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On the scientific work of Olga Oleinik

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PRESENTAZIONE: *Questo articolo è tratto da una conferenza tenuta durante il Convegno Internazionale "Non linear evolution problems" (Roma – Università "La Sapienza" e C.N.R. – 2-6 ottobre 1995), nella giornata dedicata ad Olga Oleinik in occasione del suo settantesimo compleanno.*

In esso si passa in rassegna la produzione scientifica della Oleinik con particolare riferimento a quattro settori, molto importanti nel campo delle equazioni a derivate parziali, che presentano tuttora un notevole interesse, e nei quali il suo contributo è stato fondamentale. Essi sono:

1) *la teoria delle soluzioni discontinue delle equazioni e dei sistemi di equazioni iperbolici delle leggi di conservazione;*
2) *il problema di Stefan relativo ai cambiamenti di fase;*
3) *le equazioni a derivate parziali lineari del secondo ordine con forma caratteristica non negativa;*
4) *l'individuazione di classi di equazioni e di sistemi di equazioni a derivate parziali le cui soluzioni sono analitiche.*

ABSTRACT: *The paper is a survey of scientific work of Olga Oleinik with particular regard to four arguments:*

1) *discontinuous solutions of hyperbolic conservation equations or systems;*
2) *Stefan's phase-transitions problem;*
3) *partial linear differential equations with non negative characteristic form;*
4) *determination of classes of partial differential equations or systems, with analytic solutions.*

KEY WORDS AND PHRASES: *Partial differential equations – Conservation equations – Stefan's problem*
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1 – Introduction

First of all I would like to thank the “Organizing Committee” for the invitation to give a lecture on the scientific work of *Olga Oleinik*. I am very glad and honored for that, not only because she is a worldwide outstanding mathematician, but also because of her friendly relations with the Italian mathematical community.

O. Oleinik was born in Matusov, a town in the region of Kiev, on the second of july, 1925. After one year at the Perm’ University, she moved to the Moscow State University, where she completed her mathematical studies and obtained the Candidate’s degree in 1950 and the Doctor’s degree in 1954, having as scientific advisor I. Petrowsky. From 1950 and up to now she has been working in the Chair of differential equations at the Moscow State University, first as an assistent professor, afterwards as an associate professor and finally, from 1955, as a full professor. Since 1973 she is the head of the Chair succeeding I. Petrowsky.

The mathematical work of O. Oleinik is very wide and deep: more than 340 papers and 8 books; it is impossible to give a satisfactory description of it in a short time. Therefore I shall limit myself to describe with some details only some of her researches and to mention the others. The choice I made depends on my mathematical knowledge and it does not mean that these researches are the most important or the deepest. Anyway I hope to succeed in giving you an idea of the exceptional quality of the scientific work of O. Oleinik both for the importance of the results and for the deepness and the difficulty of the used techniques. For a more extensive exposition and a complete list of the publications I refer to the two following papers: the first one by V.I. Arnold, M.J. Vishik, Ju. V. Egorov, A.S. Kalashnikov, A.V. Kolmogorov, S.P. Novikov, S.L. Sobolev in the “Uspehi” 1985, the second by N.S. Bakhvalov, S.P. Novikov, A.T. Fomenco in “Uspehi” 1995.

The first works of O. Oleinik, some of them carried out in collaboration with I. Petronsky, concern some topological problems related to the sixteenth Hilbert’s problem and they give remarkable results (the estimates for the Euler characteristic of real algebraic manifolds, so called the Petrowsky-Oleinik inequalities [2], the estimates for the Betti numbers [3], the topology of real algebraic curves on an algebraic surface [4]), which have served as a base for subsequent studies, in particular of V.I. Arnold.

But all the remaining part of her production concerns the field of partial differential equations and its applications. The title of her Doctoral thesis was "Boundary-value problems for partial differential equations with a small parameter by the highest derivatives and the Cauchy problem in the large for nonlinear equations" (a summary was published in [8]). Moreover one of her first and meaningful results was an elementary proof of a famous lemma (usually called Hopf's lemma) on the sign of the derivative, along a direction non tangent to the boundary, of the solution of a second order linear elliptic equation, at a boundary extremum point [5].

In the following sections we shall consider O. Oleinik researches concerning:

1) the theory of discontinuous solutions of nonlinear hyperbolic equations of conservations laws (§ 2);
2) the Stefan problem (§ 3);
3) the linear second order P.D.E. with non negative characteristic form (§ 4);
4) the analiticity of the solutions of linear P.D.E. and systems of P.D.E. (§ 5);

Finally in § 6 we shall list, with some remarks, the other researches.

2 – Discontinuous solutions of P.D.E. and systems of P.D.E. of conservation laws

Equations and systems of conservation laws were studied since a long time not only by mathematicians (S.D. Poisson, E.S. Stokes, B. Riemann, W.J.U. Rankine, H. Hugoniot are some of the most famous names). But the interest of the mathematicians for a rigorous mathematical theory dates from the period during and immediately after the second world war, thanks to scientists as R. Courant, K. Friedrichs, J. Von Neumann, H. Weyl. Starting from a paper of E. Hopf (in Comm. Pure Appl. Math. 1950), the theory has been developed in the years 54-57 by O. Oleinik and P.D. Lax, who can be consider as its "founders".

In order to explain Oleinik's work in this field (see in particular [6]-[13], [16], [17], [26]), I shall limit myself to consider the simplest but in any case very meaningful case of the Cauchy problem for a first order equation in one space variable

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0\,,\quad u(x,0)=u_0(x) \tag{2.1}$$

in the domain

$$\Omega=\{0\le t\le T<\infty\,,\ \ -\infty<x<+\infty\}\,.$$

First let me suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$, sufficiently smooth and convex and u_0 is a measurable bounded function from $\mathbb{R}$ to $\mathbb{R}$. Since the beginning of the "fifthies" it was clear that the classical theory of the characteristics curves was inadeguate to obtain solutions in "large", with possible discontinuities, as required for the applications, in particular to gas-dynamics. It was necessary to overcome the idea of classical solutions, as it was made for the linear P.D.E. Therefore if we want to write the equation in the distribution sense in Ω, it is natural to introduce the following definition of weak solution of (2.1).

Definition 1. *A bounded measurable function u in Ω is a weak solution of* (2.1) *if we have*

$$\int_\Omega\Big(u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}\Big)dx\,dt+\int_{-\infty}^{+\infty}u_0(x)\varphi(x,0)\,dx=0 \tag{2.2}$$

for any function φ sufficiently smooth in $\overline{\Omega}$, with compact support and $\varphi(x,T)=0$.

Definition 1 allows to consider discontinuous solutions and also to find again the "Rankine-Hugoniot" jump condition, still shown for some concrete gas-dynamics problems. More precisely: if $u(x,t)$ is a classical solution in $\overline{\Omega}$, except along a regular curve $x=x(t)$ $(0\le t\le T)$, where u can have a jump (as function of x, for every fixed t) given by u_+-u_-, $u_+=u(x(t)-0,t)$, $u_-=u(x(t)+0,t)$, the jump condition says that

$$\frac{dx}{dt}=\frac{f(u_+)-f(u_-)}{u_+-u_-} \tag{2.3}$$

and gives the speed of propagation of the discontinuity along the curve as function of the jump.

Nevertheless, even if we consider only discontinuous solutions of this type, condition (2.3) does not imply the uniqueness of the solution of (2.2) (a simple example is given by the Burgers equation). Therefore we have to find a new "admissibility condition", significant from the physical point of view. This condition is exactly "Oleinik's entropy condition", which express the growth of the entropy along the discontinuity on the curve $x = x(t)$

$$f'(u_+) \leq \frac{dx}{dt} \leq f'(u_-)\,. \tag{2.4}$$

Conditions (2.3) and (2.4) characterize the so called "shocks" and can be used for a new definition of "weak solution" of problem (2.1). The "Oleinik" condition indeed can also be written in the following form, which can be applied to every measurable bounded function u; there exists a suitable continuous function $K(t)$ for $0 < t \leq T$ such that

$$\int_{-\infty}^{+\infty} \Big[\frac{\partial \psi}{\partial t} u + K(t)\psi\Big] dx \geq 0 \tag{2.5}$$

for any $t > 0$ and any smooth and non negative function ψ in $\overline{\Omega}$. Therefore following O. Oleinik, we can give the following new definition:

Definition 2. *A bounded measurable function $u(x,t)$ in Ω is called weak solution of* (2.1) *if it satisfies* (2.2) *and* (2.5)*.*

Using this new definition O. Oleinik proves the uniqueness theorem for (2.1) and moreover she proves that, if u and v are two weak solutions related to two initial data u_0 and v_0, we have that

$$\int_{-\infty}^{+\infty} |u(x,t) - v(x,t)| dx \quad \text{is a decreasing function of} \quad t \tag{2.6}$$

(stability in L^1 – norm for the solution of (2.1)).

Concerning the existence theorem it is possible to follow two methods. First the method of "artificial viscosity", which was suggested for the gas dynamics problems by J. Von Neumann and R.D. Rictmyer and by E. Hopf in his paper of 1950, still quoted. Let us indeed consider the parabolic problem for $\varepsilon > 0$

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial f(u_\varepsilon)}{\partial x} - \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2} = 0\,, \quad u_\varepsilon(x,0) = u_0(x)\,. \tag{2.7}$$

This problem has a unique solution u_ε, with some regularity properties. By using the maximum principle and some suitable estimates and compactness arguments, O. Oleinik proves that u_ε tends to a weak solution u of (2.1) in the following sense

$$\lim_{\varepsilon \longrightarrow o} \int_{-a}^{a} |u_\varepsilon(x,t) - u(x,t)| dx = 0 \quad \forall t > 0 \quad \text{and} \quad \forall a > 0\,. \tag{2.8}$$

Moreover she proves that this solution (unique for the uniqueness theorem) also verifies an explicit representation formula, introduced for (2.1) by P. Lax (in Comm. Pure Appl. Math. 1954) which is an extension to (2.1) of the Hopf's formula for the Burgers equation. This formula given in [6]-[11] in the most general case, allows to prove many other important properties for the solution u:

a) first, that the discontinuity points of u are on a finite or countable number curves of equation $x = x_i(t)$ $(i = 1, 2, \dots)$ and they are all "shocks";
b) the initial condition $u(x,0) = u_0(x)$ is verified in the following sense:

$$\lim_{t \longrightarrow 0+} \int_{-\infty}^{+\infty} \varphi(x)[u(x,t) - u_0(x)] dx = 0 \tag{2.9}$$

for any function φ continuous, with compact support in $\mathbb{R}$;

c) for any $t > 0$, $u(x,t)$ is a BV function in any bounded interval (let us remark that the BV space is fundamental for equations in more than one space variable and for systems).

A second method to prove an existence theorem is the method of finite difference introduced by P. Lax in the following way:

for any positive numbers h and l let us define the algorithm:

$$u_n^{k+1} = \frac{1}{2}[u_{n+1}^k + u_{n-1}^k] - [f(u_{n+1}^k) - f(u_{n-1}^k)]\frac{h}{2l} \tag{2.10}$$

where $k = 0, 1, 2, \dots$ and $n = \pm 1, \pm 3, \pm 5, \dots$ for $k = 1, 3, 5 \dots$ and $n = 0, \pm 2, \pm 4, \dots$ for $k = 0, 2, 4, \dots$; moreover $u_n^0 = u_0(nl)$.

Then let us define the function

$$u_{h,l}(x,t) = u_n^k \quad \text{for} \quad (k-1)h < t \leq kh\,, \quad (k \geq 1), \quad (n-1)l < x \leq nl\,.$$

Then O. Oleinik in [11], [26] and her student N.D. Vvedenskaja proved the almost-everywhere convergence in Ω, when h and l tend to zero in a suitable way, of $u_{h,l}$ to a weak solution of (2.1).

Let us now come back to the hypothesis that f is convex (or, with obvious modifications, concave). O. Oleinik in [17] also studied the general case, by replacing the "entropy" condition (2.4) with the following one: under the same hypotheses on u and the same notation for u_+ and u_-, let us introduce the function

$$l(u) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}(u - u_+)f(u_+)\,;$$

then the new "entropy" condition is written as:

$$\text{(2.11)} \qquad \begin{cases} l(u) \leq f(u) \quad \text{for} \quad u \in [u_-, u_+] \quad \text{if} \quad u_- < u_+ \quad \text{and} \\ l(u) \geq f(u) \quad \text{for} \quad u \in [u_+, u_-] \quad \text{if} \quad u_+ < u_-\,. \end{cases}$$

Under these more general hypotesis on f, O. Oleinik proved the uniqueness theorem and A.S. Kalashnikov, a student of O. Oleinik, proved the existence theorem by the method of "artificial viscosity". We also remark that the theory has been developed by O. Oleinik by the same techniques for the more general equation

$$\text{(2.12)} \qquad \frac{\partial u}{\partial t} + \frac{\partial f(x,t,u)}{\partial x} + g(x,t,u) = 0\,.$$

In conclusion we can say that O. Oleinik developed a general and complete theory for the first order equation of conservation laws in one space variable; this theory is now a standard chapter in the books on hyperbolic equations.

Finally let us remark that also for systems of equations the contributions of O. Oleinik have been very important: in [12] she proved the uniqueness theorem for the "so called" p-systems, that is of the following form:

$$
(2.13)\qquad \begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial x} = 0\,, \quad \dfrac{\partial u}{\partial t} + \dfrac{\partial p(v)}{\partial x} = 0 \quad \text{in} \quad \Omega \\[2ex] u(x,0) = u_0(x)\,, \quad v(x,0) = v_0(x) \end{cases}
$$

under suitable assumptions on u_0, v_0, p and on the definitions of weak solution (u,v) of (2.13).

3 – The Stefan problem

One of the peculiarities of the scientific work of O. Oleinik has been the constant interest in mathematical problems important for applications, as we have seen in § 2 for the gas-dynamics. Also the Stefan problem comes from a very important application: the phase-change phenomena. Stefan problem is the oldest model of these phenomena (Stefan's paper was published a century ago). Nevertheless until the last years 1950 a satisfactory general mathematical formulation didn't exist. It was thanks to O. Oleinik (see [18] and also [28]) and her student S. Kamin (this one in her candidate's degree thesis and in "Mat. Sbornik" 1961) that the "good" concept of weak solution was introduced for that problem. I'm going to explain here the results of [18] in the simplest case, the classical two-phases Stefan problem (solid-liquid) in one space variable: let Ω be the rectangular domain $\Omega = \{(x,t) : 0 < x < l,\ 0 < t < T\}$; we have to find a sufficiently smooth function $s(t)$, $0 \le t \le T$, such that $x = s(t)$ represents the line N (the "free boundary") separating at each time t the two phases, and two sufficiently smooth functions $\theta_i(x,t)$, $i = 1,2$, representing the temperature in the two phases, such that:

$$
(3.1)\qquad \begin{cases} \rho_1 c_1 \dfrac{\partial \theta_1}{\partial t} - k_1 \dfrac{\partial^2 \theta_1}{\partial x^2} = 0 \ \text{ in } \Omega_1 = \{(x,t) :\ 0 < t < T,\ 0 < x < s(t)\} \\[2ex] \rho_2 c_2 \dfrac{\partial \theta_2}{\partial t} - k_2 \dfrac{\partial^2 \theta_2}{\partial x^2} = 0 \ \text{ in } \Omega_2 = \{(x,t) :\ 0 < t < T,\ s(t) < x < l\}; \end{cases}
$$

$$
s(0) = b \in]0, l[;
$$

$$(3.2)\quad \begin{cases} \theta_1 = \theta_2 (= 0) & \text{on } N \text{ (temperature's continuity)} \\ -k_1\Big(\dfrac{\partial \theta_1}{\partial x}\Big)^- + k_2\Big(\dfrac{\partial \theta_2}{\partial x}\Big)^+ = \rho_2 L \dfrac{ds}{dt} & \text{on } N \text{ ("Stefan" energy balance)} \end{cases}$$

$$(3.3)\quad \begin{cases} \theta_1(x,0) = h_1(x),\ 0 \le x \le b; & \theta_2(x,0) = h_2(x)\,,\ b \le x \le l \\ \theta_1(0,t) = g_1(t)\,, & \theta_2(l,t) = g_2(t)\,,\ 0 \le t \le T \end{cases}$$

where the positive constants ρ_i, c_i, k_i $(i = 1,2)$ are respectively the density, the heat capacity and the conductivity of each phase ($i = 1$ for liquid, $i = 2$ for solid), L is the latent heat and h_1, h_2, g_1, g_2 are suitable given functions; moreover we suppose that the melting temperature is equal to zero and we denote by f_1^- (resp. f_2^+) the boundary value on N of a function f_1 defined in Ω_1 (resp. f_2 defined in Ω_2). This is the classical formulation of the problem.

Assume that a solution exists, set $\theta = \theta_1$ in Ω_1, $\theta = \theta_2$ in Ω_2, $h = h_1$ in $[0,b]$, $h = h_2$ in $[b,l]$ and suppose that $k_1 = k_2 = 1$ (there is no loss of generality in assuming this hypothesis). Then let us define the new quantity $E(\theta)$ (which represents the "enthalpy" of the phenomenon):

$$E(\theta(x,t)) = \int_0^{\theta(x,t)} a(\xi)d\xi + \rho_2 L H(\theta)$$

where $a(\xi) = \rho_2 c_2$ if $\xi < 0$, $a(\xi) = \rho_1 c_1$ if $\xi > 0$ and H is the Heaveside graph. We have to remark that, since N is supposed smooth (therefore with area equal to zero), we can take for $H(\theta)$ on N any value between 0 and 1. Then we can multiply (3.1) by any smooth "test" function $\varphi(x,t)$ such that $\varphi(0,t) = \varphi(l,t) = \varphi(x,T) = 0$, integrate over Ω_i, $i = 1,2$ and add the two contributions. Using the Green formula and (3.2), (3.3) we obtain that θ satisfies the following relation

$$(3.4)\quad \begin{aligned} &\int_0^T \int_\Omega \Big[E(\theta)\frac{\partial \varphi}{\partial t} + \theta \frac{\partial^2 \varphi}{\partial x^2}\Big] dx\,dt + \int_0^\ell E(h(x))\varphi(x,0)dx + \\ &\quad + \int_0^T \Big[g_1(t)\frac{\partial \varphi(0,t)}{\partial x} - g_2(t)\frac{\partial \varphi(l,t)}{\partial x}\Big] dt = 0 \end{aligned}$$

for any of such φ.

Now (3.4) suggests the following definition of a weak solution:

Definition Given $u_0 \in L^\infty(0,l)$, $g_i \in L^\infty(0,T)$, $i = 1,2$ find the functions u (the "enthalpy") and θ (the "temperature") belonging to $L^\infty(\Omega)$ such that $u \in E(\theta)$ and

$$\begin{aligned}\int_0^T \int_\Omega \Big[u\frac{\partial\varphi}{\partial t} + \theta\frac{\partial^2\varphi}{\partial x^2}\Big] dx\, dt + \int_0^\ell u_0\varphi(x,0)dx+ \\ + \int_0^T \Big[g_1\frac{\partial\varphi(0,t)}{\partial x} - g_2\frac{\partial\varphi(l,t)}{\partial x}\Big]dt = 0\end{aligned} \tag{3.5}$$

for any smooth function φ, such that $\varphi(0,t) = \varphi(l,t) = \varphi(x,T) = 0$.

Under suitable assumptions on $u_0.g_1, g_2$ O. Oleinik proved in [18], (see also [28]) the existence and the uniqueness of such a weak solution; in particular, for the construction of approximate solutions and the proof of existence theorem, she suggested a new approach based on the consideration of a quasi-linear parabolic equation with discontinuous coefficients and a family of equations obtained by smoothing these coefficients.

This definition was indeed the "good" one: its introduction provided the opportunity of dealing with more general phase-change models: multiphases problems, not constant density, heat capacity and conductivity, possible heat sources and convection terms, existence of "mushy regions" (when the "free boundary" N, defined by $N = \{(x,t) : \theta(x,t) = 0\}$ is an irregular curve or a set with positive area), the problems in several space variables (studied first by S. Kamin), the regularity properties of the solutions and of the "free boundary", the numerical and computational approach to the solution and to the free boundary,.... After the papers of O. Oleinik and S. Kamin a very interesting field of researches has been opened and many mathematicians have done important and deep results on it. Till now the mathematical model of phase-change phenomena are intensively studied, also in Italy.

4 – Second order linear equations with non-negative characteristic form (elliptic-parabolic equations)

Many important papers have been dedicated from O. Oleinik, partially in collaboration with her student E.V. Radkevich, to the elliptic-parabolic equations theory. Let us recall some definitions and notations:

we shall consider the P.D.E.:

$$L(u) \equiv \sum_{h,k}^{1,\dots,n} a_{hk} \frac{\partial^2 u}{\partial x_h \partial x_k} + \sum_{h=1}^{n} b_h \frac{\partial u}{\partial x_h} + cu = f \tag{4.1}$$

in an open bounded set Ω of $\mathbb{R}^n$, with boundary Σ (for simplicity connected), where a_{hk}, b_k, c, f and Ω shall be suitably smooth according to the results to be obtained. Equation (4.1) will be called *elliptic-parabolic* if the characteristic form is non-negative in $\overline{\Omega}$, namely if $\forall x \in \Omega \cup \Sigma$ we have

$$\sum_{h,k}^{1,\dots,n} a_{hk} \xi_h \xi_k \geq 0 \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n . \tag{4.2}$$

The interest of the mathematicians for this kind of P.D.E. was not too great in the first part of our century, when the studies of the three classical equations of mathematical-physics were deeply developed. But we have to recall at least some papers of M. Picone (in particular the one of 1913 in Mem. Acc. Lincei), the famous paper of F. Tricomi (in Mem. Acc. Lincei 1923) on the equations of mixed type (which contains an elliptic problem degenerate on the "parabolic boundary") and a paper of M.V. Keldys (in Dokl. Akad. Nauk 1951) on the elliptic equations in two variables, which degenerate on the boundary.

The interest became evident after a paper of G. Fichera (in Mem. Acc. Lincei 1955), where he considered in a general form the question of finding the boundary value problems well-posed for (4.1) and he suggested and studied the most important ones, that we are going to describe. Let $\nu = (\nu_1, \dots, \nu_n)$ be the inward unit normal vector to Σ. We denote by Σ_3 the noncharacteristic part of Σ, i.e. the set of points of Σ where the condition $\sum_{h,k}^{1,\dots,n} a_{hk} \nu_h \nu_k > 0$ holds. On the set $\Sigma - \Sigma_3$ we examine the function

$$b = \sum_{h=1}^{n} \Big(b_h - \sum_{k=1}^{n} \frac{\partial a_{hk}}{\partial x_k} \Big) \nu_h \tag{4.3}$$

and we denote by $\Sigma_0, \Sigma_1, \Sigma_2$ respectively the subset of $\Sigma - \Sigma_3$ where we have $b = 0$, $b > 0$, $b < 0$; we obtain that $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. The boundary value problems considered by Fichera are the ones in which on Σ_2 u is given and on Σ_3 a linear combination $\alpha u + \beta \frac{du}{dl}$ is given, where α and β are suitable functions and l is a suitable direction. In particular if $\alpha = 1$ and $\beta = 0$ the boundary conditions become

$$u = g \quad \text{on} \quad \Sigma_2 \cup \Sigma_3 \quad (g \text{ given}) \tag{4.4}$$

and the problem is called the first boundary value problem for (4.1) (analogous of the Dirichlet problem for the elliptic equations). We will suppose for simplicity $g = 0$. Therefore we are going to consider the problem

$$L(u) = f \quad \text{on} \quad \Omega\,, \quad u = 0 \quad \text{on} \quad \Sigma_2 \cup \Sigma_3\,. \tag{4.5}$$

Formally using the Green formula, the following definition of a weak solution of (4.5) is natural:

DEFINITION 1 . *Let assume $f \in L^p(\Omega)$, $1 < p < +\infty$; we shall say that $u \in L^p(\Omega)$ is a weak solution of (4.5) if u verifies*

$$\int_\Omega u L^*(v) dx = \int_\Omega f v \, dx \tag{4.6}$$

for any function v sufficiently smooth and zero on $\Sigma_1 \cup \Sigma_3$, where L^* is the formal adjoint operator of L^*.

Then, under suitable hypothesis of smoothness of the data (i.e. Ω, Σ_2, Σ_3, a_{hk}, b_k, c and f), in particular if $c < 0$ and $c^* < 0$, where c^* is the coefficient of ν in $L^*(v)$, Fichera showed the existence of at least one weak solution of (4.5), by a method which is essentially based on the proof of the following estimate

$$\|v\|_{L^q(\Omega)} \leq C \|L^*(v)\|_{L^q(\Omega)} \tag{4.7}$$

(C positive constant, $\frac{1}{p} + \frac{1}{q} = 1$) for any function v sufficiently smooth and zero on $\Sigma_1 \cup \Sigma_3$. In this paper Fichera posed the question of the uniqueness of the weak solution, and this question was solved by O. Oleinik among other important results for the equation (4.1), in the papers [23], [24], [32], [41], [42] and in the book [40]-[51], published in four languages (russian, english, italian and chinese). First of all she showed again the existence of a weak solution by using an elliptic "regularization" of (4.5), i.e. considering the Dirichlet problem for any $\varepsilon > 0$:

$$\varepsilon \Delta u_\varepsilon + L(u_\varepsilon) = f \quad \text{in} \quad \Omega\,, \quad u_\varepsilon = 0 \quad \text{on} \quad \Sigma\,. \tag{4.8}$$

In order to prove the existence of a limit u of u_ε as $\varepsilon \longrightarrow 0$, suitable estimates (uniform in ε) are proved for u_ε; then it is proved that u is a weak solution of (4.5). After that O. Oleinik studied the difficult uniqueness problem and, by using also a suitable elliptic regularization for the adjoint equation of (4.1), she showed the uniqueness under suitable assumptions on the data (in particular if the $(n-1)$ – dimensional measure of the boundary of Σ_2 on Σ is zero) in the case where $p \geq 3$, while the uniqueness does not hold for $p < 3$.

Another definition of a weak solution of (4.5) of variational type, in the Hilbert space defined by the scalar product (depending on L)

$$(u, v) = \int_\Omega \Big(\sum_{h,k}^{1,\dots,n} a_{h,k} \frac{\partial u}{\partial x_h} \frac{\partial v}{\partial x_k} + uv \Big) dx + \int_{\Sigma_2 \cup \Sigma_1} |b| uv \, d\sigma \tag{4.9}$$

has been introduced by Fichera (see his paper in "Boundary problems and differential equations" Univ. of Wisconsin press, 1960). Problem (4.5), with this definition, has been studied by Fichera himself and by R.S. Phillips and L. Sarason (in J. Math. Mech., 1967-68); Oleinik and Radkevich in [23], [32], [51] chp. I, § 6, have also proved uniqueness and existence theorem for this kind of weak solutions.

Afterwards the interest of mathematicians was addressed to the regularity properties of the weak solutions of (4.5) and also on these questions, besides the names of J. Kohn and L. Nirenberg (see Comm. Pure Appl. Math. 1967), we have to put again the name of O. Oleinik. She has found sufficient conditions on the data in order that if $f \in W^{k,\infty}(\Omega)$, the solution $u \in W^{k,\infty}(\Omega)$ (k positive integer) (see [24, [32], [51] chp. I, § 7, 8).

Also the "local" regularity of the solutions of (4.1) and the related question of the hypoellipticity of (4.1) have been studied by O. Oleinik and Radkevich. It is well known that the hypoellipticity for P.D.E. and systems was deeply studied by L. Hormander, starting with his famous paper in Act. Mat., 1955. O. Oleinik and Radkevich in the papers [24], [32], [42] and [51] chp. II, § 5, obtained new results: in particular, sufficient conditions in order that a distribution in Ω, such that $\varphi L(u) \in H^s(\mathbb{R}^n)$, $s \geq 0$, for any $\varphi \in C_0^\infty(\Omega)$, satisfies also $\varphi u \in H^s(\mathbb{R}^n)$; therefore also conditions in order that L is hypoelliptic. Moreover if the coefficients of L are analytic in Ω, they obtained in [41] [43] and [51] chp. II, § 8, necessary and sufficient conditions for the hypoellipticity of L.

Also the maximum principle for smooth solutions of (4.1) has to be reminded here. Many authors proved important results on this principle in different forms (A.D. Alexandrov, J.M. Bony, G. Fichera, C. Pucci,...). No less interesting are the results obtained by O. Oleinik and Radkevich in [23], [32], [51] chp. I, § 1, 2, 5 and chp. III, § 1.

Finally the papers [36], [31], [38], [51] chp. III, § 2, are also related to the researches of the present section, since they concern the hyperbolic degenerate equations of the type

$$\frac{\partial^2 u}{\partial t^2} = L(u) + f\,, \quad u = u(t, x_1, \dots, x_n)$$

where $L(u)$ is a second order linear elliptic-parabolic operator in the space variables $x_1, \dots, x_n$ as (4.1), with coefficients depending also on t. They were among the first papers about hyperbolic equations with multiple characteristic.

5 – The analyticity of the solutions of linear equations and systems

This group of researches is also devoted to an important subject on the field of P.D.E. and contains many deep results. The analyticity of the solutions of the linear elliptic equations or systems with analytic coefficients is a classical result. O. Oleinik has considered the problem to find other linear equations or systems, still with analytic coefficients, which have only analytic solutions, and she has dedicated to this problem many papers most of which in collaboration with E.V. Radkevich (see in particular [45-50], [52], [56], [57], [60], [61]). The approach to the problem consists in obtaining an “a priori” estimate of the solution’s analytic continuation to the complex domain in order to describe classes of equations and systems possessing at least one nonanalytic solution, and then to find new classes for which all the solutions are analytic.

Let us consider a system of linear equations with analytic coefficients in the vector valued unknown $u = (u_1, \dots, u_N)$

$$L(u) = 0 \tag{5.1}$$

in an open bounded set Ω of $\mathbb{R}^{n+1}$ and let denote by $x = (x_0, x_1, \dots, x_n)$ the points of Ω. A weak solution of (5.1) (i.e. a solution belonging to the space of distributions $(\mathcal{D}'(\Omega))^N$) is called analytic in Ω with respect to x_j if for any open set G with $\overline{G} \subset \Omega$, there exists a positive constant $\delta(G)$ such that u can be extended in the domain

$$Q_{\delta,j}(G) = \{(x, y_j) : x \in G, \quad |y_j| < \delta(G)\}$$

to a function $u(x_0, \dots, x_j + iy_j, \dots, x_n)$ analytic with respect to $x_j + iy_j$ and u and $\frac{\partial u}{\partial x_k}$, $k = 0, 1, \dots, n$, are bounded by modulus in $Q_{\delta,j}(G)$. The class of the functions u analytic with respect to x_j in Ω will be denoted by $A_j(\Omega)$. Let now $B(\Omega)$ be a Banach space consisting of weak solution of (5.1) such that the convergence of a sequence in the norm $\|\cdot\|_B$ of $B(\Omega)$ implies the convergence in $(\mathcal{D}'(\Omega))^N$ and moreover $B(\Omega) \subset A_j(\Omega)$. Then O. Oleinik proves that for any subdomain G, with $\overline{G} \subset \Omega$, there exist constants δ_0 and C_0, depending only on G, such that for any solution $u \in B(\Omega)$ of (5.1) the following estimate holds

$$\sup_{Q_{\delta_0,j}} |u| \leq C_0 \|u\|_B \,. \tag{5.2}$$

This estimate is the fundamental point of the theory. Indeed, by using suitable P.D.E. techniques, O. Oleinik finds many sufficient conditions on $L(u)$ in order to construct a family of solutions, analytic with respect to x_j, which do not satisfy (5.2) for a suitable space $B(\Omega)$ and therefore there exists at least a solution nonanalytic with respect to x_j.

We just list here only some corollaries referring in particular to the paper [49], [61] for the general results.

a) Let us consider the equation of the first order with complex-valued analytic coefficients

$$\frac{\partial u}{\partial x_0} + \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} + c(x)u = 0 \tag{5.3}$$

and suppose that $\Im a_j(x_0, x' + iy')x_0$ does not change the sign in a neighbourhood of the origin of the space $\mathbb{R}^{2n+1}$, whose point is denoted by $(x_0, x', y')(x' = (x_1, \dots, x_n),\ y' = (y_1, \dots, y_n))$. Then (5.3) has a nonanalytic solution with respect to x_j in a neighbourhood of the origin of $\mathbb{R}^{n+1}((x_0, x'))$. In particular the equation considered by S. Mizohata in two variables

$$\frac{\partial u}{\partial x_0} + i x_0^s \frac{\partial u}{\partial x_1} = 0 \quad (s \text{ odd})$$

is a particular case of (5.3).

b) Parabolic equations in the sense of Petrowsky have nonanalytic solutions with respect to the "time" variable.

c) Let us consider with the usual notations the operator

$$P(u) = \sum_{|\alpha|=m, \alpha_n=0} a_\alpha(x) D^\alpha u + \sum_{|\alpha|\le m-1} b_\alpha(x) D^\alpha u = P_m(u) + P_{m-1}(u)$$

and suppose that there exist $\tilde{x} \in \Omega$ and $\tilde{\xi} \in \mathbb{R}^{n+1}$ with $|\tilde{\xi}| \neq 0$ such that $P_m(\tilde{x}, \tilde{\xi}) \neq 0$. Then the equation has at least one nonanalytic solution with respect to x_n in any sufficiently small neighbourhood of $\tilde{x}$.

d) Let us introduce the notations $t = (x_0, \dots, x_k, 0, \dots, 0)$, $y = (0, \dots, x_{k+1}, \dots, x_l, 0, \dots, 0)$, $z = (0, \dots, x_{l+1}, \dots, x_n)$ so that $x = t + y + z$. Let us consider the equation

$$P_1(D_t)u + |t|^{2s} P_2(D_y)u + |t|^{2d} P_3(D_z)u = 0$$

where P_1, P_2, P_3 are homogeneous elliptic operators of order $2p$ with constant coefficients and s and d are integers, $d \geq 0$. O. Oleinik shows that a necessary and sufficient condition for the analyticity of the equation (i.e. all its solutions are analytic) in a neighbourhood of the origin of $\mathbb{R}^{n+1}$ is that $s = d$. In particular we find again the result of S. Baouendi and C. Goulaouic for the equation

$$\frac{\partial^2 u}{\partial {x_0}^2} + x_0^2 \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} = 0\,.$$

e) The general second order equations still quoted in section 4, for which O. Oleinik and Radkevich in [41] and [51] found necessary and sufficient conditions for the hypoellipticity, were again considered in [60], in order to describe the ones which have only analytic solutions. In particular the equation

$$a(x)\frac{\partial^2 u}{\partial {x_0}^2} + \sum_{k,j}^{1,\dots,n} a_{kj}(x)\frac{\partial^2 u}{\partial x_k \partial x_j} + b(x)\frac{\partial u}{\partial x_0} + c(x)u = 0$$

with

$$\sum_{k,j}^{1,\dots,n} a_{kj}(x)\xi'_k\xi'_j \geq c_0|\xi'|^2\,, \quad a(x) = |x|^{2s}\tilde{a}(x)\,, \quad \tilde{a}(x) > 0$$

and c_0 a positive constant, s an integer, is analytic in a neighbourhood of the origin.

We have to relate to this group of researches also the papers of O. Oleinik about the behavior of the solutions of linear elliptic and parabolic systems in unbounded domains and the extensions of the classical "Phragmen-Lindelölf" and "Liouville" theorems (see f.i. [53], [55], [58], [61]), proved using the analyticity properties of some auxiliary equations.

6 – Other researches

6.1 – Elasticity theory

Many papers by O. Oleinik, some in collaboration with V.A. Kondratiev and some with her students G.A. Yosifian and J. Kopacek, deal with the elasticity theory. The most important and deepest of them are the ones dedicated to "Saint–Venant principle" [63], [62], [82], [80]. A new and very general formulation of this principle allows many interesting applications, which are obtained by very delicate techniques. In particular the boundary value problems of two dimensional elasticity, the "Navier-Stokes system, the Von Karman system and the biharmonic equation in unbounded domains or in domains with irregular boundary and irregular "date' are deeply studied in the papers [63], [73], [78], [79], [81], [69],

[99], [70], [97], [87], and optimal estimates are obtained, which precise the behavior of the solutions at the ∞ or near the irregular boundary points. The Korn inequalities have been also extended in many directions, together with the dependence of constants in the inequalities on the shape and size to the domain and with its application to the Dirichlet or Neumann or mixed boundary value problems for the elasticity system in unbounded domains [94], [103], [100], [97], [104], [90], [120] or for the classical Boussinesq and Cerruti problems ([102], [101]).

6.2 – G-convergence and homogenization

The theory of G-convergence and its application to homogenization for P.D.E. has been introduced and studied by the School of E. De Giorgi, starting from the papers of S. Spagnolo. In collaboration with V.V. Zhikov, S.M. Kozlov and her students G.A. Yosifian, A.S. Shamaev, T.A. Shaposhnikova, O. Oleinik developed actively this new theory in many papers. We just quote some of them: [67] where the theory of the strong G-convergence is developed, [71], [74] where the G-convergence for parabolic operators is deeply studied, [75] where a new approach for the homogenization of parabolic operators with almost periodic coefficients is given, [77] where is similarly solved the homogenization problem for the elasticity system with almost periodic coefficients. But especially we have to quote the books [111], [114], [115], published by Springer, North Holland and Cambridge Univ. Press., where in particular a complete reference of the Oleinik School is given (the last one is based on the lectures delivered at the Accademia dei Lincei in 1993). The homogeneization problems in perforated domains and partially perforated domains are considered in [111], [114], [115], [118], [121], [124].

6.3 – Nonlinear degenerate parabolic equations

The paper of O. Oleinik [14] (see also [15]) became basic for the mathematical theory of nonstationary filtration of fluids in porous media. O. Oleinik introduced a definition of a weak solution, with a physical relevant meaning, of the related nonlinear parabolic degenerate equation and proved existence, uniqueness and regularity of the solution to the Cauchy problem and some of its properties. Many other important results on the quasi-linear degenerate parabolic equations are contained in the papers

[19], [27], [30]. This subject has attracted the interest of many mathematicians and physicists. Now it is one of the most actively developing branches of P.D.E., which has many applications.

6.4 – Mathematical theory of boundary layer

In 1962-1970 O. Oleinik developed the mathematical theory of boundary layer. The equations of such phenomena were proposed by J. Prandtl since 1904, but the main fundamental questions were left open. O. Oleinik proved the existence, the uniqueness and the stability of the solution of the Prandtl boundary layer system, studied its properties and gave new approaches for constructing approximate solutions (see e.g. [21], [22], [29], [34], [35] and particular the book [37] and the lecture [44]).

6.5 – General parabolic systems

Also important are the results obtained (see [64], [65], [59]) on evolution equations, especially on general parabolic systems, by the method of introducing a parameter into the equation and on "a priori" estimates of its solution, which she suggested. Using this method, she proved the uniqueness theorems and asymptotic properties of solutions of the Cauchy problem and general boundary value problems in unbounded domains for general parabolic systems and for some other classes of evolution equations.

6.6 – Spectral problems

In collaboration with her students G.A. Yosifian and A.S. Shamaev, O. Oleinik proposed in [95] (see also [96], [114]) a new approach to the spectral problems, singularly depending on a parameter and she applied this approach to many problems concerning highly non homogeneous media, or perforated domains, or domains with rapidly oscillating boundaries and other problems, (see [96], [114] chp 3). This approach was used also by many mathematicians to study spectral problems of mathematical physics.

6.7 – Boundary value problems for elliptic equations with rapidly alternating type of boundary conditions

In papers [116], [115], [111] boundary value problems for second order elliptic equations and the elasticity system with alternating type of boundary conditions are considered. On a part Γ of the boundary $\partial\Omega$ of a domain Ω the Dirichlet condition is given and on the rest γ of the boundary $\partial\Omega$ the Neumann or the mixed type of the boundary conditions are prescribed. It is assumed that γ consists of pieces of size ε and their number tends to ∞ as $\varepsilon \to 0$. The behavior of the solution u of such boundary value problems is studied as $\varepsilon \to 0$.

6.8 – Boundary value problems in non smooth domains

Many papers of O. Oleinik are dedicated to boundary value problems in nonsmooth domains (see the survey [79] and the papers [80], [81], [85], [99], [88]), where linear elliptic second order elliptic equations, the biharmonic equation and the elasticity system are considered and sharp estimates for solutions of boundary value problems near a non regular point of the boundary are obtained. In papers [63], [72], [73], [76], [78], [70], [84], [86], [89], [90], [68]) asymptotic behavior at infinity of solutions of boundary value problems for elliptic and parabolic second order equations, the biharmonic equation and the elasticity system are studied.

6.9 – Eigenvibrations of bodies with concentrated masses

In papers [91]-[93] O. Oleinik studied problems, which arise in engineering and physics and are connected with vibrations of bodies with concentrated masses. In particular, an eigenvalue problem for an elliptic equation, depending on a parameter ε, which defines the density of a body is considered and the asymptotic behavior of eigenfunctions and eigenvalues as the parameter $\varepsilon \longrightarrow 0$ is studied.

6.10 – Nonlinear elliptic equations in unbounded domains

Many papers concern nonlinear elliptic equations in unbounded domains, interesting for applications, as the Gauss equation $\Delta u + q(x)e^{-u} = 0$ (see [66]), or the equation $\Delta u + k\frac{\partial u}{\partial x_n} - |u|^{p-1}u = 0$, $p > 1$, $k \in \mathbb{R}^n$ (see [105]-[107]), or $\Delta u - e^u = 0$ (see [109]). A good reference for nonlinear equations in unbounded domains is also the monograph [115] where general second order semilinear elliptic equations in cylindrical domains are studied.

I hope to have succeded in giving you an idea of the exceptional quality and wideness of the scientific work done by O. Oleinik. Moreover O. Oleinik is a very good teacher, with a large number of students. Certainly she is one of the leaders of the Russian School in P.D.E., which has been and still is one of the best in the world.

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