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On Plücker transformations of generalized elliptic spaces

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RIASSUNTO: *Si riconosce che le trasformazioni Plückeriane degli spazi ellittici generalizzati, esclusi quelli tridimensionali, sono indotte da collineazioni che preservano l'ortogonalità. Si descrivono tutte le trasformazioni Plückeriane degli spazi ellittici tridimensionali appartenenti ad una certa classe, che comprende gli spazi reali.*

ABSTRACT: *Plücker transformations of generalized elliptic spaces with dimensions other than three are induced by orthogonality-preserving collineations. For certain three-dimensional elliptic spaces (including real spaces) all Plücker transformations will be described.*

1 – Introduction

If we are given a linear space with point set $\mathcal{P}$, line set $\mathcal{L}$ and an orthogonality relation on its set of lines then call two lines related if they are concurrent and orthogonal or if they are identical. A bijection of $\mathcal{L}$ that preserves this relation in both directions is called Plücker transformation.

Plücker transformations of Euclidean spaces are under discussion in [1], [2], [5]. Cf. also the survey in [14]. The crucial result, due to W. BENZ and E.M. SCHRÖDER, is that 3-dimensional spaces are very ex-

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ceptional, since only here Plücker transformations are intimately connected with derivations of the ground field. For higher dimensions all Plücker transformations arise from orthogonality-preserving collineations, whereas Plücker transformations of Euclidean planes cannot deserve interest.

In this paper we discuss Plücker transformations of generalized elliptic spaces, i.e. (not necessarily finite dimensional) projective spaces with orthogonality based upon an elliptic absolute quasipolarity. For dimensions 2, 4, 5, 6, ... there are no Plücker transformations other than those arising from orthogonality-preserving collineations. In every 3-dimensional generalized elliptic space there exist Plücker transformations that cannot be induced by collineations or dualities; under certain restrictions (projective absolute polarity, Fano's postulate, existence of Clifford parallel lines) all Plücker transformations will be described via the ambient space of the associated Klein quadric.

2 – Basic concepts and first results

Let $(\mathcal{P},\mathcal{L})$ be a projective space $2 \le \dim(\mathcal{P},\mathcal{L}) \le \infty$. Assume that π is an elliptic quasipolarity [12], [13]. Thus π assigns to every point X of $\mathcal{P}$ a hyperplane X^π such that $X \notin X^\pi$. We define a mapping from the lattice of subspaces of $(\mathcal{P},\mathcal{L})$ into itself by setting

$$\mathcal{J} \mapsto \bigcap (X^\pi | X \in \mathcal{J}) \quad \text{for all subspaces} \quad \mathcal{J} \neq \emptyset \quad \text{and} \quad \emptyset \mapsto \mathcal{P}\,.$$

This mapping is again written as π and is also called a quasipolarity. Hence $(\mathcal{P},\mathcal{L},\pi)$ is a generalized elliptic space with absolute quasipolarity π [12], [13]. Every subspace $\mathcal{J}$ of $(\mathcal{P},\mathcal{L})$ is skew to $\mathcal{J}^\pi$. If $\mathcal{J}$ is finite-dimensional then $\mathcal{J}^\pi$ is even a complement of $\mathcal{J}$ and putting $\mathcal{X} \mapsto \mathcal{X}^\pi \cap \mathcal{J}$ for all subspaces $\mathcal{X}$ of $\mathcal{J}$ yields an elliptic polarity of $\mathcal{J}$.

We are going to define three binary relations on $\mathcal{L}$. Given $a, b \in \mathcal{L}$ then put

$$\begin{array}{lll} a \perp b :\iff a \cap b^\pi \neq \emptyset & & \text{(orthogonal lines)},\\ a \approx b :\iff a \perp b \quad \text{and} \quad a \cap b \neq \emptyset & & \text{(orthogonally intersecting lines)},\\ a \sim b :\iff a \approx b \quad \text{or} \quad a = b & & \text{(related lines)}. \end{array}$$

These three relations are symmetric. This follows from an axiomatic description of the relation $\approx$ in [13, p. 370ff]; cf. also [12, p. 58ff].

Given lines $a, b \in \mathcal{L}$ then there is always a finite sequence

$$a \sim a_1 \sim \ldots \sim a_n \sim b\,. \tag{1}$$

This is trivial when $a = b$. If a and b meet at a unique point X, say, then $X^\pi \cap (a \vee b) =: a_1$ is a line satisfying $a \sim a_1 \sim b$. If a and b are skew then there exists a common transversal line of a and b, say c, whence repeating the previous construction for a, c and c, b gives the required sequence. Thus $(\mathcal{L}, \sim)$ is a Plücker space [1, p. 199].

If μ is a collineation of $(\mathcal{P}, \mathcal{L})$ commuting with π then μ is preserving orthogonality of lines in both directions. A Plücker transformation is a bijective mapping $\varphi : \mathcal{L} \to \mathcal{L}$ preserving the relation $\sim$ (or, equivalently, the relation $\approx$) in both directions.

Lemma 1. *Let $\mu : \mathcal{P} \to \mathcal{P}$ be a collineation such that $a \sim b$ implies $a^\mu \sim b^\mu$ for all $a, b \in \mathcal{L}$. Then $\pi\mu = \mu\pi$, whence μ yields a Plücker transformation by its action on the set $\mathcal{L}$.*

Proof. Choose any line $a \in \mathcal{L}$. A point $X \in \mathcal{P} \setminus a$ is in a^π if, and only if, there are two distinct lines through X that are related to a. Hence $a^{\pi\mu} \subset a^{\mu\pi}$ for all lines $a \in \mathcal{L}$. But $a^{\pi\mu}$ as well as $a^{\mu\pi}$ is a co-line, so that actually $a^{\pi\mu} = a^{\mu\pi}$ and therefore $\pi\mu = \mu\pi$. Now the last assertion is obviously true. □

If $\dim(\mathcal{P}, \mathcal{L}) = 3$ then there are dualities that induce Plücker transformations, e.g., the absolute polarity π. However, there are also Plücker transformations which are by no means induced by collineations or dualities. Let $\mathcal{L}_1$ be any subset of $\mathcal{L}$ such that $x \in \mathcal{L}_1$ implies $x^\pi \in \mathcal{L}_1$. Then define

$$\delta : \mathcal{L} \to \mathcal{L}\,, \begin{cases} x \mapsto x & \text{if} \quad x \in \mathcal{L} \setminus \mathcal{L}_1\,, \\ x \mapsto x^\pi & \text{if} \quad x \in \mathcal{L}_1\,. \end{cases} \tag{2}$$

Such a bijection δ will be called partial π-transformation (with respect to $\mathcal{L}_1$); it is a Plücker transformation of $(\mathcal{L}, \sim)$, since

$$a \approx b \iff a \approx b^\pi \iff a^\pi \approx b \iff a^\pi \approx b^\pi \quad \text{for all} \quad a, b \in \mathcal{L}\,.$$

The identity on $\mathcal{L}$ and the restriction of π to $\mathcal{L}$ are partial π-transformations, as follows when setting $\mathcal{L}_1 := \emptyset$ and $\mathcal{L}_1 := \mathcal{L}$, respectively. For every other choice of $\mathcal{L}_1$ (e.g., $\mathcal{L}_1 := \{a, a^\pi\}$) it is easily seen that there exist two non-orthogonal concurrent lines $x \in \mathcal{L} \setminus \mathcal{L}_1$, $y \in \mathcal{L}_1$. Then $x^\delta = x$ and $y^\delta = y^\pi$ are skew lines, whence δ cannot arise from a collineation or duality.

In contrast to these examples we establish

Theorem 1. *Let* $\dim(\mathcal{P}, \mathcal{L}) = 2$ *or* $4 \leq \dim(\mathcal{P}, \mathcal{L}) \leq \infty$. *Then every bijection* $\varphi : \mathcal{L} \to \mathcal{L}$ *such that* $a \sim b$ *implies* $a^\varphi \sim b^\varphi$ *for all* $a, b \in \mathcal{L}$ *is induced by a collineation* μ *of* $(\mathcal{P}, \mathcal{L})$ *such that* $\pi\mu = \mu\pi$. *Hence* φ *is already a Plücker transformation.*

Proof. (a) Let $\dim(\mathcal{P}, \mathcal{L}) = 2$. Given a point X and a line a then $X \in a$ is equivalent to $X^\pi \approx a$. Hence

$$\mu : \mathcal{P} \to \mathcal{P}, \quad X \mapsto X^{\pi\varphi\pi},$$

is a collinearity preserving bijection and therefore a collineation. By construction $a^\mu = a^\varphi$ for all $a \in \mathcal{L}$ and, by Lemma 1, $\pi\mu = \mu\pi$.

(b) Let $\dim(\mathcal{P}, \mathcal{L}) \geq 4$. We claim that φ is mapping concurrent lines to concurrent lines. Assume, to the contrary, that there exist two distinct concurrent lines $a, b \in \mathcal{L}$ such that a^φ, b^φ are skew. By $\dim(\mathcal{P}, \mathcal{L}) \geq 4$ there exist two π-conjugate points $C_1, C_2 \in (a \vee b)^\pi$, i.e. $C_1 \in C_2^\pi$ so that also $C_2 \in C_1^\pi$.

Let $c_j := C_j \vee (a \cap b)$ $\big(j \in \{1, 2\}\big)$. Therefore $c_1 \approx c_2$ and $a \approx c_j \approx b$ $\big(j \in \{1, 2\}\big)$. Since a^φ and b^φ are skew, c_1^φ and c_2^φ are common transversal lines of a^φ and b^φ. But $c_1^\varphi \approx c_2^\varphi$, so that the point $c_1^\varphi \cap c_2^\varphi$ is either on a^φ or on b^φ; say $c_1^\varphi \cap c_2^\varphi \in a^\varphi$. There exists a line $b_1 \subset (a \vee b)$ such that $b \perp b_1$ and $b \cap b_1 = a \cap b$. Therefore $\{b, b_1, c_1, c_2\}$ is a set of mutually related lines. However, $\{b^\varphi, c_1^\varphi, c_2^\varphi\}$ is already a maximal set of mutually related lines, since these three lines are coplanar and not concurrent. Hence $b_1 \in \{b, c_1, c_2\}$, a contradiction.

By $\dim(\mathcal{P}, \mathcal{L}) \geq 4$ and [4, p. 328-329], every bijection of $\mathcal{L}$ that takes intersecting lines to intersecting lines is induced by a mapping $\mu : \mathcal{P} \longrightarrow \mathcal{P}$ as follows:

$$(A \vee B) \longmapsto A^\mu \vee B^\mu \quad \text{for all} \quad A, B \in \mathcal{P}, \quad A \neq B.$$

Moreover, this μ is injective and is preserving collinearity and non-collinearity of points. (In [4] it is stated that μ is also surjective and hence a collineation. There is, however, a gap in the proof of surjectivity.)

We are applying this result on the given bijection φ in order to show that φ is mapping skew lines to skew lines: Assume, to the contrary, that there exist skew lines $a, b \in \mathcal{L}$ with $a^\varphi \vee b^\varphi =: \mathcal{E}$ being a plane. Each point $X \in a \vee b$ is on some line x intersecting a and b at distinct points. Thus $X^\mu \in x^\varphi \subset \mathcal{E}$, since x^φ is meeting a^φ and b^φ at distinct points, too. We read off from this that $(a \vee b)^\mu \subset \mathcal{E}$. By $\dim(\mathcal{P}, \mathcal{L}) \geq 4$, there exist four concurrent lines $c_1, \ldots, c_4$ such that $c_i \approx c_j$ $(i \neq j)$ with $c_1, c_2, c_3 \subset a \vee b$. But $c_1^\varphi, c_2^\varphi, c_3^\varphi$ are coplanar, mutually related and distinct, so that they form a trilateral. This contradicts the existence of $c_4^\varphi \approx c_i^\varphi$ $(i = 1, 2, 3)$.

Next we establish that μ is surjective: Since φ is surjective, for each point Y of $\mathcal{P}$ there exist lines $a, b \in \mathcal{L}$ such that $Y = a^\varphi \cap b^\varphi$. But then a and b are not skew, so that $Y = (a \cap b)^\mu$. Thus μ is a collineation. Finally, by Lemma 1, $\pi\mu = \mu\pi$. □

3 – Orthogonal transversal lines in elliptic 3-spaces

In discussing 3-dimensional spaces we shall assume that $(\mathcal{P}, \mathcal{L})$ is a Pappian projective space satisfying Fano's postulate and that its absolute polarity π is projective(1). It will be convenient to let $(\mathcal{P}, \mathcal{L})$ be a projective space on a 4-dimensional vector space V over a commutative field F. We shall emphasize this by writing $\mathcal{P}(V)$ and $\mathcal{L}(V)$ rather than $\mathcal{P}$ and $\mathcal{L}$, respectively. The absolute polarity π is induced by a non-degenerate symmetric bilinear form $\beta : V \times V \to F$ satisfying $(a, a)^\beta \neq 0$ for all $a \in V \setminus \{0\}$. But β is determined by π only up to a non-zero factor in F, so we may assume that $(b_0, b_0)^\beta = 1$ for some $b_0 \in V$. There exists an ordered basis (b_0, b_1, b_2, b_3) of V such that

$$((b_i, b_j)^\beta) = \operatorname{diag}(1, e_1, e_2, e_3) \in GL(4, F)\,. \tag{3}$$

We observe that $(\mathcal{P}(V), \mathcal{L}(V), \pi)$ fits into the concept of an elliptic spaces as defined in [17] via a metric vector space(2). The group of all

(1) Cf., however, Remark 3 at the end of this section.

(2) Elliptic spaces on metric vector spaces over fields of characteristic 2 are not within our discussion, since their orthogonality is symplectic (possibly even degenerate).

collineations of $(\mathcal{P}, \mathcal{L})$ commuting with π will be written as $P\Gamma O(\mathcal{P}, \pi)$.

Denote by $V \wedge V$ the 2-fold wedge product of V with itself and write $\mathcal{P}(V \wedge V)$ for the (5-dimensional) projective space on $V \wedge V$. The well known Klein mapping

$$\gamma : \mathcal{L}(V) \to \mathcal{P}(V \wedge V)\,, \quad Fa \vee Fb \mapsto F(a \wedge b)$$

is injective and $\mathcal{L}(V)^\gamma =: \Gamma$ is the Klein quadric. The quadratic form

$$q : V \wedge V \to F\,, \quad \sum_{i<j} x_{ij} b_i \wedge b_j \mapsto x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12} \quad (x_{ij} \in F)$$

determines the Klein quadric(3). The polarity associated to the Klein quadric will be denoted by κ. The absolute polarity π of $(\mathcal{P}, \mathcal{L})$ yields a projective collineation α of $\mathcal{P}(V \wedge V)$ characterized by $a^{\gamma\alpha} = a^{\pi\gamma}$ for all $a \in \mathcal{L}(V)$. We shall refer to α as the antipodal collineation(4). No point of Γ is α-invariant and α is an involution.

Given lines $a, b \in \mathcal{L}(V)$ then define

$$\mathcal{O}(a, b) := \{x \in \mathcal{L} \mid a \approx x \quad \text{and} \quad b \approx x\}\,.$$

The set $\mathcal{O}(a, b)$ is formed by all common transversal lines of a, b, a^π, b^π so that $\mathcal{O}(a, b) = \mathcal{O}(b, a) = \mathcal{O}(a, b^\pi)$. It is easily seen that $\#\mathcal{O}(a, b) \geq 3$ holds if, and only if, either a, a^π, b, b^π are four distinct lines of a regulus or $b \in \{a, a^\pi\}$.

Lemma 2. *Given two lines $a \in \mathcal{L}(V)$ and $b \in \mathcal{L}(V) \setminus \{a, a^\pi\}$ then $\#\mathcal{O}(a, b) \geq 3$ holds if, and only if, $a^\gamma \vee b^\gamma$ carries an α-invariant point.*

Proof. (a) Suppose that $\#\mathcal{O}(a, b) \geq 3$. As a, a^π, b, b^π are four distinct lines of a regulus, $\{a^\gamma, a^{\pi\gamma}, b^\gamma, b^{\pi\gamma}\}$ is plane quadrangle and

$$I := (a^\gamma \vee b^\gamma) \cap (a^{\pi\gamma} \vee b^{\pi\gamma})$$

turns out to be an α-invariant point.

(3)See, e.g., [1], [7], [8] or [16] for results on the Klein quadric that will be used without further references.
(4)See footnote 7 for a motivation of this name.

(b) Let $I \in a^\gamma \vee b^\gamma$ be an α-invariant point. We deduce $(a^\gamma \vee b^\gamma) \cap \Gamma = \{a^\gamma, b^\gamma\}$ from $I \notin \Gamma$. Thus, by $b \notin \{a, a^\pi\}$, the line $a^\gamma \vee b^\gamma$ cannot be α-invariant. Hence $\{a^\gamma, a^{\gamma\alpha}, b^\gamma, b^{\gamma\alpha}\}$ is a plane quadrangle with I being one of its diagonal points. The intersection of the Klein quadric Γ with the plane $a^\gamma \vee a^{\gamma\alpha} \vee b^\gamma$ cannot be a cross of lines, since no point of Γ is fixed under α; thus $\{a^\gamma, a^{\gamma\alpha}, b^\gamma, b^{\gamma\alpha}\}$ is part of a regular conic on the Klein quadric. This in turn shows that a, a^π, b, b^π are in one regulus, whence $\#\mathcal{O}(a,b) \geq 3$, as required. $\square$

The antipodal collineation α is induced by an $f \in GL(V \wedge V)$ such that

$$\begin{aligned} b_0 \wedge b_1 &\mapsto e_1 b_2 \wedge b_3\,, & b_2 \wedge b_3 &\mapsto e_2 e_3 b_0 \wedge b_1\,,\\ b_0 \wedge b_2 &\mapsto -e_2 b_1 \wedge b_3\,, & b_1 \wedge b_3 &\mapsto -e_1 e_3 b_0 \wedge b_2\,,\\ b_0 \wedge b_3 &\mapsto e_3 b_1 \wedge b_2\,, & b_1 \wedge b_2 &\mapsto e_1 e_2 b_0 \wedge b_3\,. \end{aligned}$$

The characteristic polynomial of f is $(X^2 - e_1e_2e_3)^3 \in F[X]$. Hence α has an invariant point if, and only if, $e_1e_2e_3 \in F^{(2)}$, i.e. the set of squares in F.

Up to Remark 2 at the end of this section it is assumed that there exists a square root $\sqrt{e_1e_2e_3} \in F$ of $e_1e_2e_3$. Then

$$\begin{aligned} c_0 &:= e_2e_3 b_0 \wedge b_1 + \sqrt{e_1e_2e_3}\, b_2 \wedge b_3\,,\\ c_1 &:= -e_1e_3 b_0 \wedge b_2 + \sqrt{e_1e_2e_3}\, b_1 \wedge b_3\,,\\ c_2 &:= e_1e_2 b_0 \wedge b_3 + \sqrt{e_1e_2e_3}\, b_1 \wedge b_2\,,\\ c_3 &:= e_2e_3 b_0 \wedge b_1 - \sqrt{e_1e_2e_3}\, b_2 \wedge b_3\,,\\ c_4 &:= -e_1e_3 b_0 \wedge b_2 - \sqrt{e_1e_2e_3}\, b_1 \wedge b_3\,,\\ c_5 &:= e_1e_2 b_0 \wedge b_3 - \sqrt{e_1e_2e_3}\, b_1 \wedge b_2\,, \end{aligned}$$

is an eigenbasis of $V \wedge V$ with respect to f and the distinct planes[5]

$$\mathcal{E}_L := Fc_0 \vee Fc_1 \vee Fc_2\,, \quad \mathcal{E}_R := Fc_3 \vee Fc_4 \vee Fc_5$$

are fixed pointwise under α. There are no α-invariant points other than those in $\mathcal{E}_L$ or $\mathcal{E}_R$. We shall frequently use the projections

$$\begin{aligned} \lambda &: \mathcal{P}(V \wedge V) \setminus \mathcal{E}_L \to \mathcal{E}_R\,,\ X \mapsto (X \vee \mathcal{E}_L) \cap \mathcal{E}_R\,,\\ \rho &: \mathcal{P}(V \wedge V) \setminus \mathcal{E}_R \to \mathcal{E}_L\,,\ X \mapsto (X \vee \mathcal{E}_R) \cap \mathcal{E}_L\,. \end{aligned} \tag{5}$$

[5] The indices L and R stand for "left" and "right", respectively, and are arbitrarily assigned to these two planes.

They are induced by $f-\sqrt{e_1e_2e_3}\,\mathrm{id}$ and $f+\sqrt{e_1e_2e_3}\,\mathrm{id}$, respectively. If X, X^α are distinct antipodal points in $\mathcal{P}(V\wedge V)$ then $X\vee X^\alpha = X^\lambda \vee X^\rho$. By (5) and Lemma 2,

$$(\#\mathcal{O}(a,b)\geq 3) \iff (a^{\gamma\lambda}=b^{\gamma\lambda} \text{ or } a^{\gamma\rho}=b^{\gamma\rho}) \quad \text{for all} \quad a,b\in\mathcal{L}(V)\,.$$

Lines $a,b\in\mathcal{L}(V)$ will be called Clifford parallel $(a \parallel b)$ if $\mathcal{O}(a,b)\geq 3$, left parallel $(\|_L)$ if $a^{\gamma\lambda}=b^{\gamma\lambda}$ and right parallel $(\|_R)$ if $a^{\gamma\rho}=b^{\gamma\rho}$. Left (right) parallelism is an equivalence relation on $\mathcal{L}(V)$. We adopt the notations

$$\mathcal{S}_L(a) := \{x\in\mathcal{L}(V)|x\|_L a\}\,,\quad \mathcal{S}_R(a) := \{x\in\mathcal{L}(V)|x\|_R a\}\,.$$

In subsequent results the terms "left" and "right" may be interchanged without further notice. We start with an almost trivial

Lemma 3. *Let $a\in\mathcal{L}(V)$. Then $\mathcal{S}_L(a)$ is an elliptic linear congruence of lines (regular spread).*

Proof. The subspace $a^\gamma\vee\mathcal{E}_L\subset\mathcal{P}(V\wedge V)$ is 3-dimensional and $\mathcal{S}_L(a)^\gamma = (a^\gamma\vee\mathcal{E}_L)\cap\Gamma$ is a quadric that cannot contain a line, since distinct left parallel lines are skew. The point $a^\gamma\in\mathcal{S}_L(a)^\gamma$ is regular(6), because of $(a^\gamma\vee a^{\pi\gamma})\cap\mathcal{E}_L\notin\Gamma$. Hence quadric $\mathcal{S}_L(a)^\gamma$ is elliptic so that $\mathcal{S}_L(a)$ is an elliptic linear congruence. □

We infer from $\mathcal{E}_L^\kappa=\mathcal{E}_R$ and $\mathcal{E}_L\cap\Gamma=\emptyset$ that κ induces an elliptic projective polarity κ_L in $\mathcal{E}_L$ by setting $\mathcal{X}\mapsto\mathcal{X}^\kappa\cap\mathcal{E}_L$ for all subspaces $\mathcal{X}\subset\mathcal{E}_L$. Hence $\mathcal{E}_L$ becomes an elliptic plane(7). Two points of $\mathcal{E}_L$ are κ_L-conjugate if, and only if, they are κ-conjugate.

Lemma 4. *Given lines $a,b\in\mathcal{L}(V)$ then $a\approx b$ holds if, and only if, $a^{\gamma\lambda}$, $b^{\gamma\lambda}$ are κ_R-conjugate and $a^{\gamma\rho}$, $b^{\gamma\rho}$ are κ_L-conjugate.*

(6) A point X of a quadric is regular if the tangent space at this point is a hyperplane. This will be true if at least one line through X meets the quadric in exactly two points.
(7) The elliptic quadric $\mathcal{S}_L(a)^\gamma$ discussed in the proof of Lemma 3 may be seen as a sphere in the Euclidean 3-space that arises from $a^\gamma\vee\mathcal{E}_L$ by regarding $\mathcal{E}_L$ as its plane at infinity and κ_L as its absolute polarity. The midpoint of $\mathcal{S}_L(a)^\gamma$ is $(a^\gamma\vee\mathcal{E}_L)\cap\mathcal{E}_R$, whence antipodal points on $\mathcal{S}_L(a)^\gamma$ (with respect to α) are antipodal points in the usual sense.

PROOF. In $\mathcal{P}(V \wedge V)$ we shall consider the lines

$$m_a := a^{\gamma\lambda} \vee a^{\gamma\rho} = a^{\gamma} \vee a^{\pi\gamma}\,, \quad m_b := b^{\gamma\lambda} \vee b^{\gamma\rho} = b^{\gamma} \vee b^{\pi\gamma}\,.$$

If $a \approx b$ then each of a, a^{π} is a common transversal line of b and b^{π}. Hence each of $a^{\gamma}, a^{\pi\gamma}$ is κ-conjugate to b^{γ} and $b^{\pi\gamma}$ which in turn implies that $m_a \subset m_b^{\kappa}$ so that $a^{\gamma\lambda} \in b^{\gamma\lambda\kappa}$ and $a^{\gamma\rho} \in b^{\gamma\rho\kappa}$, as required.

Since each point of $\mathcal{E}_L$ is κ-conjugate to all points of $\mathcal{E}_R$, the first part of the proof is easily reversed. ∎

LEMMA 5. *Suppose that $a \approx b$ holds for two lines.*

(I) *If $x \in \mathcal{S}_L(a)$ and $y \in \mathcal{S}_L(b)$ are concurrent then $x \approx y$.*

(II) *Define*

$$\mathcal{R}_L := \{x \in \mathcal{S}_L(a) | x \approx y' \quad \text{for some} \quad y' \in \mathcal{S}_R(b)\}\,,$$
$$\mathcal{R}_R := \{y \in \mathcal{S}_R(b) | y \approx x' \quad \text{for some} \quad x' \in \mathcal{S}_L(a)\}\,.$$

Then $\mathcal{R}_L$ and $\mathcal{R}_R$ are mutually opposite reguli. Therefore $x \in \mathcal{R}_L$ and $y \in \mathcal{R}_R$ imply that $x \approx y$.

PROOF. (I) We infer $x \neq y$ from $a \neq b$. Write $\mathcal{F} \subset \mathcal{L}(V)$ for the only ruled plane (or, dually, the only star of lines) containing x and y. This $\mathcal{F}$ may be regarded as a projective plane with "point set" $\mathcal{F}$ and the pencils in $\mathcal{F}$ being the "lines". From this point of view $\gamma\lambda|\mathcal{F} : \mathcal{F} \to \mathcal{E}_R$ is a collineation. By Lemma 4 and by slight modification of Lemma 1, we obtain that $x' \approx y'$ is equivalent to $x'^{\gamma\lambda}$, $y'^{\gamma\lambda}$ being κ-conjugate for all $x', y' \in \mathcal{F}$. But $x^{\gamma\lambda} = a^{\gamma\lambda}$ and $y^{\gamma\lambda} = b^{\gamma\lambda}$ are κ-conjugate by Lemma 4, whence $x \approx y$.

(II) We ask for all pairs $(x, y) \in \mathcal{S}_L(a) \times \mathcal{S}_R(b)$ satisfying $x \approx y$. By Lemma 4 this is equivalent to

$$x^{\gamma\rho} \in y^{\gamma\rho\kappa} = b^{\gamma\rho\kappa} \quad \text{and} \quad y^{\gamma\lambda} \in x^{\gamma\lambda\kappa} = a^{\gamma\lambda\kappa}\,.$$

This in turn may be written as

$$x^{\gamma} \in \mathcal{S}_L(a)^{\gamma} \cap b^{\gamma\rho\kappa} = \mathcal{R}_L^{\gamma} \quad \text{and} \quad y^{\gamma} \in \mathcal{S}_R(b)^{\gamma} \cap a^{\gamma\lambda\kappa} = \mathcal{R}_R^{\gamma}\,,$$

where $\mathcal{R}_L \subset \mathcal{S}_L(a)$ and $\mathcal{R}_R \subset \mathcal{S}_R(b)$ are reguli, since their γ-images are regular conics containing $\{a^{\gamma}, a^{\pi\gamma}\}$ and $\{b^{\gamma}, b^{\pi\gamma}\}$, respectively. In order to establish that $\mathcal{R}_L$ and $\mathcal{R}_R$ are mutually opposite, it is sufficient to show that the planes spanned by $\mathcal{R}_L^\gamma$ and $\mathcal{R}_R^\gamma$ are polar with respect to κ. Using the law of modularity in the lattice of subspaces of $\mathcal{P}(V \wedge V)$ yields

$$\mathrm{span}(\mathcal{R}_L^\gamma)^\kappa = \big((a^{\gamma\lambda} \vee \mathcal{E}_L) \cap b^{\gamma\rho\kappa}\big)^\kappa = a^{\gamma\lambda\kappa} \cap (\mathcal{E}_R \vee b^{\gamma\rho}) = \mathrm{span}(\mathcal{R}_R^\gamma)\,. \quad \square$$

Remark 1 Instead of the mapping $\gamma\lambda : \mathcal{L}(V) \to \mathcal{E}_R$ one could also assign to every line $a \in \mathcal{L}(V)$ its only left parallel line through some fixed point of $\mathcal{P}(V)$ or, dually, in a fixed plane of $\mathcal{P}(V)$. This gives a non-injective idempotent mapping $\mathcal{L}(V) \to \mathcal{L}(V)$ that is preserving the relation $\sim$. Cf. [20], where this is discussed for real elliptic 3-spaces.

Remark 2 Recall the settings at the beginning of section 3. The discriminant of the form β (with respect to any basis) is a square if, and only if, $e_1e_2e_3 \in F^{(2)}$. If $e_1e_2e_3 \in F^{(2)}$ then we shall speak of a 3-dimensional classical elliptic space. One may easily show that $(\mathcal{P}(V), \mathcal{L}(V), \|_L, \|_R)$ is a projective double space, whence it can be described in terms of a quaternion skew field with centre F; cf. [9, p. 75].

Remark 3 We discuss some modifications of the settings stated at the beginning of section 3 when $\big(\mathcal{P}(V), \mathcal{L}(V)\big)$ is a 3-dimensional Pappian projective space over F.

If π is a non-projective elliptic polarity or if F is a field with characteristic 2, then a (possibly non-projective) antipodal collineation α may be defined in an analogous way and Lemma 2 remains true.

If π is projective and $\operatorname{char} F = 2$ then all results on π, β, α and f up to formula (4) remain true. However, (4) is non longer basis of $V \wedge V$. If α has an invariant point then all α-invariant points form a plane $\mathcal{E}$, say. The linear mapping

$$\vartheta : \mathcal{P}(V \wedge V) \setminus \mathcal{E} \to \mathcal{E}\,, \quad X \mapsto (X \vee X^\alpha) \cap \mathcal{E}\,,$$

induced by $f + \sqrt{e_1e_2e_3}\,\mathrm{id}$, is replacing the projections (5). Defining Clifford parallel lines as before yields that $a\|b$ is equivalent to $a^{\gamma\vartheta} = b^{\gamma\vartheta}$, whence $\|$ is an equivalence relation. We mention without proof that $\big(\mathcal{P}(V), \mathcal{L}(V), \|\big)$ is a projective parallelogram space so that it permits an algebraic description in terms of a pure separable extension field of F [9, p. 75].

4 – Plücker transformations of classical elliptic 3-spaces

At first we are going to discuss the invariance of left and right parallelism under Plücker transformations:

THEOREM 2. *Let* $(\mathcal{P}(V), \mathcal{L}(V), \pi)$ *be a* 3*-dimensional classical elliptic space. Every Plücker transformation* $\varphi : \mathcal{L}(V) \to \mathcal{L}(V)$ *has the following properties:*

(I) *Clifford parallelism of lines is preserved.*

(II) $a^{\pi\varphi} = a^{\varphi\pi}$ *for all* $a \in \mathcal{L}(V)$.

(III) *If two left parallel lines* $a \in \mathcal{L}(V)$, $b \in \mathcal{L}(V) \setminus \{a, a^\pi\}$ *go over to left parallel lines, then left and right parallelism of lines is invariant; otherwise left and right parallelism is interchanged.*

PROOF. (I) This is trivial by definition.

(II) Note that a^π can be characterized as the only line $y \in \mathcal{L}(V) \setminus \{a\}$ such that $x \approx a$ is equivalent to $x \approx y$ for all $x \in \mathcal{L}(V)$.

(III) We shall confine our attention to $a^\varphi \|_L b^\varphi$. Choose any line $x \in \mathcal{S}_L(a) = \mathcal{S}_L(b)$. Hence x^φ is parallel to a and b. There are two possibilities: If $x^\varphi \|_L a^\varphi$ or $x^\varphi \|_L b^\varphi$ then $x^\varphi \in \mathcal{S}_L(a^\varphi)$, as required. Otherwise $x^\varphi \|_R a^\varphi$ and $x^\varphi \|_R b^\varphi$, whence $a^\varphi \|_R b^\varphi$, so that $a^{\pi\varphi} = b^\varphi$, an absurdity. Repeating these arguments for φ^{-1} establishes that $\mathcal{S}_L(a^\varphi) = \mathcal{S}_L(a)^\varphi$ is a left parallel class.

Next suppose $c \approx a$. The arguments from above show that $\mathcal{S}_L(c)^\varphi$ is either a left or right parallel class. For every line $x \in \mathcal{S}_L(a)$ there exists a concurrent line $y \in \mathcal{S}_L(c)$ by Lemma 3, whence $x \approx y$ follows from Lemma 4. But this property carries over to $\mathcal{S}_L(a)^\varphi$ and $\mathcal{S}_L(c)^\varphi$. Thus, by Lemma 4, $\mathcal{S}_L(c)^\varphi$ is a left parallel class. Finally, by virtue of (1), we may drop the assertion $c \approx a$, whence $\mathcal{S}_L(c)^\varphi$ is a left parallel class for all $c \in \mathcal{L}(V)$. □

By Theorem 2 (III), a Plücker transformation $\varphi : \mathcal{L}(V) \to \mathcal{L}(V)$ is either direct or opposite, i.e. preserving or interchanging left and right parallelism, respectively. All partial π-transformations are direct. The product of an elliptic reflection (a harmonic homology with centre $C \in \mathcal{P}(V)$, say, and axis C^π) with an opposite Plücker transformation yields a direct Plücker transformation. Thus we may restrict our attention to direct transformations.

We introduce a definition: Two collineations $\zeta : \mathcal{E}_L \to \mathcal{E}_L$ and $\eta : \mathcal{E}_R \to \mathcal{E}_R$ are called admissible if they satisfy the following conditions:

(Ad1) ζ and η are commuting with κ_L and κ_R, respectively.

(Ad2) Whenever a line $X \vee Y$ ($X \in \mathcal{E}_L$, $Y \in \mathcal{E}_R$) has non-empty intersection with the Klein quadric Γ then $(X^\zeta \vee Y^\eta) \cap \Gamma$ is non-empty too.

Theorem 3. *Let $(\mathcal{P}(V), \mathcal{L}(V), \pi)$ be a 3-dimensional classical elliptic space.*

(I) *If $\varphi : \mathcal{L}(V) \to \mathcal{L}(V)$ is a direct Plücker transformation then*

$$\begin{aligned} &\varphi_L : \mathcal{E}_L \to \mathcal{E}_L\,,\ a^{\gamma\rho} \mapsto a^{\varphi\gamma\rho} \quad \text{and} \\ &\varphi_R : \mathcal{E}_R \to \mathcal{E}_R\,,\ a^{\gamma\lambda} \mapsto a^{\varphi\gamma\lambda} \qquad (a \in \mathcal{L}(V)) \end{aligned}$$

are admissible collineations.

(II) *A homomorphism from the group of direct Plücker transformations of $(\mathcal{L}(V), \sim)$ into $P\Gamma L(\mathcal{E}_L) \times P\Gamma L(\mathcal{E}_R)$ is given by $\varphi \mapsto (\varphi_L, \varphi_R)$. The kernel of this homomorphism is formed by all partial π-transformations.*

(III) *Let $\zeta : \mathcal{E}_L \to \mathcal{E}_L$ and $\eta : \mathcal{E}_R \to \mathcal{E}_R$ be two admissible collineations. Then there exists a direct Plücker transformation $\varphi : \mathcal{L}(V) \to \mathcal{L}(V)$ such that $\varphi_L = \zeta$, $\varphi_R = \eta$.*

Proof. (I) The γ-image of a ruled plane is a plane on the Klein quadric. Thus $\Gamma^\rho = \mathcal{E}_L$. The fibres of $\rho|\Gamma$ are exactly the Klein images of the equivalence classes with respect to right parallelism so that φ_L is a well-defined bijection by Theorem 2 (III). Applying Lemma 4 yields that φ_L takes κ_L-conjugate points to κ_L-conjugate points. Finally, by the dual of Theorem 1, φ_L is a collineation commuting with κ_L. These results carry over to φ_R.

If we are given points $X \in \mathcal{E}_L$, $Y \in \mathcal{E}_R$ such that $(X \vee Y) \cap \Gamma \neq \emptyset$ then

$$X = a^{\gamma\rho} = a^{\pi\gamma\rho}\,, \quad Y = a^{\gamma\lambda} = a^{\pi\gamma\lambda}$$

for some line $a \in \mathcal{L}(V)$, say. Joining the φ_L-image of X and the φ_R-image of Y yields a line carrying the points $a^{\varphi\gamma}$, $a^{\pi\varphi\gamma}$ of Γ.

(II) This is obviously true.

(III) By the axiom of choice there exists a subset $\overline{\mathcal{L}} \subset \mathcal{L}(V)$ such that

$$\overline{\mathcal{L}} \cup \overline{\mathcal{L}}^{\pi} = \mathcal{L}(V)\,, \quad \overline{\mathcal{L}} \cap \overline{\mathcal{L}}^{\pi} = \emptyset\,.$$

We define a mapping φ: $\mathcal{L}(V) \to \mathcal{L}(V)$ as follows:

- If $a \in \overline{\mathcal{L}}$ then there exists a unique line in $\overline{\mathcal{L}}$, say b, such that

$$(a^{\gamma\rho\varsigma} \vee a^{\gamma\lambda\eta}) \cap \Gamma = \{b^{\gamma}, b^{\pi\gamma}\}\,.$$

Let $a^{\varphi} := b$.

- If $a \in \mathcal{L}(V) \setminus \overline{\mathcal{L}}$ then $a^{\pi} \in \overline{\mathcal{L}}$ and we set $a^{\varphi} := a^{\pi\varphi\pi}$.

This mapping φ has the required properties by Lemma 4. □

Theorem 3 is a generalization of a result from the kinematics of real elliptic 3-spaces[(8)] or, in algebraic terms, a result on an isomorphism of classical groups; cf., e.g., [3, p. 6ff], [6, p. 107ff], [7, p. 252], [11, p. 323], [19, p. 18ff] for details and further references.

To sum up, we have shown that for a 3-dimensional classical elliptic space all Plücker transformations of $(\mathcal{L}(V), \sim)$ can be obtained according to the proof of Theorem 3, possibly followed by an elliptic reflection.

Now we are going to express the previous results in algebraic terms. Recall the eigenbasis (4) of the linear mapping f inducing the antipodal collineation α. Write

$$E_L := \operatorname{span}\{c_0, c_1, c_2\}\,, \quad E_R := \operatorname{span}\{c_3, c_4, c_5\}\,.$$

In the sequel we shall represent the Klein quadric by the quadratic form

$$Q := e_1(\sqrt{e_1e_2e_3})^{-1}q$$

and we shall change to the new basis

$$\begin{aligned} d_0 &:= (\sqrt{e_1e_2e_3})^{-1}c_0\,, & d_3 &:= (\sqrt{e_1e_2e_3})^{-1}c_3\,,\\ d_1 &:= e_1^{-1}c_1\,, & d_4 &:= e_1^{-1}c_4\,,\\ d_2 &:= e_1^{-1}c_2\,, & d_5 &:= e_1^{-1}c_5 \end{aligned}$$

(8) In that context instead of two elliptic planes two Euclidean spheres are used and the lines of $\mathcal{L}(V)$ are subject to orientation. Dropping orientation of lines forces to identify antipodal points on those spheres and yields two elliptic planes.

of $V \wedge V$. Then

$$\Big(\sum_{j=0}^{5} x_j d_j\Big)^Q = x_0^2 + e_3 x_1^2 + e_2 x_2^2 - x_3^2 - e_3 x_4^2 - e_2 x_5^2 \quad (x_j \in F)\,.$$

LEMMA 6. *Two collineations $\zeta : \mathcal{E}_L \to \mathcal{E}_L$ and $\eta : \mathcal{E}_R \to \mathcal{E}_R$ are admissible if, and only if, they can be induced by mappings $g \in \Gamma L(E_L)$ and $h \in \Gamma L(E_R)$, respectively, such that the following conditions hold true for some constant $k \in F \setminus \{0\}$:*

(I) $x^{gQ} = k(x^{QG})$ *for all* $x \in E_L$.

(II) $x^{hQ} = k(x^{QH})$ *for all* $x \in E_R$.

(III) *If* $Fa \in \mathcal{E}_L$, $Fb \in \mathcal{E}_R$ *and* $-(b^Q)^{-1}a^Q \in F^{(2)}$ *then* $-(b^{QH})^{-1}a^{QG} \in F^{(2)}$.

Here $G, H \in \mathrm{Aut}(F)$ denote the companion automorphisms of g, h, respectively.

PROOF. (a) Let ζ and η be admissible. Choose any semilinear mapping g inducing ζ assume that g belongs to $G \in \mathrm{Aut}(F)$. Since ζ is commuting with κ_L, there exists a constant $k \in F \setminus \{0\}$ such that (I) holds true. Similarly η can be induced by a mapping $h' \in \Gamma L(E_R)$ with companion automorphism $H \in \mathrm{Aut}(F)$ satisfying

$$x^{h'Q} = l(x^{QH}) \quad \text{for all} \quad x \in E_R$$

and some constant element $l \in F \setminus \{0\}$.

Given points $Fa \in \mathcal{E}_L$ and $Fb \in \mathcal{E}_R$ then $(Fa \vee Fb) \cap \Gamma \neq \emptyset$ holds if, and only if, the equation $a^Q + b^Q x^2 = 0$ has a solution in F, which in turn is equivalent to

$$-(b^Q)^{-1} a^Q \in F^{(2)}\,. \tag{6}$$

Applying this condition to Fd_0 and Fd_3 yields $-(d_3^Q)^{-1} d_0^Q = 1$. Since ζ and η are admissible, $(Fd_0)^\zeta \vee (Fd_3)^\eta$ too has common points with Γ or, equivalently,

$$-(d_3^{h'Q})^{-1} d_0^{gQ} = l^{-1}k \in F^{(2)}\,.$$

But this allows to replace h' by the semilinear mapping

$$h := \sqrt{l^{-1}k}h'$$

so that (II) is fulfilled. Finally, (6) implies $((Fa)^\zeta \vee (Fb)^\eta) \cap \Gamma \neq \emptyset$, whence

$$-(b^{hQ})^{-1}a^{gQ} = -k^{-1}(b^{QH})^{-1}ka^{QG} \in F^{(2)}\,.$$

(b) If g and h are given subject to (I), (II), (III), then the induced collineations are easily seen to be admissible. ∎

Theorem 4. *Let $(\mathcal{P}(V), \mathcal{L}(V), \pi)$ be a 3-dimensional classical elliptic space and let $\varphi : \mathcal{L}(V) \to \mathcal{L}(V)$ be a direct Plücker transformation. There exists a partial π-transformation $\delta : \mathcal{L}(V) \to \mathcal{L}(V)$ such that $\varphi\delta$ is induced by a collineation $\mu \in P\Gamma O(\mathcal{P}(V), \pi)$ if, and only if, the collineations $\varphi_L : \mathcal{E}_L \to \mathcal{E}_L$ and $\varphi_R : \mathcal{E}_R \to \mathcal{E}_R$ (described in Theorem 3) belong to the same automorphism of the ground field*[9].

Proof. (a) If δ and μ are existing then, by Lemma 1, $\mu \in P\Gamma O(\mathcal{P}(V), \pi)$. There is a unique automorphic collineation σ of the Klein quadric such that $a^{\mu\gamma} = a^{\gamma\sigma}$ for all $a \in \mathcal{L}(V)$. Moreover, σ and the antipodal collineation α are commuting. We infer $\mathcal{E}_L^\sigma = \mathcal{E}_L$ and $\mathcal{E}_R^\sigma = \mathcal{E}_R$ from $\varphi\delta$ being direct. Hence

$$\varphi_L = (\varphi\delta)_L = \sigma|\mathcal{E}_L \quad \text{and} \quad \varphi_R = (\varphi\delta)_R = \sigma|\mathcal{E}_R$$

so that φ_L and φ_R belong to the same automorphism of the ground field.

(b) Assume that φ_L and φ_R belong to the automorphism of the ground field. Thus, according to Lemma 6, we may choose semilinear mappings $g \in \Gamma L(E_L)$ and $h \in \Gamma L(E_R)$ with the same companion automorphism $G = H$. There exists a unique semilinear mapping $s \in \Gamma L(V \wedge V)$ extending both g and h. Since $V \wedge V = E_L \oplus E_R$ is an orthogonal direct sum (with respect to the bilinear form associated to Q), we obtain

$$x^{sQ} = k(x^{QG}) \quad \text{for all} \quad x \in V \wedge V\,.$$

[9] An example of two admissible collineations with different companion automorphisms is given at the end of the paper.

Therefore the mapping s gives rise to an automorphic collineation σ, say, of the Klein quadric. We observe that σ and α are commuting. Hence there exists either a collineation or a duality, say ω, of $(\mathcal{P}(V), \mathcal{L}(V))$ commuting with the absolute polarity π such that $a^{\omega\gamma} = a^{\gamma\sigma}$ for all $a \in \mathcal{L}(V)$. If ω is a collineation then set $\mu := \omega$, else put $\mu := \pi\omega$. So under all circumstances we obtain a collineation $\mu \in P\Gamma O(\mathcal{P}(V), \pi)$ such that

$$\{a^{\mu}, a^{\pi\mu}\} = \{a^{\varphi}, a^{\pi\varphi}\} \quad \text{for all} \quad a \in \mathcal{L}(V)\,.$$

Thus there exists a partial π-transformation δ with required properties. □

Since every opposite Plücker transformation equals the product of an elliptic reflection and a direct Plücker transformation, Theorem 4 immediately implies

THEOREM 5. *Let $(\mathcal{P}(V), \mathcal{L}(V), \pi)$ be a 3-dimensional classical elliptic space such that every automorphism of the ground field F is trivial. Then for every Plücker transformation $\varphi : \mathcal{L}(V) \to \mathcal{L}(V)$ there exists a partial π-transformation $\delta : \mathcal{L}(V) \to \mathcal{L}(V)$ such that $\varphi\delta$ is induced by a* (*necessarily projective*) *collineation $\mu \in P\Gamma O(\mathcal{P}(V), \pi)$.*

Theorem 5 describes, e.g., the Plücker group of the real elliptic 3-space.

REMARK 4 Let F be a commutative Pythagorean field. Then every sum of non-zero squares in F is again a non-zero square in F. Following [1, p. 73ff] we discuss the elliptic space on $V = F^4$ with an absolute polarity π induced by the standard bilinear form. Thus (b_0, b_1, b_2, b_3) can be chosen as the standard basis, so that $e_1 = e_2 = e_3 = 1$. Hence

$$\Big(\sum_{j=0}^{5} x_j d_j\Big)^{Q} = x_0^2 + x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 \quad (x_j \in F)\,,$$

$$x^Q \in F^{(2)} \quad \text{for all} \quad x \in E_L\,, \quad -y^Q \in F^{(2)} \quad \text{for all} \quad y \in E_R\,.$$

Now, by Lemma 6, condition (Ad2) in the definition of admissible collineations is automatic.

We note the following consequence: Write $\mathcal{L}_\pi(V)$ for the set of unordered pairs $\{a, a^\pi\}$ where $a \in \mathcal{L}(V)$. Then

$$\iota : \mathcal{L}_\pi(V) \to \mathcal{E}_L \times \mathcal{E}_R\,, \quad \{a, a^\pi\} \mapsto (a^{\gamma\rho}, a^{\gamma\lambda})$$

is a bijection. $\mathcal{E}_L \times \mathcal{E}_R$ may be regarded as the Corrado Segre product space of the projective planes $\mathcal{E}_L$ and $\mathcal{E}_R$ (cf. [18, p. 211], [15]) and, by virtue of ι^{-1}, we obtain an isomorphic partial line space with "point set" $\mathcal{L}_\pi(V)$. Two "points" $\{a, a^\pi\}, \{b, b^\pi\} \in \mathcal{L}_\pi(V)$ are "collinear" if, and only if, the lines $a, b \in \mathcal{L}(V)$ are Clifford parallel. It is immediate that all Plücker transformations of $\big(\mathcal{L}(V), \sim\big)$ induce automorphisms of this partial line space by their action on $\mathcal{L}_\pi(V)$.

Remark 5 The field $\mathbb{R}((T))$ of formal Laurent series with real coefficients is Pythagorean (cf., e.g., [10, p. 204]) and admits a non-trivial automorphism G taking T to $T+1$. Defining

$$g : E_L \to E_L\,, \quad \sum_{j=0}^{2} x_j d_j \mapsto \sum_{j=0}^{2} x_j^G d_j \quad \big(x_j \in \mathbb{R}((T))\big)$$

and letting h be the identity on E_R, yields a direct Plücker transformation that does not permit a factorization into an orthogonality-preserving collineation and a partial π-transformation.

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