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New linear relationships of hypergeometric-type functions with applications to orthogonal polynomial

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RIASSUNTO: *Vengono ricavate due relazioni ricorrenti contenenti anche le derivate* $y_\nu^{(k+1)}(z), y_\nu^{(k)}(z)$ *e* $y_{\nu+1}^{(k+1)}(z)$ *oppure* $y_{\nu-1}^{(k+1)}(z)$ *di una funzione* $y_\mu(z)$ *di tipo ipergeometrico. Le funzioni* $y \equiv y_\nu(z)$ *sono soluzioni della equazione differenziale* $\sigma(z)y''+\tau(z)y'+\lambda y = 0$, *dove* $\sigma(z)$ *e* $\tau(z)$ *sono polinomi di grado rispettivamente non superiore a 2 e a 1 e* $\lambda = -\nu\tau' - \frac{1}{2}\nu(\nu-1)\sigma''$. *Sono di questo tipo le funzioni di Gauss e di Kummer, i polinomi ortogonali classici e molte altre funzioni della matematica e della fisica. In particolare queste due relazioni sono applicate a polinomi di tipo ipergeometrico, che formano un'ampia classe di funzioni* $y_n(z)$, *dove* n *è un numero intero positivo. Infine vengono date le corrispondenti relazioni per i polinomi ortogonali classici, che danno nuove caratterizzazioni per i polinomi di Jacobi, Laguerre, Hermite e Bessel.*

ABSTRACT: *Here, two general differential-difference relations among the functions* $y_\nu^{(k+1)}(z)$, $y_\nu^{(k)}(z)$ *and* $y_{\nu+1}^{(k+1)}(z)$ *or* $y_{\nu-1}^{(k+1)}(z)$ *are found. The symbol* $y_\mu^{(k)}(z)$ *denotes the kth-derivative of the function of hypergeometric type* $y_\mu(z)$. *The functions* $y \equiv y_\nu(z)$ *are solutions of the differential equation* $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$, *where* $\sigma(z)$ *and* $\tau(z)$ *are polynomials of degrees not higher than 2 and 1, respectively, and* $\lambda = -\nu\tau' - \frac{1}{2}\nu(\nu-1)\sigma''$. *Instances of these functions are the Gauss and Kummer functions, the classical orthogonal polynomials and many other functions of mathematics and physics. Then, these two relations are applied to the polynomials of hypergeometric type, which form a broad class of functions* $y_n(z)$, *where* n *is a positive integer number. Finally, the corresponding relationships for the classical orthogonal polynomials, which supply new characterizations for the Jacobi, Laguerre, Hermite and Bessel polynomials, are given.*

KEY WORDS: *Hypergeometric functions – Orthogonal Polynomials – Recurrence relations.*

A.M.S. CLASSIFICATION: 33C05 – 33C45 – 42C05

1 – Introduction

The Gauss or hypergeometric equation, the Kummer or confluent hypergeometric equation, the Hermite equation and the second order linear, homogeneous differential equation with constant coefficients are instances of the so-called equation of hypergeometric type, which has the form

$$\sigma(z)\,y'' + \tau(z)\,y' + \lambda y = 0 \tag{1}$$

where $\sigma(z)$ and $\tau(z)$ are polynomials of degrees not higher than 2 and 1, respectively, and λ is a constant.

In this paper we will deal with the solutions of Eq. (1) of the form [1]

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\,\rho(s)}{(s-z)^{\nu+1}}\,ds \tag{2}$$

where ν is a root of the equation

$$\lambda + \nu\tau' + \frac{1}{2}\nu(\nu-1)\,\sigma'' = 0 \tag{3}$$

C_ν is a normalization constant, $\rho(z)$ is a solution of the equation

$$(\sigma\rho)' = \tau\rho \tag{4}$$

and C is a contour whose endpoints s_1 and s_2 satisfy the condition

$$\left.\frac{\sigma^{\nu+1}(s)\,\rho(s)}{(s-z)^{\nu+2}}\right|_{s\,=\,s_1}^{s_2} = 0 \tag{5}$$

Particular instances of these functions abound in all fields of mathematics and physics [1,2] such as, for example, the Gauss, Kummer and Hermite functions, all the classical orthogonal polynomials and many other functions encountered in different fields of mathematics and physics.

Recently, a systematic study of the structural and spectral properties of the functions $y_\nu^{(k)}(z)$ has been initiated from the differential equation (1) which they verify. In the first two papers [3,4] we have calculated the fundamental recurrence relations (i.e. the three-term recurrence relation and the so-called ladder or structure relations) for them and their derivatives of any order. These relations considerably extend the corresponding characterizations for the classical orthogonal polynomials, nicely reviewed by W.A. AL-SALAM [5] in 1990.

Here we find two new characterizations for the functions of hypergeometric type and for their derivatives of any order by means of two novel differential-difference relations which involve the functions $y_\nu^{(k+1)}(z)$, $y_\nu^{(k)}(z)$ and $y_{\nu+1}^{(k+1)}(z)$ or $y_{\nu-1}^{(k+1)}(z)$, being $k \geq 0$. Remark that for $k = 0$ one has a relationship among $y'_\nu(z)$, $y_\nu(z)$ and $y'_{\nu+1}(z)$ or $y'_{\nu-1}(z)$. All these relationships are derived in Section II. Then, in Section III, we show how these relationships get reduced for polynomials of hypergeometric type and the specific relationships for all classical orthogonal polynomials are tabulated.

2 – Main results

Here we will present and prove two differential-difference relations which involve the functions $y_\nu^{(k+1)}(z)$, $y_\nu^{(k)}(z)$ and $y_{\nu+1}^{(k+1)}(z)$ or $y_{\nu-1}^{(k+1)}(z)$. The coefficients of both relations are given in terms of the polynomials $\sigma(z)$ and $\tau(z)$ of the differential equation (1). They are as follows:

First relation.- It is fulfilled that

$$A_{1k}(z)\ y_\nu^{(k+1)}(z) + A_{2k}\, y_\nu^{(k)}(z) + A_{3k}\, y_{\nu+1}^{(k+1)}(z) = 0 \tag{6}$$

where the coefficients A_{ik}, $i = 1, 2, 3$, are given by

$$\begin{aligned} A_{1k}(z) &= \frac{1}{2}\left[\tau_\nu(z)\,\sigma'' - 2\tau'_\nu\sigma'(z)\right] \\ &= -\frac{1}{2}\left[\tau'_\nu\sigma'' z + 2\tau'_\nu\sigma'(0) - \tau_\nu(0)\,\sigma''\right] \\ A_{2k} &= -\tau'_\nu\tau'_{\frac{\nu+k-1}{2}} \\ A_{3k} &= \frac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu-1}{2}} \end{aligned} \tag{7}$$

Second relation.- It is fulfilled that

$$\overline{A}_{1k}(z)\ y_\nu^{(k+1)}(z) + \overline{A}_{2k}\, y_\nu^{(k)}(z) + \overline{A}_{3k}\, y_{\nu-1}^{(k+1)} = 0 \tag{8}$$

where the coefficients $\overline{A}_{ik}$, $i = 1, 2, 3$, are given by

$$\begin{aligned} \overline{A}_{1k}(z) &= \tau_{\nu-1}(z) \\ &= \left[\tau'_{\nu-1} z + \tau_{\nu-1}(0)\right] \\ \overline{A}_{2k} &= (k-\nu)\,\tau'_{\nu-1} \\ \overline{A}_{3k} &= \frac{1}{2\tau'_{\frac{k}{2}-1}} \frac{C_\nu}{C_{\nu-1}} \left\{\sigma'' \tau^2_{\nu-1}(0) + 2\tau'_{\nu-1}\left[\sigma(0)\,\tau'_{\nu-1} - \sigma'(0)\,\tau_{\nu-1}(0)\right]\right\} \end{aligned} \tag{9}$$

In writing both relationships, we have used the notation

$$\tau_\mu = \tau + \mu\sigma'$$

It is interesting to point out that the only coefficients of the two relations (6) and (8) which depend on k are those which multiply $y_\nu^{(k)}(z)$.

Let us consider two particular cases:

A.- In case that $k = 0$, one has

$$A_{10}(z)\; y'_\nu(z) + A_{20}\, y_\nu(z) + A_{30}\, y'_{\nu+1}(z) = 0 \tag{10}$$

with

$$\begin{gathered} A_{10}(z) = A_{1k}(z) \quad , \quad A_{30} = A_{3k} \\ A_{20} = -\tau'_\nu \tau'_{\frac{\nu-1}{2}} \end{gathered} \tag{12}$$

and

$$\overline{A}_{10}(z)\; y'_\nu(z) + \overline{A}_{20}\, y_\nu(z) + \overline{A}_{30}\, y'_{\nu-1}(z) = 0 \tag{13}$$

with

$$\begin{gathered} \overline{A}_{10}(z) = \overline{A}_{1k}(z) \quad , \quad \overline{A}_{30} = \overline{A}_{3k} \\ \overline{A}_{20} = -\nu\tau'_{\nu-1} \end{gathered} \tag{14}$$

B.- if $\sigma'' = 0$, Eqs. (7) get reduced as

$$A_{1k}(z) = -\sigma'(0)\ , \qquad A_{2k} = -\tau'\ , \qquad A_{3k} = \frac{C_\nu}{C_{\nu+1}}$$

and Eqs. (9) simplify similarly as

$$\overline{A}_{1k} = [\tau' z + \tau_{\nu-1}(0)]$$
$$\overline{A}_{2k} = (k-\nu)\,\tau'$$
$$\overline{A}_{3k} = \frac{C_\nu}{C_{\nu-1}}\,[\sigma(0)\,\tau' - \tau_{\nu-1}(0)\,\sigma'(0)]$$

PROOF. We will prove only the first relation given by Eqs. (6)-(7) because the second one can be obtained similarly. Following a procedure analogous to that developed in Refs. 1, 3 and 4, we begin with the summation

$$S = A_{1k}(z)\,y_\nu^{(k+1)}(z) + A_{2k}(z)\,y_\nu^{(k)}(z) + A_3(z)\,y_{\nu+1}^{(k+1)}(z) \tag{15}$$

where $y_\nu^{(k)}(z)$ denotes the kth-derivative of $y_\nu(z)$, and $A_{ik}(z)$ ($i = 1, 2$ and 3) are arbitrary functions of z. By means of the integral representation of $y_\nu^{(k)}(z)$ found by NIKIFOROV and UVAROV [1] one can easily find the following expressions for the functions involved in the above written summation:

$$y_\nu^{(k+1)}(z) = \frac{C_\nu^{(k+1)}}{\sigma^{k+1}(z)\,\rho(z)} \int_C \frac{\sigma^\nu(s)\,\rho(s)}{(s-z)^{\nu-k}}\,ds \tag{16}$$

$$y_\nu^{(k)}(z) = \frac{C_\nu^{(k)}}{\sigma^{k}(z)\,\rho(z)} \int_C \frac{\sigma^\nu(s)\,\rho(s)}{(s-z)^{\nu-k+1}}\,ds \tag{17}$$

$$y_{\nu+1}^{(k+1)}(z) = \frac{C_{\nu+1}^{(k+1)}}{\sigma^{k+1}(z)\,\rho(z)} \int_C \frac{\sigma^{\nu+1}(s)\,\rho(s)}{(s-z)^{\nu-k+1}}\,ds \tag{18}$$

where the normalization constant $C_\nu^{(k)}$ is given by [1,4]

$$C_\nu^{(k)} \equiv \left(\tau'_{k-1} + \frac{\nu-k}{2}\sigma''\right) C_\nu^{(k-1)} = C_\nu \prod_{s=0}^{k-1}\left(\tau' + \frac{\nu+s-1}{2}\sigma''\right)$$

Now, we replace the expressions (16)-(18) in Eq. (15). We obtain that

$$S = \frac{1}{\sigma^{k+1}(z)\,\rho(z)} \int_C \frac{\sigma^\nu(s)\,\rho(s)}{(s-z)^{\nu-k+1}}\,P(s)\,ds \tag{19}$$

with

$$P(s) = A_{1k}(z)\, C_\nu^{(k+1)}(s-z) + A_{2k}(z)\, C_\nu^{(k)} \sigma(z) + + A_{3k}(z)\, C_{\nu+1}^{(k+1)} \sigma(s) \tag{20}$$

Let us choose the coefficients A_{ik}, $(i = 1, 2, 3)$ so that

$$\frac{\sigma^\nu(s)\,\rho(s)}{(s-z)^{\nu-k+1}} P(s) = \frac{d}{ds}\left[\frac{\sigma^{\nu+1}(s)\,\rho(s)}{(s-z)^{\nu-k}} Q(s)\right] \tag{21}$$

where $Q(s)$ is a polynomial. Then, Eqs. (19) and (20) show that $S = 0$ provides that the contour C is such that

$$\left.\frac{\sigma^{\nu+1}(s)\,\rho(s)}{(s-z)^{\nu-k}} s^m\right|_{S_1}^{S_2} = 0 \quad \text{for each m=0,1,2,3, \ldots} \tag{22}$$

which is a condition similar to (5).

Making the derivation contained in Eq. (21) and taking into account that $[\sigma^{\nu+1}(s)\,\rho(s)]' = \tau_\nu(s)\,\sigma^\nu(s)\,\rho(s)$, which is a consequence of Eq. (4), one obtains

$$P(s) = [\tau_\nu(s)(s-z) - (\nu-k)\,\sigma(s)]\, Q(s) + \sigma(s)(s-z)\, Q'(s) \tag{23}$$

From the identity of Eqs. (20) and (23) one notices that $Q(s)$ must be a constant, which we choose equal to unity without any loose of generality. Then, by taking into account the Taylor's development of $\tau_\nu(s)$ and $\sigma(s)$ around the position $s = z$ and by equating the coefficients of the corresponding powers of $(s - z)$ in both sides of the identity one finally obtain a system of three equations with the three unknowns A_{ik} $(i = 1, 2, 3)$. The solution of this system of equations produces easily the wanted values (7) for the coefficients A_{ik}.

3 – Application to polynomials of hypergeometric type

The polynomials of hypergeometric type $y_n(z)$ are [1] particular instances of the functions of hypergeometric type given by Eq. (2), where $\nu = n$ (positive integer), $C_n = B_n n!/(2\pi i)$ and the contour C is closed.

B_n is a normalization factor which, for monic polynomials (i.e. when the leading coefficient is unity) $\hat{y}_n(z)$, has the value

$$\widehat{B_n} = \left\{ \prod_{k=0}^{n-1} \left[\tau' + \frac{1}{2}(n+k-1)\sigma'' \right] \right\}^{-1}$$

Keeping all this in mind and since the conditions (5) and (22) are automatically satisfied for any closed contour, the two main differential-difference relationships (6)-(7) and (8)-(9) take on the following simpler forms for the polynomials of hypergeometric type $y_n(z)$, respectively:

First

$$(\alpha_n z + \beta_n)\; y_n^{(k+1)}(z) + \gamma_{nk}\, y_n^{(k)}(z) + \delta_n\, y_{n+1}^{(k+1)}(z) = 0 \tag{24}$$

where the constants α, β, γ and δ are given by

$$\begin{aligned} \alpha_n &= -\tfrac{1}{2}\tau_n'\sigma'', \qquad & \beta_n &= \tfrac{1}{2}\tau_n(0)\sigma'' - \tau_n'\sigma'(0) \\ \gamma_{nk} &= -\tau_n'\tau'_{\frac{n+k-1}{2}}, \qquad & \delta_n &= \frac{B_n}{B_{n+1}}\frac{\tau'_{\frac{n-1}{2}}}{n+1} \end{aligned} \tag{25}$$

Second

$$\left(\overline{\alpha}_n z + \overline{\beta}_n\right)\; y_n^{(k+1)}(z) + \overline{\gamma}_{nk}\, y_n^{(k)}(z) + \overline{\delta}_n\, y_{n-1}^{(k+1)}(z) = 0 \tag{26}$$

with

$$\begin{aligned} \overline{\alpha}_n &= \tau'_{n-1} \\ \overline{\beta}_n &= \tau_{n-1}(0) \\ \overline{\gamma}_{nk} &= (k-n)\,\tau'_{n-1} \\ \overline{\delta}_n &= \frac{n}{2\tau'_{\frac{n}{2}-1}}\frac{B_n}{B_{n-1}}\left\{\sigma''\tau_{n-1}^2(0) + 2\tau'_{n-1}\left[\sigma(0)\tau'_{n-1} - \sigma'(0)\tau_{n-1}(0)\right]\right\} \end{aligned} \tag{27}$$

It is important to point out how these two relationships get simplified in some particular cases

(i) In case that $\sigma'' = 0$, one has that

$$\sigma'(0)\, y_n^{(k+1)}(z) + \tau' y_n^{(k)}(z) - \frac{B_n}{(n+1)B_{n+1}} y_{n+1}^{(k+1)}(z) = 0 \tag{28}$$

and

$$
\begin{aligned}
&[\tau' z + \tau_{n-1}(0)]\, y_n^{(k+1)}(z) + (k-n)\tau' y_n^{(k)}(z) + \\
(29) \qquad &+ \frac{nB_n}{B_{n-1}}\left[\sigma(0)\tau' - \tau_{n-1}(0)\sigma'(0)\right] y_{n-1}^{(k+1)}(z) = 0
\end{aligned}
$$

Notice that the coefficients of the relation (28) have no dependence on k.

(ii) For monic polynomials $\hat{y}_n(z)$ one has that [1,4]

$$\frac{\hat{B}_n}{\hat{B}_{n+1}} = \frac{\tau'_{n-\frac{1}{2}}\tau'_n}{\tau'_{\frac{n-1}{2}}}$$

so that the two main relationships become

$$(30) \qquad \left(\hat{\alpha}_n z + \hat{\beta}_n\right) \hat{y}_n^{(k+1)}(z) + \hat{\gamma}_{nk}\, \hat{y}_n^{(k)}(z) + \hat{\delta}_n\, \hat{y}_{n+1}^{(k+1)}(z) = 0$$

with the coefficients given by

$$(31) \qquad \hat{\alpha}_n = \alpha_n\,, \quad \hat{\beta}_n = \beta_n\,, \quad \hat{\gamma}_{nk} = \gamma_{nk}\,, \quad \hat{\delta}_n = \frac{1}{n+1}\tau'_{n-\frac{1}{2}}\tau'_n$$

and

$$(32) \qquad \left(\tilde{\alpha}_n z + \tilde{\beta}_n\right) \hat{y}_n^{(k+1)}(z) + \tilde{\gamma}_{nk}\, \hat{y}_n^{(k)}(z) + \tilde{\delta}_n\, \hat{y}_{n-1}^{(k+1)}(z) = 0$$

with the coefficients given by

$$
\begin{aligned}
&\tilde{\alpha}_n = \overline{\alpha}_n\,, \quad \tilde{\beta}_n = \overline{\beta}_n\,, \quad \tilde{\gamma}_{nk} = \overline{\gamma}_{nk} \quad \text{and} \\
(33) \quad &\tilde{\delta}_n = \frac{n}{2\tau'_{n-\frac{3}{2}}\tau'_{n-1}}\left\{\sigma''\tau^2_{n-1}(0) + 2\tau'_{n-1}\left[\sigma(0)\tau'_{n-1} - \sigma'(0)\tau_{n-1}(0)\right]\right\}
\end{aligned}
$$

For the sake of illustration we will calculate explicitly the coefficients of these two relationships for the so-called classical orthogonal polynomials [1,2]: Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, Laguerre polynomials $L_n^{(\alpha)}(x)$, Hermite polynomials $H_n(x)$ and Bessel polynomials $B_n^{(\alpha)}(z)$. In Table 1 we collect the basic data of these polynomials:the domain of orthogonality Γ, the weight function $\rho(z)$ and the coefficients $\sigma(z)$, $\tau(z)$ and λ_n of the differential equation of hypergeometric type (1) which they satisfy. Then in Table 2 the $\alpha-$, $\beta-$, $\gamma-$ and $\delta-$ coefficients of the differential-difference relations (30) and (32) for all monic classical orthogonal polynomials $\hat{y}_n(z)$ are shown. Indeed, Table 2 contains (a) the coefficients $\left\{\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_{nk}, \hat{\delta}_n\right\}$ which are given by Eqs. (31) and (b) the coefficients $\left\{\tilde{\alpha}_n, \tilde{\beta}_n, \tilde{\gamma}_{nk}, \tilde{\delta}_n\right\}$ which are given by Eqs. (33).

Finally, let us consider the case $k = 0$ for the classical orthogonal polynomials of the monic nature. Table 2 gives that:

- Monic Jacobi Polynomials $\hat{y}_n(x) = P_n^{(\alpha,\beta)}(x)$

$$\left[x + \frac{\beta-\alpha}{2n+\alpha+\beta+2}\right]\hat{y}'_n(x) + (n+\alpha+\beta+1)\hat{y}_n(x) + \\ - \frac{2n+\alpha+\beta+1}{n+1}\hat{y}'_{n+1}(x) = 0$$

$$\left[x + \frac{\alpha-\beta}{2n+\alpha+\beta}\right]\hat{y}'_n(x) - n\hat{y}_n(x) + \\ + \frac{4n(n+\alpha)(n+\beta)}{(1-\alpha-\beta-2n)(2n+\alpha+\beta)^2}\hat{y}'_{n-1}(x) = 0$$

- Monic Laguerre Polynomials $\hat{y}_n(x) = L_n^{(\alpha)}(x)$

$$\begin{aligned} &\hat{y}'_n(x) - \hat{y}_n(x) + \frac{1}{n+1}\hat{y}'_{n+1}(x) = 0 \\ &[x-(n+\alpha)]\,\hat{y}'_n(x) - n\hat{y}_n(x) - n(n+\alpha)\,\hat{y}'_{n-1}(x) = 0 \end{aligned} \tag{34}$$

- Monic Hermite Polynomials $\hat{y}_n(x) = H_n(x)$

$$\begin{aligned} &(n+1)\,\hat{y}_n(x) - \hat{y}'_{n+1}(x) = 0 \\ &x\hat{y}'_n(x) - n\hat{y}_n(x) - \frac{n}{2}\hat{y}'_{n-1}(x) = 0 \end{aligned} \tag{35}$$

- Monic Bessel Polynomials $\hat{y}_n(z) = B_n^{(\alpha)}(z)$

$$\left[z - \frac{2}{2n+\alpha+2}\right]\hat{y}'_n(z) + (n+\alpha+1)\hat{y}_n(z) - \frac{2n+\alpha+1}{n+1}\hat{y}'_{n+1}(z) = 0$$

$$\left[z + \frac{2}{2n+\alpha}\right]\hat{y}'_n(z) - n\hat{y}_n(z) + \frac{4n}{(2n+\alpha-1)(2n+\alpha)^2}\hat{y}'_{n-1}(z) = 0$$

Some of these expressions for classical orthogonal polynomials were previously found by means of a completely different technique [2].

Table 1

$y_n(x)$	$P_n^{(\alpha,\beta)}(x)$; $\alpha > -1$, $\beta > -1$	$L_n^{(\alpha)}(x)$; $\alpha > -1$	$H_n(x)$	$B_n^{(\alpha)}(z)$
Γ	$(-1,1)$	$(0,\infty)$	$(-\infty,\infty)$	unit disk
$\rho(x)$	$(1-x)^\alpha(1+x)^\beta$	$x^\alpha e^{-x}$	e^{-x^2}	$z^\alpha e^{-\frac{2}{z}}$
$\sigma(x)$	$1-x^2$	x	1	z^2
$\tau(x)$	$\beta-\alpha-(\alpha+\beta+2)x$	$1+\alpha-x$	$-2x$	$(\alpha+2)z+2$
λ_n	$n(n+\alpha+\beta+1)$	n	$2n$	$-n(n+\alpha+1)$

Basic data of the classical orthogonal polynomials. The domain of orthogonality Γ, the weight function $\rho(x)$ and the coefficients $\sigma(x)$, $\tau(x)$ and λ_n of the second order differential equation of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the Hermite polynomials $H_n(x)$ and the Bessel polynomials $B_n^{(\alpha)}(z)$ are shown.

Table 2

$y_n(x)$	$P_n^{(\alpha,\beta)}(x)$	$L_n^{(\alpha)}(x)$	$H_n(x)$	$B_n^{(\alpha)}(x)$
$\hat{\alpha}_n$	$-(2n+\alpha+\beta+2)$	0	0	$-(2n+\alpha+2)$
$\hat{\beta}_n$	$\alpha-\beta$	1	0	2
$\hat{\gamma}_{nk}$	$-(n+k+\alpha+\beta+1)(2n+\alpha+\beta+2)$	-1	-1	$-(n+k+\alpha+1)(2n+\alpha+2)$
$\hat{\delta}_n$	$\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{(2n+\alpha+1)(2n+\alpha+2)}{n+1}$
$\bar{\alpha}_n$	$2n+\alpha+\beta$	1	1	$2n+\alpha$
$\bar{\beta}_n$	$\alpha-\beta$	$-(n+\alpha)$	0	2
$\bar{\gamma}_{nk}$	$(k-n)(2n+\alpha+\beta)$	$k-n$	$k-n$	$(k-n)(2n+\alpha)$
$\bar{\delta}_n$	$\frac{4n(n+\alpha)(n+\beta)}{(1-\alpha-\beta-2n)(2n+\alpha+\beta)}$	$-n(n+\alpha)$	$-\frac{n}{2}$	$\frac{4n}{(2n+\alpha-1)(2n+\alpha)}$

The coefficients $\left\{\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_{nk}, \hat{\delta}_n\right\}$ and $\left\{\bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_{nk}, \bar{\delta}_n\right\}$ of the differential-difference relations expressed by Eqs. (30) and (32), respectively. The coefficients are given by Eqs. (31) and (33), also respectively.

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