conditions on $\phi_n(0)$ such as those given by ABRAMYAN in [1], may modify the behaviour of the zeros.

In this paper some connections between the behaviour of Szegö parameters and the location of limit points of the set of zeros of the polynomials $\phi_n(z)$ are revealed (Prop. 2.1, 2.2, 2.9). Likewise, some results dealing with the zeros of the orthogonal polynomials are deduced (Prop. 2.3, and Prop. 2.4, 2.8 on the other hand).

2 – The results and their proofs

Let $\{\alpha_{in}\}_{i=1}^{n}$ be, for every n, the zeros of the polynomial $\phi_n(z)$ ordered as in (1).

It is well known that

$$|a_{n-1}| = \prod_{i=1}^{n} |\alpha_{in}| \tag{2}$$

and, consequently,

$$|a_{n-1}| < |\alpha_{in}| < 1 \qquad (1 \le i \le n) \tag{3}$$

$$|\alpha_{nn}| \le |a_{n-1}|^{1/n} \le |\alpha_{1n}| \tag{4}$$

We deduce from Prop. 1.1 that any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ is a limit point of the set of the zeros of a system $\{\phi_n(z)\}_{n=0}^{\infty}$. Even more, Prop. 1.1 can be applied to a couple of interesting situations:

i) When $\alpha_n = \alpha \in D$ $(D = \{z\colon |z| < 1\})$ for $n = 1, 2, \dots$, we can build a system $\{\phi_n(z)\}_{n=0}^{\infty}$ of orthogonal polynomials on T verifying $\phi_n(\alpha) = 0$ $\forall n \ge 1$. In particular, in the case $\phi_n(z) = z^n$ for $n = 1, 2, \dots$, the zeros of all the polynomials $\phi_n(z)$ lie on the origin.

ii) If $\{\alpha_n\}_{n=1}^{\infty}$ is a set dense in D, the derived set of zeros of the family $\phi_n(z)$ is the closed unit circle.

In short, if we denote with A the set of zeros of the polynomials $\phi_n(z)$ $(n = 1, 2, \dots)$ and with A' the derived set of A, for A' two extreme situations appear, namely:

$$A' = \{0\} \quad \text{and} \quad A' = \overline{D}$$

The behaviour of Szegö parameters in certain intermediate situations is considered in the next propositions. Besides, some information about the location of the zeros is deduced from them.

Proposition 2.1. *If* $\limsup_{n\to\infty} |a_n| > 0$*, then* $T \cap A' \neq \phi$

In particular,

i) *If* $\lim_n |a_n| = 1$ *all the zeros of* $\phi_n(z)$*, for* n *large enough, are arbitrarily close to the unit circle.*

ii) *If* $\lim_n |a_n| = a \in (0,1)$*, then* $\lim_n |\alpha_{1n}| = 1$.

Besides, $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ *such that* $\alpha_{in} \in \{z\colon a - \varepsilon < |z| < 1\}$ *if* $n \geq n_0$ *and* $1 \leq i \leq n$.

PROOF. If $\limsup_{n\to\infty} |a_n| = r > 0$ we deduce from (2) that

$$r = \limsup_{n\to\infty} \Big(\prod_{i=1}^{n} |\alpha_{in}\Big) > 0\,; \quad \forall \varepsilon > 0 \quad (r > \varepsilon)$$

there are infinitely many indexes n such that $r - \varepsilon < \prod_{i=1}^{n} |\alpha_{in}|$ and, consequently, $|\alpha_{1n}| > (r - \varepsilon)^{1/n}$. Otherwise, there exists a subsequence $\{\phi_{n_h}(z)\}_{h=0}^{\infty}$ of $\{\phi_n(z)\}_{n=0}^{\infty}$ such that $\lim_h |\alpha_{1n_h}| = 1$; if α is a limit point of $\{\alpha_{in_h} | h \geq 0\}$, then $|\alpha| = 1$ and $T \cap A' \neq \phi$.

The statements i), ii) are deduced directly from (2) and (3). □

The converse theorem is not true. The family $\{\phi_n(z)\}_{n=0}^{\infty}$ of polynomials orthogonal on the unit circle with respect to the measure $d\mu = \sin^2 \frac{\theta}{2} d\theta$, provides a very nice example (see [11] p.542) of polynomials whose zeros tend towards one, being $\phi_n(0) = \frac{1}{n+1}$ and so, $\lim_n a_n = 0$.

PROPOSITION 2.2. *If* $0 \in A'$*, then* $\liminf_{n\to\infty} |a_n| = 0$

PROOF. If $0 \in A', \forall \varepsilon > 0$ there are infinitely many indexes n such that $|\alpha_{in}| < \varepsilon$. Hence $\prod_{i=1}^{n} |\alpha_{in}| < \varepsilon$ holds for infinitely many indexes n, and so $\liminf_{n\to\infty} |a_n| = 0$. □

As a consequence of Prop. 2.2, if $\liminf_{n\to\infty} |a_n| > 0$ there is a neigborhood of the origin $B_\varepsilon(0)$ such that $B_\varepsilon(0) \cap A = \phi$. Because necessarily $\limsup_{n\to\infty} |a_n| > 0$, $T \cap A' \neq \phi$ also holds, and so, whenever $0 \in A'$, then $T \cap A' \neq \phi$, perhaps excluding the case $\lim_n |a_n| = 0$. The zeros of the polynomials lie on a circular crown with radii $\varepsilon, 1$.

PROPOSITION 2.3. *If for every n there exists i $(1 \leq i \leq n)$ such that $\lim_n |\alpha_{in}| = 0$, then $\{a_n\} \in l^1$. In fact, it suffices that $\limsup_{n\to\infty} |\alpha_{1n}| < 1$ holds.*

PROOF. Suppose $\lim_n |\alpha_{in}| = 0$, $(1 \leq i \leq n)$. Then, $\{a_n\} \in l^2$ is an obvious consequence from (3). The result by NEVAI-TOTIK given in section 1, proves easily that $\{a_n\} \in l^1$. Indeed, in our hipothesis, Prop. 1.2 assures that

$$0 = r_2(d\mu) = r_1(d\mu)$$

and, consequently,

$$\limsup_{n\to\infty} \left|\phi_n(0)\right|^{1/n} = \lim_{n\to\infty} \left|a_{n-1}\right|^{1/n} = 0$$

So, $\{a_n\} \in l^1$.

With respect to the second assertion, the result follows immediately from the inequalities

$$0 \leq \left|a_{n-1}\right|^{1/n} \leq \left|\alpha_{1n}\right| < 1\,. \tag{5}$$

□

PROPOSITION 2.4. *Let $\{\alpha_{in}\}_{n=1}^{\infty}$ $(1 \leq i \leq n)$ be a sequence of complex numbers whose n-th term is just a zero of $\phi_n(z)$. If $\{a_n\} \in l^2$ and $A' \cap T = \phi$, then $\lim_{n\to\infty} \phi_n(\alpha_{i,n-1}) = 0$ $(i = 1, 2, \ldots, n-1)$.*

PROOF. If $T \cap A' = \phi$, there exists $D_r = \{z \in \mathbb{C}\colon |z| < r < 1\}$ with $A \subset \overline{D}_r$. Since $\overline{D}_r$ is a compact subset of D and $\{a_n\} \in l^2$, $\phi_n^*(z) \longrightarrow D(d\mu, z)^{-1}$ uniformly on $\overline{D}_r$. For every n, $\phi_n^*(z)$ is a continuous function on $\overline{D}_r$ and, so, bounded on $\overline{D}_r$. Therefore $\{\phi_n^*(z)\}_{n=0}^{\infty}$ is uniformly bounded on $\overline{D}_r$ ([5], p.221).

Consequently, writing (R.II) with $z = \alpha_{i,n-1}$ $(i = 1, 2, \ldots, n-1)$

$$\phi_n(\alpha_{i,n-1}) = -\overline{a_{n-1}}\phi_n^*(\alpha_{i,n-1})$$

and there exists $\lim_{n\to\infty} \phi_n(\alpha_{i,n-1}) = 0$. □

COROLLARY 2.5. *The function $\phi_n(z)/\phi_n^*(z)$ takes the same value at all the zeros of $\phi_{n-1}(z)$.*

REMARK 2.6. The situation is a bit different for the zeros $\{\alpha_{i,n+1}\}_{i=1}^{n+1}$ of $\phi_{n+1}(z)$. From (R.I), we obtain

$$\alpha_{i,n+1}\frac{\phi_n(\alpha_{i,n+1})}{\phi_n^*(\alpha_{i,n+1})} = \bar{a}_n \quad (i = 1, 2, \dots, n+1),$$

and if

$$1 > \left|\alpha_{1,n+1}\right| \geq \left|\alpha_{2,n+1}\right| \geq \dots \geq \left|\alpha_{n+1,n+1}\right| > 0$$

then

$$0 < \left|\frac{\phi_n(\alpha_{1,n+1})}{\phi_n^*(\alpha_{1,n+1})}\right| \leq \left|\frac{\phi_n(\alpha_{2,n+1})}{\phi_n^*(\alpha_{2,n+1})}\right| \leq \dots \leq \left|\frac{\phi_n(\alpha_{n+1,n+1})}{\phi_n^*(\alpha_{n+1,n+1})}\right| < 1 \tag{6}$$

We deduce from (6) that $|\alpha_{i,n+1}| = r \; \forall i$ is the only situation in which the ratios $\frac{\phi_n(\alpha_{i,n+1})}{\phi_n^*(\alpha_{i,n+1})}$ have constant modulus.

REMARK 2.7. From (6) and Corollary 2.5 it follows that for any zero $\alpha_{in} (i = 1, 2, \dots, n)$ of $\phi_n(z)$ and any zero $\alpha_{j,n-1} (j = 1, 2, \dots, n-1)$ of $\phi_{n-1}(z)$,

$$\left|\frac{\phi_{n-1}(\alpha_{in})}{\phi_{n-1}^*(\alpha_{in})}\right| > \left|\frac{\phi_n(\alpha_{j,n-1})}{\phi_n^*(\alpha_{j,n-1})}\right| \tag{7}$$

holds.

Expression (7) is mentioned here only because zeros of consecutive orthogonal polynomials are involved in it. We must remember that the interlacing property for zeros of consecutive polynomials orthogonal with respect to a measure supported in an infinite real set, has not an analogous property for orthogonal polynomials on the unit circle. In this last case, the only known separation property is for the zeros of nth polynomials corresponding to different terminal extensions of the set $\{\phi_h(z)\}_{h=0}^{n-1}$ ([2], th. 6.5, p.51).

PROPOSITION 2.8. *Let $\{\alpha_{in}\}_{n=1}^{\infty}$ $(1 \leq i \leq n)$ be a sequence in the hypothesis of Prop. 2.4. If $\{a_n\} \in l^1$, there exists $\lim_{n\to\infty} \phi_n(\alpha_{i,n-1}) = 0$ $(i = 1, 2, \dots, n-1)$.*

PROOF. If $\{a_n\} \in l^1, \phi_n^*(z) \longrightarrow D(d\mu, z)^{-1}$ uniformly on $\overline{D}$ ([7], th. 8.5, p.163). Since $A \subset \overline{D}$, the argument of the above proposition can be used. □

We want to point out that Prop. 2.4, and Prop. 2.8 lead to the same conclusion under different hypotheses. This situation seems to suggest that a relationship between the conditions $T \cap A' = \phi$ and $\{a_n\} \in l^1$ may exist. This relationship is formulated in the next proposition.

PROPOSITION 2.9. *If $T \cap A' = \phi$, then $\{a_n\} \in l^1$.*

PROOF. If $T \cap A' \neq \phi$, then it is obvious that $\limsup_{n\to\infty} |\alpha_{1n}| = 1$. Conversely, if $\limsup_{n\to\infty} |\alpha_{1n}| = 1$ there exists a subsequence $\{\alpha_{1n_h}\}_{h=1}^{\infty}$ such that $\lim_{h\to\infty} |\alpha_{1n_h}| = 1$ and following the same argument as in Prop. 2.1, we deduce $T \cap A' \neq \phi$.

Therefore, the condition $T \cap A' = \phi$ is equivalent to the condition that $\limsup_{n\to\infty} |\alpha_{1n}| < 1$. Now then, $\limsup_{n\to\infty} |\alpha_{1n}| < 1$ implies $\{a_n\} \in l^1$ as a consequence of (5). □

The converse proposition is not true. Indeed, from the inequalities (5) we also deduce that the condition $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ is sufficient to get $\limsup_{n\to\infty} |\alpha_{1n}| = 1$ and therefore, $T \cap A' \neq \phi$. But $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ holds even if $\{a_n\} \notin l^1$.

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